Notes on Quantum Field Theory

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Caution: these are preliminary notes in progress.
They do NOT cover all the topics discussed in the course.

Contents

Chapter 1

Introduction

Quantum Field Theory (QFT) is the fundamental tool that is currently used for the description of physics at very short distances and high energies. Since high energy implies relativistic motion, QFT has to nicely combine special relativity with quantum mechanics. The description in terms of "fields", indeed, arise to have a manifestly relativistic invariant description, where time and space are treated (almost) on equal footing. One might think that including special relativity in quantum mechanics should be possible without drastic consequences. This is not true, and its reason is intuitively clear. At very short time scales, the energy-time uncertainty principle tells us that particles with energy $E \geq mc^2$ could be created from the vacuum for a time $t \sim \hbar/E$ (virtual particles), before disappearing again in the vacuum. This effect is totally negligible in studying physical systems at low energies and long time scales, but it becomes relevant for physical processes whose time scale is $t \sim \hbar/E$ or less. A quantum description in terms of single-particle wave function is then inadequate and a more powerful description is needed. This formulation is in fact Quantum Field Theory. It leads to striking consequences, such as the prediction of anti-particles and an understanding of the spin-statistic relation between particles. The experimental successes of QFT are impressive, in particular when applied to the description of electrodynamics, giving rise to the Quantum Electro Dynamics (QED).

Most of the considerations in these lectures are devoted to the study of fields which are weakly interacting, namely in which the interactions can be studied in a perturbative fashion, starting from the description of free fields.

These notes *do assume* that the reader has a basic knowledge of Quantum Field Theory. Quantization of spin 0, spin 1/2 and abelian spin 1 fields are assumed, as well as basic notions of the path integral formulation of QFT (including Berezin integration for fermions), Feynman rules, basic knowledge of renormalization and the notion of functional generators of disconnected, connected and one-particle irreducible (1PI) Green functions.

These notes cover roughly two thirds of the QFT course given at SISSA by the author, in collaboration with Prof. R. Iengo until the academic year 2010-2011, and since then with Prof. Andrea Gambassi. We hope to be able to write the remaining part in the near future. The topics marked with a $*$ in the text are optional for students in the Astroparticle curriculum.

These notes are *very preliminary*, they surely contain many typos, imprecisions, etc. I have decided to omit most of the references to the original literature for simplicity.

I hope that the students will help me in improving the notes and in spotting the many mistakes in there.

Chapter 2

General Renormalization Theory

Chapter 3

Non-Abelian Gauge Theories

3.1 Introduction and Classical Analysis

Non-abelian gauge theories are at the base of our current understanding of particle physics. Both the strong and the electroweak interactions are described in terms of them. These theories are based on a generalization of the QED $U(1)$ gauge symmetry, where two transformations do not necessarily commute with each other (hence the name non-abelian). Before describing them, let us quickly review the role of the $U(1)$ symmetry in QED. We assume an invariance of the Lagrangian under local (i.e. space-time dependent) transformations under which any field ψ carrying charge q transforms as

$$\psi(x) \to e^{iq\alpha(x)}\psi(x) \,. \tag{3.1.1}$$

Due to the space-time dependence of the transformation, the derivative of the field $\partial_\mu\psi$ does not transform covariantly:

$$\partial_\mu\psi(x) \to e^{iq\alpha(x)}\Big(\partial_\mu\psi(x) + iq\psi(x)\partial_\mu\alpha(x)\Big) \,. \tag{3.1.2}$$

We then add a gauge field (photon) A_μ that transforms inhomogeneously:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x) \,, \tag{3.1.3}$$

that allows us to define a new (covariant) derivative transforming nicely covariantly under $U(1)$ gauge transformations:

$$D_\mu\psi(x) \equiv \partial_\mu\psi(x) - iqA_\mu(x)\psi(x) \to e^{i\alpha(x)}D_\mu\psi(x) \,. \tag{3.1.4}$$

Under infinitesimal $U(1)$ transformations parametrized by $\epsilon(x) \ll 1$, we obviously have

$$\delta_\epsilon\psi(x) = iq\epsilon(x)\psi(x) \,, \qquad \delta_\epsilon A_\mu(x) = \partial_\mu\epsilon(x) \,. \tag{3.1.5}$$

The QED Lagrangian is then constructed by forming gauge invariant combinations of ψ, $D_\mu\psi$ and of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3.1.6}$$

Non-abelian gauge theories are constructed by generalizing the above construction to a set of fields ψ_l, with l labeling the different fields. We now assume that under an infinitesimal gauge transformations, the ψ_l are rotated among each other:

$$\delta_\epsilon \psi_l(x) = i\epsilon^\alpha (t_\alpha)_l^m \psi_m(x) \,, \tag{3.1.7}$$

where t_α are a set of constant matrices, labelled by the index α, satisfying the relation

$$[t_\alpha, t_\beta] = iC_{\alpha\beta}^\gamma t_\gamma \,. \tag{3.1.8}$$

The coefficients $C_{\alpha\beta}^\gamma$ are a set of real parameters denoted the structure constants. They are manifestly antisymmetric in the two lower indices: $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$. We have written the index γ upstairs, because in principle there might be a non-trivial metric $g_{\alpha\beta}$ in group space, to raise and lower the group indices. In the physically most interesting cases, this metric can be chosen to be the identity. From now on, we will assume a trivial group metric. Correspondingly, the position of the indices α, β, etc. will be irrelevant. In this case, it can be shown that the structure constants $C_{\alpha\beta\gamma}$ become antisymmetric in all three indices.

Matrices satisfying (3.1.8) form a so called Lie algebra, namely define infinitesimal transformations of a so called Lie group. Differently from the QED case, the group of transformations (3.1.7) is multidimensional and this explains the origin of the index α, that runs from 1 up to $\dim G$, the number of dimensions of the group. The matrices t_α are nothing else than a set of generators of the Lie group, meaning that any group transformation can be written in terms of these matrices, that are linearly independent from each other. Like any set of matrices, the t_α's also satisfy the so called Jacobi identity

$$[[t_\alpha, t_\beta], t_\gamma] + [[t_\gamma, t_\alpha], t_\beta] + [[t_\beta, t_\gamma], t_\alpha] = 0 \,. \tag{3.1.9}$$

Using (3.1.8), this identity implies the following constraints among the structure constants:

$$C_{\alpha\beta}^\omega C_{\omega\gamma}^\delta + C_{\gamma\alpha}^\omega C_{\omega\beta}^\delta + C_{\beta\gamma}^\omega C_{\omega\alpha}^\delta = 0 \,. \tag{3.1.10}$$

Depending on how many fields ψ_l we have, the matrices t_α are said to be in different representations of the Lie algebra. In general, there is an infinite set of matrices (of different size), satisfying (3.1.8) and (3.1.9). Among these, a special role is played by the "adjoint" representation. This is the representation in which the generators t_α^{Adj} are $\dim G \times \dim G$ matrices, i.e. the indices l, m coincide with the indices α, β. An explicit form of this representation is given by

$$(t_\alpha^{Adj})_\gamma^\beta = iC_{\alpha\gamma}^\beta \,. \tag{3.1.11}$$

Indeed, it is straightforward to check that the Jacobi identity (3.1.10) can be rewritten as

$$[t_\alpha^{Adj}, t_\beta^{Adj}] = C_{\alpha\beta}^\gamma t_\gamma^{Adj} \,. \tag{3.1.12}$$

