Stability analysis of diagonally equipotent matrices

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Abstract

Diagonally equipotent matrices are diagonally dominant matrices for which dominance is never strict in any coordinate. They appear e.g. as Laplacian matrices of signed graphs. We show in this paper that for this class of matrices it is possible to provide a complete characterization of the stability properties based only on the signs of the entries of the matrices.

Key words: Stability Analysis; Linear Systems; Sign pattern matrices.

1 Introduction

At the boundary of the set of diagonally dominant matrices lies a special class of matrices which we call *diagonally equipotent*, meaning that for them the diagonal dominance is never strict in any coordinate. This class includes as special cases the Laplacian matrices of directed graphs used in studying the consensus problem (Mesbahi & Egerstedt, 2010; Ren et al., 2007) but also those that can be obtained when the consensus problem is relaxed to include competing interactions (modeled as negative weights of the adjacency matrix), see Altafini (2012). It also occurs for example in chemical reaction network theory, where biochemical reactions are represented as mass-action ODEs (Jayawardhana et al., 2012) (although in this case only the positive orthant is usually of interest). More generally, it occurs whenever in a linear system the off-diagonal terms (of any sign) are exactly compensated (in absolute value) by the diagonal entry of each row.

The scope of this paper is to show that the class of diagonally equipotent matrices admits a complete classification for what concerns their stability properties. In particular, irreducible diagonally equipotent matrices with negative diagonal entries are H-matrices (Berman & Plemmons, 1994; Horn & Johnson, 1991), and as such they are at least diagonally semistable (Hershkowitz & Schneider, 1985; Hershkowitz, 1992). For them, nonsingularity corresponds to asymptotic stability, while singularity corresponds to critical stability. In order to discriminate the two cases, it is not possible to use any of the standard criteria for diagonally dominant matrices (Huang, 1998; Hershkowitz, 1992; Kaszkurewicz & Bhaya, 2000), nor arguments inspired by Geršgorin theorem (Horn & Johnson, 1985). In fact, diagonally equipotent matrices have always singular comparison matrices, because by construction the latter have always zero row sums. However, this does not imply that diagonally equipotent matrices are singular. In Kolotilina (2003) a necessary and sufficient condition for nonsingularity of such matrices is provided. Combining this condition with diagonal stability, for the case of negative diagonal entries (most common in the applications mentioned above) we show in this paper that diagonally equipotent matrices are asymptotically stable if and only if all the cycles of length > 1 formed by their graph have positive sign (i.e., have an even number of negative edges).

Such a condition on the sign of the cycles appears under different names in different domains. For example it is used in linear algebra (Fiedler & Ptak, 1969; Engel & Schneider, 1973), in the theory of signed graphs (Zaslavsky, 1982), in systems and control (Sezer & Siljak, 1994; Willems, 1976) and in the theory of monotone dynamical systems (Smith, 1988). It is also used in different contexts in other disciplines, spanning from social network theory (where it is known under the name of "structural balance", see Cartwright & Harary (1956)) to statistical physics (where it has to do with the presence or less of "frustration" in spin glasses (Binder & Young, 1986)).

The use of this cycle sign property in the context of stability analysis is however new. Most remarkably, for the diagonally equipotent matrices, asymptotic stability does not depend on the numerical value of the entries of a matrix but only on their sign. As such it can be considered a "qualitative" condition (Brualdi & Shader, 1995; Maybee & Quirk, 1969), i.e., it determines the (singularity and stability) properties of the entire class of matrices having the same sign pattern, under the additional constraint of diagonal equipotence. When compared to the classical results for sign-pattern matrices (Brualdi & Shader, 1995; Hall & Li, 2006; Maybee & Quirk, 1969), the resulting qualitative conditions are remarkably different. For example, if for unconstrained matrices *sign non-singularity* (i.e., non-singularity of the class of matrices carrying a given sign pattern) corresponds to having all negative cycles on the graph of the matrix, for diagonally equipotent matrices this reduces to at least one negative cycle of length > 1. The same condition is necessary and sufficient for *qualitative stability* (i.e., asymptotic stability of the entire class of matrices carrying a given sign pattern), under the constraint of diagonal equipotence.

The extension of the conditions to generalized diagonally equipotent matrices (i.e., matrices that are diagonally equipotent up to a right multiplication by a positive diagonal matrix) is straightforward.

2 Basic notations and properties

2.1 Graphs associated to a matrix

Given a matrix $A \in \mathbb{R}^{n\times n}$, consider the directed graph $\Gamma(A)$ of A: $\Gamma(A) = \{\mathcal{V}, \mathcal{E}, A\}$ where $\mathcal{V} = \{v_1, \dots, v_n\}$ is the set of n nodes, $\mathcal{E} = \{(v_j, v_i) \text{ s. t. } a_{ij} \neq 0\}$ is the set of directed edges (we use the convention that v_i is the head of the arrow and v_j its tail) and A is its weighted adjacency matrix. A directed path $\mathcal{P}$ is a sequence of edges in $\Gamma(A)$: $\mathcal{P} = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{p-1}}, v_{i_p})\} \subset \mathcal{E}$, its length is the number of nodes it touches (i.e., p), and its sign is the sign of $a_{i_1,i_2} \dots a_{i_{p-1},i_p}$. The paths are always considered directed and simple, i.e., all nodes are distinct. When needed we indicate $\mathcal{P}_{i_1,i_p}$ the path having v_{i_1} as starting node and v_{i_p} as terminal node, and $\mathcal{V}_{\mathcal{P}}$ the set of nodes touched by a path $\mathcal{P}$. The concatenation of two paths $\mathcal{P}_{i,j}$ and $\mathcal{P}_{j,k}$ is denoted $\mathcal{P}_{i,j} \cup \mathcal{P}_{j,k}$. A matrix A is said irreducible if there does not exist a permutation matrix Π such that $\Pi^T A \Pi$ is block triangular. On $\Gamma(A)$, irreducibility of A corresponds to strong connectivity of $\Gamma(A)$, i.e., for each pair $v_i, v_j \in \mathcal{V}$ $\exists$ a path $\mathcal{P}_{ij}$ of $\Gamma(A)$ connecting them. A path $\mathcal{P}_{i_1,i_p}$ for which $v_{i_p} = v_{i_1}$ is a directed cycle. A length-1 cycle is called a loop (and corresponds to a diagonal element of A). A cycle $\mathcal{C} = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_p}, v_{i_1})\} \subset \mathcal{E}$ is negative if $a_{i_1,i_2} \dots a_{i_p,i_1} < 0$. It is positive if $a_{i_1,i_2} \dots a_{i_p,i_1} > 0$. Unless otherwise specified, also all cycles are considered directed and simple, i.e., all nodes except for v_{i_p} and v_{i_1} are distinct. Let $\mathcal{V}_{\mathcal{C}}$ be the set of nodes touched by $\mathcal{C}$.