We do not enter here in any detail concenring the definition of Lie algebras, Lie groups, etc., because all this will be extensively treated in the group theory course. However, we need to introduce another relevant representation, the fundamental. It might be useful to consider a simple example of non-abelian Lie group, $G = SU(2)$. $SU(2)$ is defined by the set of 2×2 unitary matrices $U^\dagger = U$ with unit determinant (that's why the name S(for special, with unit determinant)U(for unitary)(2)). It is straightforward to check that this set is in fact a group and it is three dimensional. Any $SU(2)$ matrix U can be written as

$$U = e^{i\omega_\alpha t_\alpha} \,, \quad \alpha = 1, 2, 3 \,, \tag{3.1.13}$$

where $t_\alpha = \sigma_\alpha/2$ and σ_α are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.1.14}$$

The structure constants are

$$C_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} \,, \tag{3.1.15}$$

where $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric tensor, with $\epsilon_{123} = +1$. The above 2×2 matrices t_α, that enter in the definition of the group $SU(2)$, form the fundamental representation. The 3×3 matrices (3.1.11) define instead the adjoint representation. We can then have field "doublets" D_l ($l = 1, 2$), that transforms as $D_l \to U_{lm} D_m$, with U as in (3.1.13) or field "triplets" T_l ($l = 1, 2, 3$), transforming as $T_l \to (U^{Adj})_{lm} D_m$, where

$$U^{Adj} = e^{i\omega_\alpha t_\alpha^{Adj}} \,, \quad \alpha = 1, 2, 3 \,. \tag{3.1.16}$$

In addition to the fundamental and adjoint representations, there are an infinite number of other representations, labelled by the angular momentum J, of dimension $2J + 1$, with J any positive integer or semi-integer number.

Similar considerations apply for more general groups, such as $SU(N)$, with $N > 2$, or $SO(N)$, with $N > 3$.[1] Instead of defining $\dim r \times \dim r$ matrices U^r, for each different

[1] The group $SO(3)$ is locally isomorphic to $SU(2)$ and is defined by the same 3 generators $t_\alpha^{Adj.}$ introduced above for $SU(2)$. The group $SO(2)$ is isomorphic to $U(1)$.

representation r, it is practically more convenient to write the transformation properties of r in terms of, say, the matrices U in the fundamental representation. All we need to know is how the representation r in question appears in the tensor product of fundamentals. For instance, for $SU(N)$ groups, we have $\mathbf{N} \otimes \bar{\mathbf{N}} = \mathbf{N^2-1} \oplus \mathbf{1}$, where $\mathbf{N}$ and $\bar{\mathbf{N}}$ are the fundamental and anti-fundamental (i.e. complex conjugate) representations, and $\mathbf{N^2-1}$ is the adjoint one. A field ψ in the adjoint representation of $SU(N)$ can correspondingly be written as an $N \times N$ matrix field ψ_{ij}, transforming as

$$\psi \to U\psi U^\dagger \,. \tag{3.1.17}$$

Coming back to physics, given the transformation (3.1.7), we add gauge fields A_μ^α, one for each independent direction in field space, so that we can form a covariant derivative. In analogy to the $U(1)$ case, the transformations of the gauge fields A_μ^α must contain a term of the form $\partial_\mu \epsilon_\alpha$. Contrary to the $U(1)$ case, this cannot be the end of the story, since α is an index in the adjoint representation. The natural guess for the infinitesimal transformation of A_μ^α is then

$$\delta A_\mu^\alpha = \partial_\mu \epsilon^\alpha + i\epsilon^\beta (t_\beta^{Adj.})^\alpha_\gamma A_\mu^\gamma = \partial_\mu \epsilon^\alpha + C^\alpha_{\beta\gamma} A_\mu^\beta \epsilon^\gamma \,. \tag{3.1.18}$$

The covariant derivative is defined as

$$D_\mu \psi_l \equiv \partial_\mu \psi_l - i A_\mu^\alpha (t_\alpha)_l^m \psi_m \,. \tag{3.1.19}$$

It is straightforward to show that this guess is in fact correct and the covariant derivative transforms as it should:

$$\delta(D_\mu\psi)_l = i\epsilon^\alpha (t_\alpha)_l^m (D_\mu \psi)_m . \tag{3.1.20}$$

It is often convenient to write the components of the gauge fields A_μ^α in matrix form by defining

$$A_\mu = A_\mu^\alpha t_\alpha \,. \tag{3.1.21}$$

For simplicity of notation we have omitted, and from now on will be done most of the time, the gauge group indices in (3.1.20). The finite form of (3.1.18) is easily found by demanding that the gauge transformed connection $A \to A^U$, is such that

$$D_\mu(A)\psi \to U D_\mu(A)\psi \,. \tag{3.1.22}$$

for any field in a given representation r, transforming as $\psi \to \psi^U = U\psi$, where U is defined as

$$U(x) = e^{i\Lambda_\alpha(x) t_\alpha} \,. \tag{3.1.23}$$

We have

$$\partial_\mu - iA_\mu \to \partial_\mu(U\psi) - iA_\mu^U U\psi = U(\partial_\mu\psi - iA_\mu\psi) + (\partial_\mu U)\psi + (iUA_\mu - iA_\mu^U U)\psi\,. \quad (3.1.24)$$

Demanding that the last three terms in (3.1.22) vanish uniquely fixes

$$A_\mu^U = UA_\mu U^{-1} - i(\partial_\mu U)U^{-1}\,. \quad (3.1.25)$$

The first term in eq.(3.1.25) is the one expected from a field in the adjoint representation (see eq.(3.1.17)), while the second is the inohomogenous one characterizing a gauge connection. For the $U(1)$ case eq.(3.1.25) trivially reduces to eq.(3.1.3), with $\Lambda_\alpha = \alpha$. For infinitesimal transformations, $\Lambda_\alpha = \epsilon_\alpha$, the transformation (3.1.25) correctly reproduces eq.(3.1.18), as it should.

The generalization of the $U(1)$ field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ can be found by recalling that for any field ψ with charge q, the field strength is proportional to the (commutator part of the) action of two covariant derivatives acting on the field itself. Taking $q = 1$, we have

$$D_\mu D_\nu\psi = (\partial_\mu - iA_\mu)(\partial_\nu - iA_\nu)\psi = (\partial_\mu\partial_\nu - i\partial_\mu A_\nu - iA_\nu\partial_\mu - iA_\mu\partial_\nu - A_\mu A_\nu)\psi\,. \quad (3.1.26)$$

Only the second term in the above equation survives when we take the antisymmetric part combination in $\mu \leftrightarrow \nu$ and we get

$$[D_\mu, D_\nu]\psi = -iF_{\mu\nu}\psi\,. \quad (3.1.27)$$

The field strength in a non-abelian gauge theory can be defined by generalizing eq.(3.1.27). Denoting ψ a field in an arbitrary representation of the gauge group, eq.(3.1.26) still applies, but now the last term does not vanish, since A_μ is a matrix. We have

$$[D_\mu, D_\nu]\psi = -i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])\psi \equiv -iF_{\mu\nu}\psi\,. \quad (3.1.28)$$

In components, $F_{\mu\nu} = F_{\mu\nu}^\alpha t_\alpha$, with

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + C_{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma\,. \quad (3.1.29)$$

Contrary to the abelian case, the field strength $F_{\mu\nu}$ is not gauge invariant. Its transformation properties can easily be found by considering the gauge transform of (3.1.28):

$$[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi = -iUF_{\mu\nu}\psi = -iF_{\mu\nu}^U\psi^U\,, \quad (3.1.30)$$

from which we immediately get

$$F_{\mu\nu}^U = UF_{\mu\nu}U^{-1}\,. \quad (3.1.31)$$

In non-abelian gauge theories the gauge field strength transforms in the adjoint representation of the gauge group. The most general gauge-invariant Lagrangian can be written as a Lorentz invariant functional of matter fields and their covariant derivatives, and of the field strengths $F_{\mu\nu}$ and their covariant derivatives. At the level of dimension 4 (or less) operators, we have

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{Matter} \tag{3.1.32}$$

where

$$\begin{aligned} \mathcal{L}_{YM} &= -\frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \,, & (3.1.33) \\ \mathcal{L}_{Matter} &= \sum_i \bar{\psi}_i (i\not{D} - m_i)\psi_i + \sum_j |D_\mu \phi_j|^2 - V(\psi_i, \phi_j)\,. & (3.1.34) \end{aligned}$$

In eq.(3.1.33), we have introduced a dimensionless parameter g that is identified as the gauge coupling constant of the non-abelian theory. By rescaling the gauge fields as

$$A_\mu^\alpha \to g A_\mu^\alpha \tag{3.1.35}$$

we get canonical kinetic terms, having normalized the generators as

$$\mathrm{Tr}\, t^\alpha t^\beta = \frac{\delta^{\alpha\beta}}{2} \tag{3.1.36}$$

in the fundamental representation. The coupling g now appears in all covariant derivatives in the matter sector and in the self-coupling of the gauge fields in the Yang-Mills Lagrangian. Gauge invariance requires that the same coupling g governs all these interactions. In eq.(3.1.34) i and j run over all fermions and scalars in the theory, D_μ are the covariant derivatives in the appropriate representations and V encodes the scalar potential of the scalar field ϕ_j and their Yukawa interactions with the fermions ψ_i. Gauge invariance requires that $\delta_\epsilon V = 0$.

The carriers of the force associated to the non-abelian group, that we generally denote by "gluons", are themselves subject to the force they carry. The equations of motion (e.o.m.) for A_μ^α deduced from the Lagrangian (3.1.32) are

$$\partial_\mu F_\alpha^{\mu\nu} = -g C_{\alpha\beta\gamma} A_\mu^\beta F_\gamma^{\mu\nu} - \frac{\delta \mathcal{L}_{Matter}}{\delta A_\nu^\alpha} \equiv -\mathcal{J}_\nu^\alpha \tag{3.1.37}$$

where $\mathcal{J}_\nu^\alpha$ are the $\dim G$ conserved currents associated to the symmetry group G:

$$\partial_\mu \mathcal{J}_\alpha^\mu = 0. \tag{3.1.38}$$

The e.o.m. (3.1.37), written in terms of $\mathcal{J}_\mu^\alpha$ are not covariant. It is convenient to shift the gluon contribution to the current, the first term in the second relation in eq.(3.1.37), to the left-hand side of that equation. In doing so, the e.o.m. read

$$D_\nu F^{\mu\nu}_\alpha = -J^\mu_\alpha \,, \tag{3.1.39}$$

where

$$\begin{aligned} D_\rho F^{\mu\nu}_\alpha &= \partial_\rho F^{\mu\nu}_\alpha - ig(t^{Adj}_\beta)_{\alpha\gamma} A^\beta_\rho F^{\mu\nu}_\gamma \,, \\ J^\mu_\alpha &= \frac{\delta \mathcal{L}_{Matter}}{\delta A^\alpha_\mu} = \frac{\delta \mathcal{L}_{Matter}}{\delta D_\mu \Psi_I}(-igt_\alpha \Psi_I)\,. \end{aligned} \tag{3.1.40}$$

In eq.(3.1.40) we have rewritten the form of the current J^α_μ to make explicit its covariant properties, in contrast to to the conserved current $\mathcal{J}^\alpha_\mu$. The field Ψ encodes both fermions and scalars and $I = (i, j)$. The current J^α_μ is covariantly conserved, namely $D_\mu J^\mu_\alpha = 0$. Indeed,

$$D_\mu J^\mu_\alpha = -D_\nu D_\mu F^{\mu\nu}_\alpha = [D_\mu, D_\nu] F^{\mu\nu}_\alpha = i F^\beta_{\nu\mu} (t^{Adj}_\beta)_{\alpha\gamma} F^{\mu\nu}_\gamma = C_{\alpha\beta\gamma} F^\beta_{\mu\nu} F^{\mu\nu}_\gamma = 0. \tag{3.1.41}$$

The field strength $F_{\mu\nu}$ satisfies another important relation, called Bianchi identity. It is a consistency relation, and can be derived starting from the Jacobi identity (3.1.9), applied to covariant derivatives:

$$[D_\mu, [D_\nu, D_\rho]] + [D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] = 0\,. \tag{3.1.42}$$

Using eq.(3.1.27), we can rewrite the above expression as

$$D_\mu F_{\nu\rho} + D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} = 0\,. \tag{3.1.43}$$

It is a straightforward exercise to show that eq.(3.1.43) is identically satisfied.

3.2 Quantum Treatment: the Faddeev-Popov Method

The quantization of gauge theories is non-trivial. The essential point is that Lorentz invariance forces up to describe helicity one fields in terms of a four-vector field, but the latter has four components, and hence more degrees of freedom than necessary. The time-component of a vector field, in addition, is problematic, because it would lead to a kinetic term with an opposite sign with respect to its spatial components. Non-physical degrees of freedom are then expected. The way in which gauge theories solve the problem is to introduce a redundancy in a theory, gauge invariance, so that we can eliminate these extra unwanted and unphysical degrees of freedom.

Functional methods based on the path integral are by far the best way to quantize gauge theories. Before considering non-abelian theories, it is very useful to recall how

QED can be quantized using the path integral. Let's fist see why a naive path integral quantization cannot work by computing the photon propagator

$$\langle A_\mu(x) A_\nu(y)\rangle = \mathcal{N}\int \mathcal{D}A_\mu \; A_\mu(x) A_\nu(y)\, e^{iS(A)}\,, \tag{3.2.1}$$

where $\mathcal{N}$ is a normalization constant and

$$S(A) = -\frac{1}{4}\int d^4x \; F_{\mu\nu}^2 = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} A^\mu(-p) D_{\mu\nu}(p) A^\nu(p) \tag{3.2.2}$$

is the usual free action, written both in configuration and momentum space. The tensor $D_{\mu\nu}$ equals

$$D_{\mu\nu}(p) = (-p^2\eta_{\mu\nu} + p_\mu p_\nu)\,. \tag{3.2.3}$$

The photon propagator is equal to the Fourier transform of the inverse of $D_{\mu\nu}(p)$:

$$\langle A_\mu(x) A_\nu(y)\rangle = \int \frac{d^4p}{(2\pi)^4} i D_{\mu\nu}^{-1}(p) e^{ip\cdot(x-y)}\,. \tag{3.2.4}$$