Lemma 1 *A strongly connected graph $\Gamma(A)$ has all cycles of length > 1 positive if and only if $\exists$ a diagonal signature matrix $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$, such that $\Sigma A \Sigma$ has all nonnegative off-diagonal entries.*

For undirected graphs the equivalence of the two conditions of Lemma 1 is well-known in the theory of signed graphs (Zaslavsky, 1982). Both conditions are in fact equivalent to what is called structural balance in Cartwright & Harary (1956) (see also Altafini (2013)). The case of directed graphs treated in Lemma 1 is discussed e.g. in Fiedler & Ptak (1969); Engel & Schneider (1973). Although the version given in the Lemma can be readily deduced from these references, we provide an independent proof in the Appendix A.

Operations like the transformation

$$A \to \hat{A} = \Sigma A \Sigma, \quad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_i \in \{\pm 1\}. \tag{1}$$

are called signature similarities in linear algebra (Hershkowitz & Schneider, 1988), switching equivalences in the theory of signed graphs (Zaslavsky, 1982) and gauge transformation in the theory of Ising spin glasses (Binder & Young, 1986; Altafini, 2013). They also correspond to changes of orthant order in $\mathbb{R}^n$ and they are used for this purpose in the theory of monotone dynamical systems (Smith, 1988), see also Willems (1976). The matrices A for which (1) holds are also sometimes called Morishima matrices (Sezer & Siljak, 1994).

2.2 Linear agebraic properties

A matrix $A \in \mathbb{R}^{n\times n}$ is said Hurwitz stable if all its eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n$, have $\mathrm{Re}[\lambda_i(A)] < 0$. It is said critically stable if $\mathrm{Re}[\lambda_i(A)] \leqslant 0$, $i = 1, \ldots, n$, and $\lambda_i(A)$ such that $\mathrm{Re}[\lambda_i(A)] = 0$ have an associated Jordan block of order one. For any $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, its *comparison matrix* is given by $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{n\times n}$ where

$$\tilde{a}_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

Matrices A_1 and A_2 such that $\tilde{A}_1 = \tilde{A}_2$ are said *equimodular*. When A is such that $a_{ij} \leqslant 0 \; \forall \; i \neq j$, it is said a *Z-matrix*. It is said a (nonsingular) *M-matrix* if it is a Z-matrix and the real part of its eigenvalues is positive. When the real part of its eigenvalues is instead nonnegative, then A is said a *singular M-matrix*. A matrix A whose comparison matrix is a (singular or nonsingular) M-matrix is said an *H-matrix*. As we will see below, for the singular comparison matrices of this paper, the H-matrices can be singular or nonsingular (i.e., they belong to the "mixed class" of general H-matrices in the terminology of Bru et al. (2008)).

The matrix A is said *diagonally stable* if $\exists$ a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_i > 0$, s.t. $DA + A^T D < 0$. It is said *diagonally semistable* if $DA + A^T D \leqslant 0$.

For H-matrices we will need the following results.

Theorem 1 *(Hershkowitz (1992), Theorem 5.6) Let A be an H-matrix. Then A is diagonally stable if and only if A is nonsingular and $a_{ii} \leqslant 0$.*

Theorem 2 *(Hershkowitz (1992), Theorem 7.1) (also Hershkowitz & Schneider (1985), Theorem 3.19) Let A be an irreducible H-matrix with $a_{ii} \leqslant 0$. Then A is diagonally semistable. Furthermore, the following are equivalent:*

(1) A is singular;
(2) A is not diagonally stable;
(3) $\exists\, \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$, such that $-A = \Sigma \tilde{A} \Sigma$ and $\tilde{A}$ singular;
(4) A has a unique Lyapunov scaling factor (up to a scalar multiplication), i.e., $\exists$ unique positive diagonal D for which $DA + A^T D \leqslant 0$;
(5) $\exists$ unique (up to a scalar multiplication) positive diagonal matrix D such that $\ker(A^T D) = \ker(A)$.