However, $\det D_{\mu\nu}(p) = 0$, no inverse exists and no propagator can be defined. Another way of looking at the problem is obtained by performing a shift of variables in the path integral: $A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$. We get

$$\langle A_\mu(x) A_\nu(y)\rangle = \langle A_\mu(x) A_\nu(y)\rangle + (\partial_\mu\lambda\partial_\nu\lambda)\,\mathcal{N}\int \mathcal{D}A_\mu \, e^{iS(A)}\,, \tag{3.2.5}$$

whose only solution is $\langle A_\mu(x) A_\nu(y)\rangle = \infty$. The problem arises from the fact that the action is gauge invariant and we are intergating over all possible field configurations, including those that are related by a gauge transformation. All pure gauge configurations, such as $A_\mu(x) = \partial_\mu\lambda$, are not dumped by the action and lead to the above divergence. The problem is solved by restricting the integration to gauge inequivalent configurations only. In other words, we have to implement a gauge-fixing condition of the form $G(A) = 0$, where $G(A)$ is a functional of the gauge field A_μ. This can be imposed inside the path integral by means of a functional generalization of the Dirac delta-function. being careful to possible Jacobian factors. These can arise by recalling the formula

$$\delta[f(x)] = \sum_i \left|\frac{df}{dx}\right|^{-1}_{x=x_0^{(i)}} \delta(x - x_0^{(i)})\,, \tag{3.2.6}$$

where $x_0^{(i)}$ are the zeros of the function f. The n-dimensional integral generalization of eq.(3.2.6) is

$$1 = \int \prod_i dx_i \; \delta^{(n)}(f_j(x_i)) \left|\det \frac{\partial f_j}{\partial x_i}\right| \tag{3.2.7}$$

where f_j are n functions of the n variables x_i and we have assumed that they all vanish at a single zero $x_{0,i}$. The further infinite dimensional generalization of eq.(3.2.7) is

$$1=\int\mathcal{D}\lambda\,\delta(G(A^\lambda))\left|\det\frac{\delta G(A^\lambda)}{\delta\lambda(x)}\right|\,, \tag{3.2.8}$$

where $A_\mu^\lambda(x)=A_\mu(x)+\partial_\mu\lambda(x)$ and $G(A^\lambda)$ is an arbitrary functional, assumed to have a single zero. A simple choice for $G(A)$ is

$$G(A)=\partial_\mu A^\mu\Longrightarrow G(A^\lambda)=\partial_\mu A^\mu+\Box\lambda\,. \tag{3.2.9}$$

Inserting eq.(3.2.8) inside the path integral gives

$$\begin{aligned}&\mathcal{N}\int\mathcal{D}A_\mu\,e^{iS(A)}\int\mathcal{D}\lambda\,\delta(G(A^\lambda))\left|\det\frac{\delta G(A^\lambda)}{\delta\lambda(x)}\right|=\mathcal{N}|\det\Box|\int\mathcal{D}A_\mu\,e^{iS(A)}\int\mathcal{D}\lambda\,\delta(G(A^\lambda))\\&=\mathcal{N}'\int\mathcal{D}\lambda\int\mathcal{D}A_\mu\,e^{iS(A)}\,\delta(G(A))=\mathcal{N}''\int\mathcal{D}A_\mu\,e^{iS(A)}\,\delta(G(A))\,,\end{aligned} \tag{3.2.10}$$

where we have included into the normalization constants the gauge-field independent factor $\det\Box$ and the integration over the gauge parameter λ. The functional delta into the last term of eq.(3.2.10) avoids to integrate over redundant field configurations. With a simple trick, we can also get rid of the functional delta and yet have a well-defined path integral. Instead of taking a single gauge fixing like the one in eq.(3.2.9) we can introduce a family of gauge fixing terms, parametrized by an arbitrary function f:

$$G_f(A)=\partial_\mu A^\mu-f\,. \tag{3.2.11}$$

Since no physical observable can depend on the gauge fixing, we can average over the different gauge fixings by introducing a phase factor

$$\exp\Big(-\frac{i}{2\xi}\int d^4x\,f^2(x)\Big) \tag{3.2.12}$$

and integrating over $f(x)$. This is useful, since in so doing we can get rid of the functional delta. In eq.(3.2.12) ξ is a positive parameter. The path integral becomes now

$$\mathcal{N}\int\mathcal{D}A_\mu\,e^{iS(A)-\frac{i}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2}\,. \tag{3.2.13}$$

The final outcome of all these manipulations is the addition of a new gauge-variant term in the action, called gauge-fixing term. It is crucial to make sense of the photon propagator. Eq.(3.2.1) is replaced by

$$\langle A_\mu(x)A_\nu(y)\rangle=\mathcal{N}\int\mathcal{D}A_\mu\;A_\mu(x)A_\nu(y)\,e^{iS(A)-\frac{i}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2}\,. \tag{3.2.14}$$

The tensor (3.2.3) becomes

$$D_{\mu\nu}(p) = -p^2\eta_{\mu\nu} + p_\mu p_\nu \frac{(\xi - 1)}{\xi}\,, \tag{3.2.15}$$

and it admits the inverse

$$D^{-1}_{\mu\nu}(p) = -\frac{1}{p^2}\Big(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\Big)\,. \tag{3.2.16}$$

Reintroducing the $i\epsilon$, we finally get

$$\langle A_\mu(x)A_\nu(y)\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+i\epsilon}\Big(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\Big)e^{ip\cdot(x-y)}\,. \tag{3.2.17}$$

The photon propagator is not a direct physical observable and depends on ξ. Common choices for ξ in explicit computations are $\xi = 0$ (Landau gauge), in which case the tree-level propagator is transverse and $\xi = 1$ (Feynman gauge), in which this simplifies considerably.

The quantization of non-abelian gauge theories proceeds along the same way as the abelian case, but it presents additional complications. The naive measure is

$$\mathcal{N}\int \mathcal{D}A_\mu^\alpha\, e^{iS(A)}\,. \tag{3.2.18}$$

It is gauge-invariant, but in a less trivial way than in abelian theories, since a gauge transformation rotates the fields. The Jacobian associated to the infinitesimal transformation (3.1.18) is

$$\mathrm{Jac}^{\alpha\beta}_{\mu\nu}(x,y) = \frac{\delta A^\alpha_\mu(x)}{\delta A^\beta_\nu(y)} = \delta^\nu_\mu\delta(x-y)(\delta_{\alpha\beta} + \epsilon^\gamma C_{\alpha\beta\gamma}). \tag{3.2.19}$$

Since $\mathrm{Det}(1+\epsilon) = 1 + \mathrm{Tr}\,\epsilon + \mathcal{O}(\epsilon^2)$, we have

$$\det \mathrm{Jac}^{\alpha\beta}_{\mu\nu}(x,y) = \delta^\mu_\mu\delta(0)(\delta_{\alpha\alpha} + \epsilon^\gamma C_{\alpha\alpha\gamma}) = \delta^\mu_\mu\delta(0)\delta_{\alpha\alpha} \tag{3.2.20}$$

that is the infinite dimensional generalization of the unit matrix. Let us now proceed like in the abelian case, introducing a delta-functional gauge-fixing in the path integral, like in eq. (3.2.8). Let us define

$$\Delta^{-1}_G(A) = \int \mathcal{D}U\,\delta\Big(G(A^U(x))\Big)\,. \tag{3.2.21}$$