Given A, consider the (reduced) row sums $R_i = \sum_{j \neq i} |a_{ij}|$, $i = 1, \ldots, n$ and the (reduced) column sums $C_i = \sum_{j \neq i} |a_{ji}|$, $i = 1, \ldots, n$. A matrix A is said *diagonally dominant* (by rows, omitted hereafter) if

$$|a_{ii}| \geqslant R_i, \quad i = 1, \ldots, n. \tag{2}$$

It is said *strictly diagonally dominant* when all inequalities of (2) are strict, and *weakly diagonally dominant* when at least one (but not all) of the inequalities (2) is strict. An irreducible weakly diagonally dominant A is often called *irreducibly diagonally dominant* in the literature. A is said *diagonally equipotent* if

$$|a_{ii}| = R_i, \quad i = 1, \ldots, n. \tag{3}$$

We also call A *doubly diagonally equipotent* if it is simultaneously rows and columns diagonally equipotent: $|a_{ii}| = R_i, \quad i = 1, \ldots, n$ and $|a_{ii}| = C_i, \quad i = 1, \ldots, n$. This obviously means $R_i = C_i$, $i = 1, \ldots, n$, a property which is called weight balance of $\Gamma(A)$ in Altafini (2013); Gharesifard & Cortés (2009) (and is called simply "balanced" in the rest of the literature on distributed consensus problems, see Mesbahi & Egerstedt (2010); Ren et al. (2007)). We indicate $\mathcal{D}$ the set of diagonally dominant matrices, with $\mathcal{DE}$ that of diagonally equipotent matrices and with $\mathcal{DDE}$ that of doubly diagonally equipotent matrices.

The following Proposition shows that diagonally equipotent matrices are H-matrices.

Proposition 1 *$A \in \mathcal{DE}$ irreducible is such that $\tilde{A}$ is a singular M-matrix with*

$$0 = \lambda_1(\tilde{A}) < \mathrm{Re}[\lambda_2(\tilde{A})] \leqslant \ldots \leqslant \mathrm{Re}[\lambda_n(\tilde{A})].$$

If in addition $a_{ii} \leqslant 0$, $i = 1, \ldots, n$, then $\mathrm{Re}[\lambda_i(A)] \leqslant 0$, $i = 1, \ldots, n$ and $\mathrm{rank}(A) \geqslant n - 1$.

Proof. For $A \in \mathcal{DE}$, $\tilde{A}$ is singular by construction and $\tilde{A}\mathbf{1} = 0$, $\mathbf{1} = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T \in \mathbb{R}^n$ i.e., $\mathbf{1}$ is a right eigenvector relative to the eigenvalue $\lambda_1(\tilde{A}) = 0$. Irreducibility implies that for $\tilde{A}$ the 0 eigenvalue has multiplicity 1. To see it, observe that $\tilde{A}$ is a singular irreducible M-matrix, hence from Theorem 6.4.16 of Berman & Plemmons (1994) $\mathrm{rank}(\tilde{A}) = n - 1$. As for the second part, for $A \in \mathcal{D}$ (and hence also for $A \in \mathcal{DE}$), the Geršgorin theorem affirms that the eigenvalues of A are located in the union of the n disks

$$\left\{ z \in \mathbb{C} \text{ s.t. } |z - a_{ii}| \leqslant \sum_{j \neq i} |a_{ij}| \right\}. \tag{4}$$

Irreducibility, combined with diagonal equipotence, implies that $a_{ii} \leqslant 0$ is in reality $a_{ii} < 0$. Diagonal dominance and $a_{ii} < 0$ guarantees that the disks are located in the left half of the complex plane and intersect the imaginary axis only in $z = 0$, which cannot be in the interior of any of the disks because of diagonal equipotence, see Horn & Johnson (1985), § 6.2. Hence $\mathrm{Re}[\lambda_i(A)] \leqslant 0$. Since A is an irreducible H-matrix, all its principal minors of order less than n are positive (see e.g. Lemma 3.20 of Hershkowitz & Schneider (1985)), hence the 0 eigenvalue of A is at most simple.

■

The classical results of the literature, combining diagonal dominance with irreducibility, are summarized by the Levy-Desplanques theorem (and its variants, see Horn & Johnson (1985)).

Theorem 3 *(Taussky, 1949) If A is strictly diagonally dominant or irreducibly diagonally dominant then it is nonsingular. If in addition $a_{ii} < 0$, then A is Hurwitz stable.*

For the specific case of diagonally equipotent matrices, we also have the following condition (whose existence was pointed out to the author by one of the reviewers of the first version of the manuscript).

Theorem 4 *(Kolotilina (2003), Theorem 2.5) $A \in \mathcal{DE}$ irreducible is nonsingular if and only if $\exists$ at least a cycle $\mathcal{C} = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_p}, v_{i_{p+1}})\}$ $(p+1=1)$ of lenght $p \geqslant 2$ such that $(-1)^p \prod_{j=1}^{p} a_{i_j i_j}^{-1} a_{i_j i_{j+1}} < 0$.*

3 Hurwitz stability of diagonally equipotent matrices

Formally, the problem investigated in the paper is the following.

Problem 1 *Determine the stability character of $A \in \mathcal{DE}$ irreducible and with negative diagonal entries.*

It follows from Proposition 1 that such $A \in \mathcal{DE}$ irreducible with negative diagonal entries is at least critically stable. However, as the following example shows, $A \in \mathcal{DE}$ may or may not be singular.

Example Consider the two matrices

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}.$$

Even if A_1, $A_2 \in \mathcal{DE}$ are equimodular, their spectra are

$$\mathrm{sp}(A_1) = \{-0.49 \pm 0.99i,\ -3.01 \pm 0.91i\}$$

$$\mathrm{sp}(A_2) = \{0,\ -2.71 \pm 1.35i,\ -3.57\}$$

hence A_1 is Hurwitz stable, while A_2 is only critically stable. ■

By construction, the Geršgorin disks (4) depend on the absolute values of the off-diagonal entries of A, hence for equimodular matrices they cannot discern properties depending on the signs of the a_{ij}. Similar considerations hold for the Cassini ovals and for all other known inclusion regions generalizing the Geršgorin disks. For example Brualdi's diagonal dominance for cycles (Brualdi, 1982) states that if all cycles $\mathcal{C} \subset \Gamma(A)$ are such that

$$\prod_{i \text{ s.t. } v_i \in \mathcal{V}_\mathcal{C}} |a_{ii}| \geqslant \prod_{i \text{ s.t. } v_i \in \mathcal{V}_\mathcal{C}} R_i$$

with strict inequality holding for at least one cycle, then A is nonsingular. This is clearly never verified for $A \in \mathcal{DE}$.

For the case of $A \in \mathcal{DE}$ with negative diagonal entries, the following Theorem provides a necessary and sufficient condition for discriminating between critically and Hurwitz stable diagonal equipotent matrices. Unlike Lemma 2 of Altafini (2013), it covers all possible cases of $A \in \mathcal{DE}$ with negative diagonal (including negative cycles of length 2).

Theorem 5 *Consider $A \in \mathcal{DE}$, irreducible with $a_{ii} < 0$, $i = 1, \ldots, n$. A is Hurwitz stable if and only if its graph $\Gamma(A)$ has at least a negative cycle of length > 1.*

The proof of Theorem 5 can be obtained by combining Theorem 1 and Theorem 4. We provide however an independent proof in Appendix B.

Notice that testing the sign of the cycles of a graph is an easy computational problem, see Iacono et al. (2010).

Since, from Proposition 1, A is at least critically stable, by applying Lemma 1, Theorem 5 has the following corollary.

Corollary 1 *$A \in \mathcal{DE}$ irreducible with $a_{ii} < 0$, $i = 1, \ldots, n$, is critically stable but not Hurwitz stable if and only if it is equivalent to a negated singular M-matrix via a change of orthant order* (1)*.*

Corollary 1 links Theorem 5 to one of the conditions of Theorem 2. Hence also the other conditions of Theorem 2 can be used for matrices having only positive cycles.

Corollary 2 *$A \in \mathcal{DE}$ irreducible with $a_{ii} < 0$, $i = 1, \ldots, n$, is critically stable but not Hurwitz stable if and only if any of the following hold:*

(1) A is diagonally semistable but not diagonally stable;
(2) $\exists$ a unique positive diagonal matrix D such that $\ker(A) = \ker(A^T D)$*.*

From Theorem 1, Theorem 5 also implies the following:

Corollary 3 *Consider $A \in \mathcal{DE}$ irreducible with $a_{ii} < 0$, $i = 1, \ldots, n$. The following conditions are equivalent:*

(1) A is nonsingular;
(2) $\Gamma(A)$ has at least a negative cycle of length > 1;
(3) A is diagonally stable.

Clearly, from Lemma 1 and Theorem 5 whenever A is Hurwitz stable it cannot be equivalent to a negated M-matrix via a change of orthant order (1).

The diagonal matrix D of Corollary 2 is also the scaling factor for the diagonal semistability (see Hershkowitz & Schneider (1985) Theorem 3.25). A straightforward consequence is the following:

Proposition 2 *Given a singular irreducible $A \in \mathcal{DE}$, $a_{ii} < 0$, $i = 1, \ldots, n$, $\exists$ a unique (up to a scalar multiplication) positive diagonal matrix D such that $DA \in \mathcal{DDE}$.*

Proof. Since $A \in \mathcal{DE}$ implies $\ker(\tilde{A}) = \mathrm{span}(\mathbf{1})$, from the last part of Corollary 2 (applied to $-\tilde{A}$) we have $\ker(\tilde{A}^T D) = \mathrm{span}(\mathbf{1})$, which can be rewritten as $\tilde{A}^T D \mathbf{1} = 0$ or $\mathbf{1}^T D \tilde{A} = 0$, i.e., $\mathbf{1}$ is the left eigenvector relative to the eigenvalue 0 for $D\tilde{A}$. Hence for $D\tilde{A}$ we have $\mathbf{1}^T D\tilde{A} = 0$ and $D\tilde{A}\mathbf{1} = 0$ i.e., $DA \in \mathcal{DDE}$. ∎

As already mentioned, double diagonal equipotence corresponds in the distributed consensus literature to weight balance (Gharesifard & Cortés, 2009), which in its turn corresponds to the symmetric part of the matrix being negative semidefinite. By definition of diagonal semistability, this is always the case for DA: $DA + A^T D \leqslant 0$. Notice from the proof of Theorem 5 in Appendix B that this is not true in general for $A \in \mathcal{DE}$ ($A \in \mathcal{DE}$ but $A \notin \mathcal{DDE}$ means $A - A^T = R - C + A_o - A_o^T \neq 0$, where A_o is the off-diagonal part of A). In the context of the distributed consensus problem, the Lyapunov scaling factor D is therefore the unique (up to scalar multiplication) right weight that renders an adjacency matrix A weight balanced.

4 Hurwitz stability for generalized diagonally equipotent matrices

A matrix B is said *generalized diagonally equipotent* (by rows) if $\exists\ P = \mathrm{diag}(p_1, \ldots, p_n)$, $p_i > 0$, $i = 1, \ldots, n$, such that BP is diagonally equipotent. In components, this can be expressed as

$$|b_{ii}| p_i = \sum_{j \neq i} |b_{ij}| p_j, \qquad i = 1, \ldots, n. \tag{5}$$

Denote $\mathcal{GDE}$ the set of generalized diagonally equipotent matrices.