In eq.(3.2.21), $\mathcal{D}U$ is the so-called invariant measure of the group G, parametrized by group elements $U(\Lambda)$. It is not, like in the abelian case, simply the integration over the Lie algebra generators, $\prod_\alpha \mathcal{D}\Lambda^\alpha(x)$, but it includes a non trivial measure $\rho(\Lambda)$. It is called invariant measure because it satisfies the following properties:

$$\int \mathcal{D}U\, f(U) = \int \mathcal{D}U\, f(U^{-1}) = \int \mathcal{D}U\, f(U\cdot U_0) = \int \mathcal{D}U\, f(U_0\cdot U) \tag{3.2.22}$$

where U_0 is a constant element of the group and $f(U)$ is an arbitrary function over the group. Although we will never need its explicit form in the following, it is worth to spend a few more words on $\rho(\Lambda)$. This can be defined starting from the metric in group space g_{ij} defined as

$$g_{\alpha\beta} = \mathrm{Tr}\left(U^{-1}(\Lambda)\Big(\frac{\partial U(\Lambda)}{\partial \Lambda_\alpha}\Big)U^{-1}(\Lambda)\Big(\frac{\partial U(\Lambda)}{\partial \Lambda_\beta}\Big)\right). \tag{3.2.23}$$

The invariant measure is

$$\rho(\Lambda) = \sqrt{\det g_{\alpha\beta}} \tag{3.2.24}$$

The parameters Λ_α in eq.(3.2.23) are space-time independent coordinates spanning the group G. They do not necessarily correspond to the x-independent version of the generators appearing in eq.(3.1.23). An example will clarify this point. The group $SU(2)$ is defined as the set of 2×2 unitary matrices U of unit determinant. Any matrix U can be written as

$$U = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}, \tag{3.2.25}$$

with $|z_1|^2 + |z_2|^2 = 1$, $z_{1,2} \in \mathbb{C}$. The group $SU(2)$ is isomorphic to the three-dimensional sphere S^3. Instead of using standard coordinates subject to a constraint, we might use radial coordinates. Denoting $z_i = x_i + iy_i$, $i = 1, 2$, we have

$$\begin{aligned} x_1 &= \sin\psi\sin\theta\cos\phi\,, \qquad & y_1 &= \sin\psi\sin\theta\sin\phi \\ x_2 &= \sin\psi\cos\theta\,, & y_1 &= \cos\psi\,, \end{aligned} \tag{3.2.26}$$

with $0 \leq \phi < 2\pi$, $0 \leq \theta < 2\pi$, $0 \leq \psi \leq \pi$. The invariant $SU(2)$ measure coincides with the standard metric of S^3: $\rho(\psi,\theta,\phi) = \sin^2\psi\sin\theta$.[2] Given the explicit form of the metric (3.2.23) it is straightforward to prove the relations (3.2.22).

The functional $\Delta_G(A)$ is gauge-invariant: $\Delta_G(A^U) = \Delta_G(A)$, that immediately follows from eq.(3.2.22). The naive measure (3.2.18) can be rewritten as (omitting Lorentz and color indices in the gauge measure)

$$\mathcal{N}\int \mathcal{D}A \int \mathcal{D}U\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A^U)\Big). \tag{3.2.27}$$

We can change variables in the path integral by defining $A_\mu = A_\mu'^{U^{-1}}$. Since $\Delta_G(A)$, the measure and the action are all gauge invariant, we get (redefine $A' \to A$)

$$\mathcal{N}\int \mathcal{D}A\int \mathcal{D}U\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A)\Big) = \mathcal{N}'\int \mathcal{D}A\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A)\Big), \tag{3.2.28}$$

[2] Notice in fact the similarity of the definition (3.2.24) with the definition of diffeomorphism-invariant measure in general relativity.

and we can reabsorb the invariant group measure into the overall path integral normalization. Let us verify that any correlation function of gauge invariant operators does not depend on the gauge-fixing, namely does not depend on the specific choice of the functional $G(A)$. If $O(A)$ schematically represent some product of gauge invariant operators, given two arbitrary gauge fixing functionals G and F, one has

$$\begin{aligned}
\langle O(A)\rangle_G &= \mathcal{N}\int \mathcal{D}A\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A)\Big)O(A) \qquad (3.2.29)\\
&= \mathcal{N}\int \mathcal{D}A\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A)\Big)\int \mathcal{D}U\delta\Big(F(A^U)\Big)\Delta_F(A)O(A)\\
&= \mathcal{N}\int \mathcal{D}A\int \mathcal{D}U\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A^{U^{-1}})\Big)\delta\Big(F(A)\Big)\Delta_F(A)O(A)\\
&= \mathcal{N}\int \mathcal{D}A\int \mathcal{D}U\, e^{iS(A)}\Delta_G(A)\,\delta\Big(G(A^{U})\Big)\delta\Big(F(A)\Big)\Delta_F(A)O(A)\\
&= \mathcal{N}\int \mathcal{D}A\, e^{iS(A)}\delta\Big(F(A)\Big)\Delta_F(A)O(A) = \langle O(A)\rangle_F\,.
\end{aligned}$$

On the contrary, the correlation function of gauge dependent quantities do depend on the choice of $G(A)$. Similarly to the abelian case, we can also take a family of gauge fixing functionals and integrate over them. The typical choice will be the non-abelian generalization of eq.(3.2.11):

$$G_f(A) = \partial_\mu A^\mu_\alpha - f_\alpha\,, \qquad (3.2.30)$$

weighted by the phase factor

$$\exp\Big(-\frac{i}{2\xi}\int d^4x\, f^2_\alpha(x)\Big)\,. \qquad (3.2.31)$$

In this way, we get

$$\langle O(A)\rangle = \mathcal{N}\int \mathcal{D}A\, e^{iS(A)-\frac{i}{2\xi}\int d^4x\,(\partial_\mu A^\mu_\alpha)^2}\Delta(A)\,, \qquad (3.2.32)$$

where

$$\Delta^{-1}(A) = \int \mathcal{D}U\,\delta\Big(\partial_\mu (A^U)^\mu_\alpha - f_\alpha\Big)\,. \qquad (3.2.33)$$

We do not need to compute $\Delta(A)$ for any A, but only for the field configurations where $G(A)$ vanishes, since these are the ones selected by the delta functional in the path integral. The integrand in eq.(3.2.33) has then a non-trivial support for values of infinitesimally close to the identity: $U = 1 + i\epsilon^\alpha t_\alpha$. We have

$$\delta\Big(\partial_\mu (A^U)^\mu_\alpha - f_\alpha\Big) = \delta\Big(\partial_\mu A^\mu_\alpha - f_\alpha + \partial_\mu D^\mu\epsilon_\alpha\Big) = \frac{\delta(\epsilon_\alpha)}{|\det \partial_\mu D^\mu|}\,. \qquad (3.2.34)$$

Modulo irrelevant constants, as usual absorbable in the path integral normalization $\mathcal{N}$, we get

$$\Delta(A) = |\det \partial_\mu D^\mu|\,. \qquad (3.2.35)$$

This is nothing else than the non-abelian generalization of the $|\det \Box|$ term appearing in eq.(3.2.10). The crucial difference with the abelian case is its gauge field dependence by means of the covariant derivative and should then be kept into the path integral. We can remove the absolute value, that would make the computation of $\Delta(A)$ quite complicated, because, loosely speaking,

$$|\det \partial_\mu D^\mu| \; = |\det \Box| \; |\det (1 + \frac{1}{\Box}\partial_\mu D^\mu)| \tag{3.2.36}$$