Proposition 3 *Consider $B \in \mathcal{GDE}$ irreducible with $b_{ii} < 0$, $i = 1, \ldots, n$, and $A = BP \in \mathcal{DE}$, for some P diagonal positive. Then the following are equivalent.*

(1) $\Gamma(A)$ *has all cycles of length* > 1 *positive;*
(2) $\Gamma(B)$ *has all cycles of length* > 1 *positive;*
(3) $\exists\ \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$, *such that* $\Sigma B \Sigma$ *has all nonnegative off-diagonal entries;*
(4) B *is singular;*
(5) B *is critically stable but not Hurwitz stable;*
(6) B *is diagonally semistable but not diagonally stable;*
(7) B *has a unique Lyapunov scaling factor (up to a scalar multiplication), i.e.,* $\exists$ *unique positive diagonal* D *for which* $DB + B^T D \leqslant 0$*;*
(8) $\exists$ *a unique positive diagonal matrix* D *such that* $\ker(B) = \ker(B^T D)$.

Proof. Notice first that since P is positive diagonal, B irreducible if and only if A irreducible, hence $\Gamma(B)$ is strongly connected. If $p = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix}^T$, then (5) corresponds to $\tilde{B} p = 0$ i.e., p is a right eigenvector relative to the eigenvalue 0. This implies that $\tilde{B}$ is a singular, irreducible M-matrix. Hence, analogously to the proof of Proposition 1, $\mathrm{rank}(\tilde{B}) = n - 1$. If we now show that condition 1 is equivalent to condition 2, then the other characterizations follow from Lemma 1, Corollary 1 and Corollary 2. But showing this equivalence is straightforward, since P diagonal positive and $A = BP$ implies $\Gamma(B)$ has the same signs on the weights as $\Gamma(A)$. ∎

Also in the case of A nonsingular the equivalence between the properties of diagonally equipotent matrices and of their generalizations in $\mathcal{GDE}$ is totally straightforward to establish.

Corollary 4 *Consider $B \in \mathcal{GDE}$ irreducible with $b_{ii} < 0$, $i = 1, \ldots, n$, and $A = BP \in \mathcal{DE}$, for some P diagonal positive. Then the following are equivalent.*

(1) $\Gamma(A)$ has at least a negative cycle of length > 1;
(2) $\Gamma(B)$ has at least a negative cycle of length > 1;
(3) B is nonsingular;
(4) B is Hurwitz stable;
(5) B is diagonally stable.

For irreducible $A \in \mathcal{DE}$ with negative diagonal and such that Lemma 1 holds, then the generalized diagonally equipotent $B = AP^{-1}$ with P diagonal positive, is such that the eigenspace relative to the 0 eigenvalue of B is span$(P\mathbf{1})$.

5 Sign pattern properties of diagonally equipotent matrices

Recall (see e.g. Brualdi & Shader (1995)) that a matrix $A \in \mathbb{R}^{n\times n}$ determines a *qualitative class* of all matrices in $\mathbb{R}^{n\times n}$ having the same sign pattern as A, denoted $\mathcal{Q}[A]$. A square matrix A is *sign nonsingular* if every matrix in $\mathcal{Q}[A]$ is non-singular, and *sign singular* if every matrix in $\mathcal{Q}[A]$ is singular. It is said *qualitatively stable* if every matrix in $\mathcal{Q}[A]$ is Hurwitz stable. A classical result concerning sign nonsingularity is the following.

Theorem 6 *(Brualdi & Shader (1995), Theorem 3.2.1) Consider $A \in \mathbb{R}^{n\times n}$ with $a_{ii} < 0$, $i = 1, \ldots, n$. Then A is sign nonsingular if and only if every cycle of $\Gamma(A)$ is negative.*

Let us define the qualitative class of matrices which are given by sign patterns but which also obey the constraint of being diagonally equipotent (and irreducible). Denote $\mathcal{DEQ}[A]$ the class of diagonally equipotent matrices in $\mathcal{Q}[A]$. A is said *Diagonally Equipotent (DE) sign nonsingular* if every matrix in $\mathcal{DEQ}[A]$ is sign nonsingular. It is said *DE sign singular* if every matrix in $\mathcal{DEQ}[A]$ is sign singular. Analogously, we will say that A is *DE qualitatively stable* if every matrix in $\mathcal{DEQ}[A]$ is Hurwitz stable. The necessary and sufficient conditions for qualitative stability are known since the Seventies (Maybee & Quirk, 1969) and essentially amount to the lack of directed cycles of length greater than 2 and to the negativity of all cycles of length 2. More precisely, denoting β a subset of the indexes $\{1, \ldots, n\}$ and $\bar{\beta}$ its complement, letting $A[\alpha, \beta]$ be the submatrix of A obtained selecting the α rows and the β columns, we have the following theorem.

Theorem 7 *(Brualdi & Shader (1995), Theorem 10.2.2) $A \in \mathbb{R}^{n\times n}$ is qualitatively stable if and only if each of the following holds:*

(1) $a_{ii} \leqslant 0 \ \forall\, i = 1, \dots, n$*;*
(2) $a_{ij}a_{ji} \leqslant 0 \ \forall\, i \neq j$*;*
(3) $\Gamma(A)$ *has no cycle of length* > 2*;*
(4) $\det(A) \neq 0$*;*
(5) $\nexists\ \beta \subseteq \{1, \dots, n\}$, $\beta \neq \emptyset$, *such that each diagonal element of $A[\beta, \beta]$ is zero, each row of $A[\beta, \beta]$ contains at least one nonzero entry and no row of $A[\bar{\beta}, \beta]$ contains exactly one nonzero entry.*

For the qualitative class $\mathcal{DEQ}[A]$ the nonsingularity condition is in stark contrast with Theorem 6.

Theorem 8 *Consider $A \in \mathcal{DE}$ irreducible with $a_{ii} < 0$, $i = 1, \dots, n$. Then A is DE sign nonsingular if and only if $\Gamma(A)$ has a least a negative cycle of length > 1.*

Proof. Given $A \in \mathcal{DE}$ irreducible with $a_{ii} < 0$, $i = 1, \dots, n$, then Theorem 4 holds, and A is nonsingular if and only if $\Gamma(A)$ contains at least a negative cycle of length > 1. Since A and any $B \in \mathcal{DEQ}[A]$ have graphs with the same sign pattern, Theorem 5 must hold for any $B \in \mathcal{DEQ}[A]$. ∎

Similarly $B \in \mathcal{DEQ}[A]$ is DE sign singular if and only if $\Gamma(B)$ has all cycles of length > 1 positive.

Also for what concerns qualitative stability, the analysis of the $\mathcal{DEQ}[A]$ class is completely straightforward, given Theorem 5 and 8.

Theorem 9 *Consider $A \in \mathcal{DE}$ irreducible with $a_{ii} < 0$, $i = 1, \dots, n$. Then A is DE qualitatively stable if and only if $\Gamma(A)$ has a least a negative cycle of length > 1.*

Whenever the condition of Theorem 9 does not hold, then the entire class $\mathcal{DEQ}[A]$ is composed of critically stable matrices. Both Theorem 8 and Theorem 9 extend straightforwardly to the qualitative classes of matrices generated by $A \in \mathcal{GDE}$.

6 Conclusion

Diagonally equipotent matrices admit a complete classification for what concerns the stability property. While the classical arguments of diagonal dominance and of location of the Geršgorin disks are inadequate in determining the asymptotically stable cases, a purely graphical criterion, the sign of the cycles of the graph associated with the given matrix, provides instead necessary and sufficient conditions. These conditions are "qualitative" i.e., they hold for the entire class of matrices having the same sign pattern, provided we impose on them the extra constraint of diagonal equipotence.

As for all matrices for which diagonal stability conditions hold, also the diagonally equipotent matrices considered in this paper are of interest in the context of decentralized systems and of distributed control, (Kaszkurewicz & Bhaya, 2000; Siljak, 1978), especially in view of their Laplacian-like structure. In addition, the signed matrices of the diagonally equipotent class still enjoy most of the properties of positive systems, such as the Perron-Frobenius theorem. It can therefore be expected that also recent decentralized synthesis results such as Rantzer (2011) can be extended beyond positively dominated systems.

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A Proof of Lemma 1

Let us first show that the sign of a directed cycle of length > 1 is invariant to the transformation (1). Assume without loss of generality that $\sigma_i = -1$ and $\sigma_j = +1$ for $j \neq i$. On the graph $\Gamma(A)$, the operation (1) corresponds to changing sign to all edges adjacent to the node v_i. Since each cycle intersects v_i in 0 or 2 edges, the parity of the cycle is unaltered by such an operation. It follows therefore that if $\exists\, \Sigma$ such that $\Sigma A \Sigma$ has all nonnegative off-diagonal entries then all cycles of A of length > 1 must be positive. Conversely, for strongly connected graphs, when all cycles are positive the existence of Σ can be shown through the following explicit construction. Consider a cycle $\mathcal{C}$ containing a nonzero (even) number of negative edges and two of its nodes $v_i, v_j \in \mathcal{V}_{\mathcal{C}}$. Consider the paths $\mathcal{P}_{ij}$ and $\mathcal{P}_{ji}$ such that $\mathcal{P}_{ij} \cup \mathcal{P}_{ji} = \mathcal{C}$. Assume for example that $\text{sgn}(\mathcal{P}_{ij}) = \text{sgn}(\mathcal{P}_{ji}) < 0$. Call $\mathcal{V}^1 = \{v_i\}$ and $\mathcal{V}^2 = \{v_j\}$. Consider

$v_k \neq v_i, v_j$. Since $\Gamma(A)$ is strongly connected, in $\mathcal{E}$ $\exists$ $\mathcal{P}_{ik}$, $\mathcal{P}_{ki}$, $\mathcal{P}_{jk}$ and $\mathcal{P}_{kj}$ such that $\mathcal{P}_{ik} \cup \mathcal{P}_{ki}$ and $\mathcal{P}_{jk} \cup \mathcal{P}_{kj}$ are positive directed cycles. Assume without loss of generality that $\mathrm{sgn}(\mathcal{P}_{ik}) = \mathrm{sgn}(\mathcal{P}_{ki}) < 0$. Then it must be necessarily $\mathrm{sgn}(\mathcal{P}_{jk}) = \mathrm{sgn}(\mathcal{P}_{kj}) > 0$, otherwise the (possibly nonsimple) directed cycles formed by $\mathcal{P}_{ik} \cup \mathcal{P}_{kj} \cup \mathcal{P}_{ji}$ and by $\mathcal{P}_{ki} \cup \mathcal{P}_{ij} \cup \mathcal{P}_{jk}$ must be negative. If the cycles $\mathcal{P}_{ik} \cup \mathcal{P}_{kj} \cup \mathcal{P}_{ji}$ and $\mathcal{P}_{ki} \cup \mathcal{P}_{ij} \cup \mathcal{P}_{jk}$ are not simple then they must be unions of directed cycles, e.g. $\mathcal{P}_{ik} \cup \mathcal{P}_{kj} \cup \mathcal{P}_{ji} = \bigcup_r \mathcal{C}_r$ for some $r > 1$. But also in this case $\mathrm{sgn}(\mathcal{P}_{ik} \cup \mathcal{P}_{kj} \cup \mathcal{P}_{ji}) < 0$ implies that at least one of the cycles $\mathcal{C}_1, \ldots, \mathcal{C}_r$ must be negative. The condition must be identical for each choice of $\mathcal{P}_{ik}$, $\mathcal{P}_{ki}$, $\mathcal{P}_{jk}$ and $\mathcal{P}_{kj}$. In fact, assume by contradiction that $\exists$ alternative paths $\mathcal{Q}_{ik}$ and $\mathcal{Q}_{ki}$ such that $\mathcal{Q}_{ik} \cup \mathcal{Q}_{ki}$ is a positive cycle and that $\mathrm{sgn}(\mathcal{Q}_{ik}) = \mathrm{sgn}(\mathcal{Q}_{ki}) > 0$. Combining with $\mathcal{P}_{ik}$, $\mathcal{P}_{ki}$, we have the two (again possibly nonsimple) directed cycles $\mathcal{P}_{ik} \cup \mathcal{Q}_{ki}$ and $\mathcal{P}_{ki} \cup \mathcal{Q}_{ik}$ of negative sign, which is a contradiction. The argument is identical for $\mathcal{P}_{jk}$ and $\mathcal{P}_{kj}$. We can therefore add v_k to one and only one of the two sets $\mathcal{V}^1$ and $\mathcal{V}^2$ (in the case above $\mathcal{V}^2 = \mathcal{V}^2 \cup v_k$). The construction can be iterated for all remaining nodes: for a node v_r it will be either:

$$\left.\begin{array}{l} \mathrm{sgn}(\mathcal{P}_{ir}) = \mathrm{sgn}(\mathcal{P}_{ri}) > 0 \ \ \forall v_i \in \mathcal{V}^1 \\ \mathrm{sgn}(\mathcal{P}_{jr}) = \mathrm{sgn}(\mathcal{P}_{rj}) < 0 \ \forall v_j \in \mathcal{V}^2 \end{array}\right\} \Longrightarrow \mathcal{V}^1 = \mathcal{V}^1 \cup v_r$$

or

$$\left.\begin{array}{l} \mathrm{sgn}(\mathcal{P}_{ir}) = \mathrm{sgn}(\mathcal{P}_{ri}) < 0 \ \ \forall v_i \in \mathcal{V}^1 \\ \mathrm{sgn}(\mathcal{P}_{jr}) = \mathrm{sgn}(\mathcal{P}_{rj}) > 0 \ \forall v_j \in \mathcal{V}^2 \end{array}\right\} \Longrightarrow \mathcal{V}^2 = \mathcal{V}^2 \cup v_r.$$

Eventually we obtain a bipartition of the nodes $\mathcal{V} = \mathcal{V}^1 \cup \mathcal{V}^2$, $\mathcal{V}^1 \cap \mathcal{V}^2 = \emptyset$. By constructing $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_i = +1$ if $v_i \in \mathcal{V}^1$ and $\sigma_i = -1$ if $v_i \in \mathcal{V}^2$, it is straightforward to verify that $\Sigma A \Sigma$ has to have all nonnegative off-diagonal entries. ∎

B Proof of Theorem 5

From Proposition 1, $\mathrm{Re}[\lambda_i(A)] \leqslant 0$ and A is at least critically stable.

"$\Longleftarrow$": Assume by contradiction that A is not Hurwitz stable, i.e., that 0 is an eigenvalue of A of eigenvector $x \neq 0$. Lemma 6.2.3 of Horn & Johnson (1985) implies that for the components of x, $|x_i| = |x_j| = \xi \neq 0 \ \forall\, i, j, = 1, \ldots, n$, and that all Geršgorin disks pass through 0. We can then write $Ax = 0$, $x^T A^T = 0$ and

$$x^T A^s x = 0, \tag{B.1}$$

where $A^s = \frac{A+A^T}{2}$ denotes the symmetric part of A. Consider the row sums R_i, $i = 1, \ldots, n$ and the column sums C_i, $i = 1, \ldots, n$. Assume $a_{ii} < 0$. $A \in \mathcal{DE}$ implies $-a_{ii} = R_i$, $i = 1, \ldots, n$ (while in general $|a_{ii}| \neq C_i$, unless we have weight balance). Denote $R = \mathrm{diag}(R_1, \ldots, R_n)$ and $C = \mathrm{diag}(C_1, \ldots, C_n)$ and let A_d and A_o be respectively the diagonal and off-diagonal parts of A. Then $A = A_d + A_o = -R + A_o$ and $A^s = -R + \frac{A_o + A_o^T}{2}$. If we rewrite A^s in the following form

$$A^s = -\frac{R-C}{2} - \frac{R+C}{2} + \frac{A_o + A_o^T}{2},$$

then (B.1) becomes

$$\begin{aligned} 0 = & -x^T \left(\frac{R-C}{2}\right) x - x^T \left(\frac{R+C}{2}\right) x \\ & + x^T \left(\frac{A_o + A_o^T}{2}\right) x. \end{aligned} \tag{B.2}$$

Expanding the first term of (B.2) and using the fact that $x_i^2 = \xi^2 \; \forall \, i = 1, \ldots, n$

$$\begin{aligned} \sum_{i=1}^n x_i^2 (R_i - C_i) = & \sum_{i=1}^n x_i^2 \left(\sum_{j \neq i} (|a_{ij}| - |a_{ji}|) \right) \\ = & \, \xi^2 \sum_{\substack{i,j=1 \\ j \neq i}}^n (|a_{ij}| - |a_{ji}|) = 0. \end{aligned}$$

The rest of (B.2) can be written as

$$\begin{aligned} 0 = & \frac{1}{2} \sum_{i=1}^n \sum_{j>i} \Big(-x_i^2 (|a_{ij}| + |a_{ji}|) \\ & - x_j^2 (|a_{ij}| + |a_{ji}|) + 2 x_i x_j (a_{ij} + a_{ji}) \Big). \end{aligned}$$