The first factor is irrelevant, while the second one is manifestly positive in perturbation theory. We can then safely remove the absolute value and write

$$\Delta(A) = \det \partial_\mu D^\mu \, . \tag{3.2.37}$$

It is not easy to directly compute determinants. It is more convenient to turn a determinant into a local action by inserting additional non-physical degrees of freedom and use the identity, valid for Grassmann variables ω and ω^*,[3]

$$\int \mathcal{D}\omega \mathcal{D}\omega^* \, e^{-i\int d^4x \, \omega^*(x) F(x) \omega(x)} \propto \det F \, , \tag{3.2.38}$$

where $F(x)$ is an arbitrary differential operator. Using eq.(3.2.37), we can write, modulo irrelevant constants,

$$\det \partial_\mu D^\mu = \int \mathcal{D}\omega_\alpha \mathcal{D}\omega^*_\alpha \, e^{iS_{ghost}} \, , \tag{3.2.39}$$

where

$$S_{ghost} = i \int d^4x \, \partial_\mu \omega^*_\alpha D^\mu \omega_\alpha = \int d^4x \, \partial_\mu \omega^*_\alpha \Big(\partial^\mu \omega_\alpha + g C_{\alpha\beta\gamma} A^\mu_\beta \omega_\gamma \Big) \, . \tag{3.2.40}$$

ω_α and ω^*_α are dim-G scalar fields with fermion statistic, transforming in the adjoint representation of G. They are not associated to physical propagating particles and for this reason they are denoted ghost fields. They cannot appear as physical external states, but they can and do appear in loops as virtual particles, by means of their interaction with the gauge field. Ghosts are crucial to restore unitarity in non-abelian gauge theories. Loosely speaking, they compensate for the (also unphysical) contribution of the longitudinal and time component of the gauge fields that, contrary to the abelian case, do not automatically decouple from scattering amplitudes.

Putting all together, the complete non-abelian Lagrangian density at the quantum level is the sum of three terms:

$$\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_{g.f.} + \mathcal{L}_{ghost} \, , \tag{3.2.41}$$

[3] Recall that inside the path integral ω and ω^* are two independent variables. In particular, ω^* is not the complex conjugate of ω.

where

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha} + \sum_i \bar{\psi}_i(i\not{D} - m_i)\psi_i + \sum_j |D_\mu\phi_j|^2 - V(\psi_i,\phi_j)\,, && (3.2.42)\\
\mathcal{L}_{g.f.} &= -\frac{1}{2\xi}(\partial_\mu A^\mu_\alpha)^2\,, && (3.2.43)\\
\mathcal{L}_{ghost} &= \partial_\mu \omega^*_\alpha D^\mu \omega_\alpha\,. && (3.2.44)
\end{aligned}
$$

The propagator of gauge and ghost field is readily found by the quadratic term of the above Lagrangian density. We have

$$
\begin{aligned}
\langle A^\alpha_\mu(x)A^\beta_\nu(y)\rangle &= \delta^{\alpha\beta}\int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+i\epsilon}\Big(\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\Big)e^{ip\cdot(x-y)}\,,\\
\langle \omega^*_\alpha(x)\omega_\beta(y)\rangle &= \delta_{\alpha\beta}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{ip\cdot(x-y)}\,.
\end{aligned}
\qquad (3.2.45)
$$

The gauge field propagator is the trivial generalization of the photon propagator (3.2.16), while the ghost propagator coincides with that of a complex massless scalar field. Recall that the ghost Lagrangian (3.2.44) depends on the specific form of $\Delta(A)$, that in turn depends on the specific gauge fixing chosen. It is invariant under a $U(1)$ symmetry (ghost number) under which the ghost fields ω_α and ω^*_α have respectively charges +1 and -1. For this reason the latter fields are commonly denoted anti-ghosts and this explains the notation ω^*_α. All the operators appearing in the Lagrangian (3.2.41) have dimensions less or equal to four, compatibly with a renormalizable theory. However, not all possible operators of dimensions $\Delta \leq 4$ appear in $\mathcal{L}_{tot}$ and hence the renormalizability of these theories is not obvious. The appearance of unphysical fields in the theory complicates also the notion of physical field in non-abelian gauge theories. All these issues are best addressed by introducing the BRST symmetry, subject of the next section.

3.3 BRST Symmetry

The Faddeev-Popov path integral quantization of gauge theories reviewed above requires to fix a gauge and hides the underlying gauge invariance of the theory. In other words, the Lagrangian $\mathcal{L}_{tot}$ cannot obviously be gauge invariant. On the other hand, it was found by Becchi, Rouet and Stora, and indepedendently by Tyutin, that $\mathcal{L}_{tot}$ is invariant under an additional symmetry, called BRST symmetry. It is useful to rewrite the gauge fixing term in a different fashion by "integrating in" an auxiliary field $H_\alpha(x)$:

$$
e^{-\frac{i}{2\xi}\int d^4x\, f^2_\alpha(x)} = \int \mathcal{D}H_\alpha\, e^{\frac{i\xi}{2}\int d^4x\, H^2_\alpha - i\int d^4x\, f_\alpha H_\alpha}\,, \qquad (3.3.1)
$$

so that

$$\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_{ghost} + \frac{\xi}{2} H_\alpha^2 - f_\alpha H_\alpha \,. \tag{3.3.2}$$

The total Lagrangian (3.3.2) is invariant under infinitesimal transformations parametrized by an anticommuting variable θ:

$$\begin{aligned} \delta_\theta \Psi &= i t_\alpha \theta \omega_\alpha \Psi \,, \\ \delta_\theta A_\mu &= \theta D_\mu \omega_\alpha \,, \\ \delta_\theta \omega_\alpha &= -\frac{1}{2} \theta C_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \,, \\ \delta_\theta \omega_\alpha^* &= -\theta H_\alpha \,, \\ \delta_\theta H_\alpha &= 0 \,. \end{aligned} \tag{3.3.3}$$

These are the BRST transformations, written in the non-canonical basis for the gauge fields where the gauge coupling does not appear in the interactions. The field Ψ represents any matter field, fermionic or bosonic. The BRST transformations are nilpotent, namely if we denote $\delta_\theta F \equiv \theta s F$, then $\delta_\theta(sF) = \theta s^2 F = 0$ for any functional of the fields Ψ, A_μ, ω_α, ω_α^* and H_α. Let us check that s^2 vanishes when acting on single fields. From eq.(3.3.3) we have $s\Psi = i t_\alpha \omega_\alpha \Psi_\alpha$. Then

$$\begin{aligned} \delta_\theta(s\Psi) =& i t_\alpha \Big((\delta_\theta \omega_\alpha)\Psi + \omega_\alpha(\delta_\theta \Psi) \Big) = i t_\alpha \Big(-\frac{1}{2} \theta C_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \Psi + \omega_\alpha i t_\beta \theta \omega_\beta \Psi \Big) \\ =& \theta \omega_\alpha \omega_\beta \Psi \Big(-\frac{i}{2} C_{\alpha\beta\gamma} t_\gamma + \frac{1}{2}[t_\alpha, t_\beta] \Big) = 0 \,. \end{aligned} \tag{3.3.4}$$

Consider now the BRST transformations of the gauge fields. We have $sA_\mu = D_\mu \omega_\alpha = \partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_\mu^\beta \omega_\gamma$. Then

$$\delta_\theta(sA_\mu^\alpha) = \partial_\mu \Big(-\frac{1}{2} \theta C_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \Big) + C_{\alpha\beta\gamma} \theta (\partial_\mu \omega_\beta + C_{\beta\rho\sigma} A_\mu^\rho \omega_\sigma) \omega_\gamma + C_{\alpha\beta\gamma} A_\mu^\beta \Big(-\frac{1}{2} \theta C_{\gamma\rho\sigma} \omega_\rho \omega_\sigma \Big) \,. \tag{3.3.5}$$

Consider first the terms proportional to A_μ in eq.(3.3.5). Reshuffling indices, one has

$$\begin{aligned} \delta_\theta(sA_\mu^\alpha)|_A &= \theta A_\mu^\beta \omega_\rho \omega_\sigma \Big(-\frac{1}{2} C_{\alpha\beta\gamma} C_{\gamma\rho\sigma} + C_{\alpha\rho\gamma} C_{\gamma\beta\sigma} \Big) \\ &= -\frac{1}{2} \theta A_\mu^\beta \omega_\rho \omega_\sigma (C_{\alpha\beta\gamma} C_{\gamma\rho\sigma} + C_{\rho\alpha\gamma} C_{\gamma\beta\sigma} + C_{\beta\rho\gamma} C_{\gamma\alpha\sigma}) = 0 \,, \end{aligned} \tag{3.3.6}$$

where the last equality follows from the Jacobi identity (3.1.10). The terms with no gauge fields give

$$\delta_\theta(sA_\mu^\alpha)|_{no\,A} = \theta C_{\alpha\beta\gamma} \Big(-\frac{1}{2} (\partial_\mu \omega_\beta) \omega_\gamma - \frac{1}{2} \omega_\beta (\partial_\mu \omega_\gamma) + (\partial_\mu \omega_\beta) \omega_\gamma \Big) = 0 \,. \tag{3.3.7}$$

Similarly, we have

$$\begin{aligned}\delta_\theta(s\omega_\alpha) = & -\frac{1}{2}C_{\alpha\beta\gamma}\Big((-\frac{1}{2}\theta C_{\beta\rho\sigma}\omega_\rho\omega_\sigma)\omega_\gamma + \omega_\beta(-\frac{1}{2}\theta C_{\gamma\rho\sigma}\omega_\rho\omega_\sigma)\Big) \\ = & \frac{1}{2}\theta\omega_\rho\omega_\sigma\omega_\beta C_{\rho\sigma\gamma}C_{\gamma\beta\alpha} = 0\,,\end{aligned} \tag{3.3.8}$$

where again the last equality follows from the Jacobi identity (3.1.10) and the antisymmetrization in the indices ρ, σ and β. The nilpotency of the BRST transformations acting on ω^*_α and H_α is trivial. We immediately have, from eq.(3.3.3),

$$\delta_\theta(s\omega^*_\alpha) = \delta_\theta s H_\alpha = 0\,. \tag{3.3.9}$$

Summarizing, we have shown that for any field $\Phi = A, \Psi, \omega, \omega^*, H$,

$$s^2\Phi = 0\,. \tag{3.3.10}$$

For two fields, we have

$$\delta_\theta(\Phi_1\Phi_2) = \theta(s\Phi_1)\Phi_2 + \Phi_1(\theta s\Phi_2) = \theta\Big((s\Phi_1)\Phi_2 \pm \Phi_1(s\Phi_2)\Big)\,, \tag{3.3.11}$$

with + or − depending whether the field Φ_1 is bosonic or fermionic.[4] Hence

$$s(\Phi_1\Phi_2) = \Big((s\Phi_1)\Phi_2 \pm \Phi_1(s\Phi_2)\Big) \tag{3.3.12}$$

Acting again with a BRST transformation gives

$$\begin{aligned}\delta_\theta(s\Phi_1\Phi_2) &= \delta_\theta(s\Phi_1)\Phi_2 + s\Phi_1\delta_\theta\Phi_2 \pm (\delta_\theta\Phi_1)s\Phi_2 \pm \Phi_1\delta_\theta(s\Phi_2) \\ &= \theta\Big((s^2\Phi_1)\Phi_2 \mp (s\Phi_1)(s\Phi_2) \pm (s\Phi_1)(s\Phi_2) + \Phi_1(s^2\Phi_2)\Big) = 0\end{aligned} \tag{3.3.13}$$

if eq.(3.3.10) is satisfied, for Φ_1 both bosonic and fermionic. Iterating the argument for more fields, we conclude that for any functional $F(\Phi)$,

$$s^2F(\Phi) = 0\,. \tag{3.3.14}$$

We now proceed to prove that the Lagrangian (3.3.2) is BRST-invariant. On the physical fields Ψ and A^α_μ, the BRST transformations (3.3.3) can be seen as infinitesimal gauge transformations with parameter $\epsilon_\alpha(x) = \theta\omega_\alpha(x)$. The BRST-invariance of the gauge and matter Lagrangian term $\mathcal{L}$ immediately follows from the fact that it is gauge invariant. Let us now turn to the remaining three terms in eq.(3.3.2). We have

$$\delta_\theta f_\alpha = \delta_\theta\partial^\mu A^\alpha_\mu = \theta\partial_\mu D^\mu\omega_\alpha = \theta s f_\alpha\,. \tag{3.3.15}$$

[4] Namely with Fermi statistics. In particular, ghosts and anti-ghosts are fermionic.

Modulo total derivatives, we then get

$$\mathcal{L}_{ghost} = -\omega^*_\alpha s f_\alpha \, . \tag{3.3.16}$$

Since $s\omega^*_\alpha = -H_\alpha$, the last three terms in eq.(3.3.2) can be rewritten as

$$\mathcal{L}_{ghost} + \frac{\xi}{2} H^2_\alpha - f_\alpha H_\alpha = s\Big(f_\alpha \omega^*_\alpha - \frac{1}{2}\xi\omega^*_\alpha H_\alpha\Big) \, . \tag{3.3.17}$$

In this way, the BRST invariance of these terms is automatically insured by the fact that $s^2 = 0$ for any functional. We conclude that the whole Lagrangian $\mathcal{L}_{tot}$ is invariant under BRST transformations.