In the case $a_{ij} a_{ji} \geqslant 0 \; \forall \, i, j = 1, \ldots, n$ (i.e., all length-2 cycles of $\Gamma(A)$ are positive), then $|a_{ij}| + |a_{ji}| = |a_{ij} + a_{ji}|$, which implies

$$\begin{aligned} 0 = & -\frac{1}{2} \sum_{i=1}^n \sum_{j>i} \Big(x_i^2 (|a_{ij} + a_{ji}|) \\ & + x_j^2 (|a_{ij} + a_{ji}|) - 2 x_i x_j (a_{ij} + a_{ji}) \Big) \\ = & -\frac{1}{2} \sum_{i=1}^n \sum_{j>i} |a_{ij} + a_{ji}| \left(x_i^2 + x_j^2 - 2 x_i x_j \mathrm{sgn}(a_{ij} + a_{ji}) \right) \\ = & -\frac{1}{2} \sum_{i=1}^n \sum_{j>i} |a_{ij} + a_{ji}| \left(x_i - \mathrm{sgn}(a_{ij} + a_{ji}) x_j \right)^2. \end{aligned} \tag{B.3}$$

This has the form of a sum of squares, implying that each term of the summation must be equal to 0 in correspondence of the eigenvector x. On the graph

$\Gamma(A)$, consider a negative cycle $\mathcal{C} = \{(v_{i_1}, v_{i_2}),\, (v_{i_2}, v_{i_3}), \ldots, (v_{i_p}, v_{i_1})\} \subset \mathcal{E}$. Denote $\mathcal{S}_+ = \{(v_j, v_i) \in \mathcal{C} \text{ s. t. } a_{ij} > 0\}$ its subset of positive edges and $\mathcal{S}_- = \{(v_j, v_i) \in \mathcal{C} \text{ s. t. } a_{ij} < 0\}$ that of negative edges. Since $\operatorname{sgn}(a_{ij}+a_{ji}) = \operatorname{sgn}(a_{ij})$ in (B.3) we have for the subset of edges in $\mathcal{C}$

$$\begin{aligned} 0 = & -\frac{1}{2} \sum_{(v_j, v_i) \in \mathcal{S}_+} |a_{ij} + a_{ji}| \left(x_i - \operatorname{sgn}(a_{ij} + a_{ji}) x_j\right)^2 \\ & -\frac{1}{2} \sum_{(v_j, v_i) \in \mathcal{S}_-} |a_{ij} + a_{ji}| \left(x_i - \operatorname{sgn}(a_{ij} + a_{ji}) x_j\right)^2 \end{aligned}$$

which implies

$$\begin{cases} x_i = x_j & (v_j, v_i) \in \mathcal{S}_+ \\ x_i = -x_j & (v_j, v_i) \in \mathcal{S}_-. \end{cases}$$

We claim that this system has no nonzero solution of the type $|x_i| = \xi \neq 0$, $i = 1, \ldots, n$, when $\mathcal{S}_-$ contains an odd number of edges. Since, in correspondence of $(v_j, v_i) \in \mathcal{S}_+$, $x_i = x_j$, it is possible to reduce the cycle dropping all edges in $\mathcal{S}_+$ and merging the corresponding node variables. If k is the number of edges in $\mathcal{S}_-$, then we are left with the subset of equations

$$\begin{aligned} x_{i_1} &= -x_{i_2} \\ x_{i_2} &= -x_{i_3} \\ &\vdots \\ x_{i_k} &= -x_{i_1}, \end{aligned} \tag{B.4}$$

from which we have, since k is odd,

$$\begin{aligned} x_{i_1} &= x_{i_3} = \ldots = x_{i_k} \\ x_{i_2} &= x_{i_4} = \ldots = x_{i_{k-1}}. \end{aligned}$$

However, from the last eq. of (B.4), we also have $x_{i_1} = -x_{i_1}$, $|x_{i_1}| = \xi \neq 0$, which is a contradiction. It implies that the terms of (B.3) belonging to a negative cycle cannot be all equal to 0 and hence that 0 cannot be an eigenvalue of A. In the case $a_{ij} a_{ji} < 0$ for at least a pair of edges in $\mathcal{E}$, then $|a_{ij} + a_{ji}| < |a_{ij}| + |a_{ji}|$, i.e., in place of (B.3) we have

$$\begin{aligned} 0 = & -\sum_{i_k, j_k \in \mathcal{K}} \epsilon_k (x_{i_k}^2 + x_{j_k}^2) \\ & -\frac{1}{2} \sum_{i=1}^{n} \sum_{j>i} |a_{ij} + a_{ji}| \left(x_i - \operatorname{sgn}(a_{ij} + a_{ji}) x_j\right)^2 \end{aligned} \tag{B.5}$$

where $\mathcal{K}$ is the set of index pairs i_k, j_k for which $a_{i_k,j_k}a_{j_k,i_k} < 0$ and $\epsilon_k > 0$. Clearly (B.5) can never be true for x such that $|x_i| = \xi \neq 0$, $i = 1, \ldots, n$ unless $\mathcal{K}$ is empty. Therefore even in presence of a single length-2 negative cycle 0 cannot be an eigenvalue of A.

"$\Longrightarrow$": Assume A Hurwitz stable and by contradiction that $\Gamma(A)$ has no negative cycle of length > 1. Then from Lemma 1 $\exists$ $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$, such that $\hat{A} = \Sigma A \Sigma$ has all non-negative off-diagonal entries and $\hat{a}_{ij} = |a_{ij}|$ $\forall i, j = 1, \ldots, n$, $i \neq j$. Since $\hat{a}_{ii} = a_{ii} < 0$, it is $\hat{A} = -\tilde{A}$. From Proposition 1, however, $\tilde{A}$ must be singular, hence we have a contradiction. ∎

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