From an operatorial point of view, the BRST transformations (3.3.3) are generated by a Grassmann Hermitian operator Q. For any field Φ, we have

$$\delta_\theta \Phi = i[\theta Q, \Phi] = i\theta [Q, \Phi]_\mp \, , \tag{3.3.18}$$

where $-$ and $+$ denote commutator and anti-commutator, respectively, depending on whether the field Φ is bosonic or fermionic. We then have

$$[Q, \Phi]_\mp = -is\Phi \, . \tag{3.3.19}$$

The nilpotency of s, $s^2 = 0$, is equivalent to

$$(-is)^2 \Phi = [Q, [Q, \Phi]_\mp]_\pm = [Q^2, \Phi]_- = 0 \Longrightarrow Q^2 = 0 \, . \tag{3.3.20}$$

The BRST operator Q allows to make the following partition of the Hilbert space in non-abelian gauge theories. Any state $|\phi\rangle$ in the Hilbert space falls in one of the following three categories:

$$\begin{aligned} & Q|\phi_1\rangle \neq 0 \\ & Q|\phi_2\rangle = 0 \, , \qquad \text{with } \; |\phi_2\rangle = Q|\phi_1\rangle \\ & Q|\phi_3\rangle = 0 \, , \qquad \text{but } \; |\phi_3\rangle \neq Q|\phi_1\rangle \, . \end{aligned} \tag{3.3.21}$$

The fields in the second class are manifestly unphysical, since they have vanishing norm:

$$||\phi_2\rangle|^2 = \langle\phi_2|\phi_2\rangle = \langle\phi_1|Q^2|\phi_1\rangle = 0 \, . \tag{3.3.22}$$

We now show that gauge invariance implies that physical fields should be annihilated by Q. More precisely, matrix elements between physical states $|\alpha\rangle$ and $|\beta\rangle$ should not depend on the choice of gauge fixing term. We have just seen that the total Lagrangian $\mathcal{L}_{tot}$ can be written as a physical gauge invariant term $\mathcal{L}$ plus a BRST variation of some functional

F: $\mathcal{L}_{tot} = \mathcal{L} + sF(\Phi)$. The specific form of F depends on the gauge-fixing chosen. If we infinitesimally deform the gauge fixing, the functional F will also be deformed $F \to F+\delta F$. Demanding that

$$\langle\alpha|\beta\rangle_F = \langle\alpha|\beta\rangle_{F+\delta F} \tag{3.3.23}$$

is equivalent to the condition

$$\langle\alpha|[Q,F]_+|\beta\rangle = 0\,, \qquad \forall \alpha,\beta \in \text{ physical} \tag{3.3.24}$$

and for any sensible choice of gauge-fixing functional F. We then conclude that Q should annihilate physical states. These are then identified with the states in the third category in eq.(3.3.21). States that are annihilated by the operator Q ($|\phi_1\rangle$) are said in the kernel of Q. The states $|\phi_2\rangle$ are said to be in the image of Q. The physical states are states in the kernel that are not in the image of Q. Such states are said to be in the cohomology of Q. It is clear that physical states are not uniquely defined. If $|\alpha\rangle$ is a given physical states, then any state of the form

$$|\widetilde{\alpha}\rangle = |\alpha\rangle + Q|\phi_1\rangle \tag{3.3.25}$$

defines the same physical state, since

$$\langle\widetilde{\alpha}|\beta\rangle = \langle\alpha|\beta\rangle \tag{3.3.26}$$

for any other physical state $|\beta\rangle$. Physical states $|phys\rangle$ correspond to equivalence classes within the class $|\phi_3\rangle$. This complicated structure of the Hilbert space in non-abelian gauge theories is a consequence of the redundancy introduced by the gauge symmetries. Like ghosts, any unphysical state, even if not present as an external line in a scattering process, can contribute as a virtual particle in loops. On the other hand, the optical theorem relates the imaginary part of a loop diagram with the square of scattering diagrams where the virtual particles become external on-shell states. Unitarity is then not obvious. BRST invariance if of great help to show us that, in fact, no problem arises because physical states are unitary by themselves, namely the contribution of unphysical states in loop diagrams always cancels. Ghosts are crucial for this cancellation to occur. This is best seen by considering

$$\langle\alpha|S^\dagger S|\beta\rangle = \sum_{i=1,2,3} \langle\alpha|S^\dagger|\phi_i\rangle\langle\phi_i|S\beta\rangle = \langle\alpha|\beta\rangle\,. \tag{3.3.27}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are two arbitrary physical states. In principle all states in the Hilbert space contribute in the completeness relation, but since the BRST operator Q commutes with the S matrix and Q annihilates both $|\alpha\rangle$ and $|\beta\rangle$, the states ϕ_i should also be annihilated by Q. Hence the states $|\phi_1\rangle$ cannot enter in eq.(3.3.28). The states $|\phi_2\rangle$ enter, but have vanishing inner product with states in both classes $|\phi_2\rangle$ and $|\phi_3\rangle$. We then conclude that effectively

$$\langle\alpha|S^\dagger S|\beta\rangle = \sum_{phys} \langle\alpha|S^\dagger|phys\rangle\langle phys|S|\beta\rangle = \langle\alpha|\beta\rangle\,, \tag{3.3.28}$$

and hence unitarity is recovered.

BRST invariance is very useful to also establish the renormalization properties of non-abelian gauge theories. Being a symmetry of the total action $S_{tot} = \int d^4x\,\mathcal{L}_{tot}$, one can show that the total quantum action Γ_{tot} is of the same form of S_{tot}, with renormalized parameters. We will not enter into this derivation. The interested reader can find a detailed description in the chapter 17 of the second volume of [2].

Chapter 4

The Renormalization Group

The renormalization group is a key concept in quantum field theory. It essentially tells us that instead of describing a physical system by some constant parameters in a Lagrangian, it is more convenient to let the parameters vary and keep only some of them, depending on the energy scale at which we are looking at the system. Intuitively this is quite obvious and is at the basis of the usual reductionism used in physics. We do not need the SM Lagrangian to study the energy levels of the hydrogen atom! The latter are well described by a much simpler Schrodinger equation, which captures the effective dynamics entering at the eV scale, namely the Coulomb potential between the electron and the proton. In general, however, the microscopic short distance behaviour of a system is not completely negligible. When this is the case, if we are interested to processes occurring at some energy scale E, we can "integrate out", rather than simply neglect, all states with higher frequencies and retain only the effective degrees of freedom of interest. The renormalization group idea, pionereed by K. Wilson, is a way to properly take into account high energy modes in low energy processes.

4.1 Relevant, Marginal and Irrelevant Couplings

In this section we consider the renormalization group in the spirit of Wilson's original idea, focusing on a particular model, the ϕ^4 theory in four space-time dimensions, Wick rotated in euclidean space. Wilson's approach has the main advantage of being conceptually very clear. It assumes the presence of a physical cut-off Λ in the theory, above which no mode can be excited. Imagine we are interested in processes occurring at scales of order $b\Lambda$, with $b<1$. We label all modes $\phi(k)$ as "heavy" and "light", depending on their momentum k. We write

$$\phi(k) = \phi_L(k)\theta(b\Lambda - |k|) + \phi_H(k)\theta(|k| - b\Lambda)\,, \quad |k| < \Lambda \tag{4.1.1}$$

and we correspondingly decompose the action of the ϕ^4 theory as follows:[1]

$$S = S_L + S_H^0 + S_{int}\,, \tag{4.1.2}$$

where

$$\begin{aligned} S_L &= \int d^4x \left(\frac{1}{2}(\partial\phi_L)^2 + \frac{1}{2}m^2\phi_L^2 + \frac{\lambda}{4!}\phi_L^4\right), \\ S_H^0 &= \int d^4x \left(\frac{1}{2}(\partial\phi_H)^2 + \frac{1}{2}m^2\phi_H^2\right), \\ S_{int} &= \frac{\lambda}{4!}\int d^4x \Big(\phi_H^4 + 4\phi_H\phi_L^3 + 4\phi_H^3\phi_L + 6\phi_H^2\phi_L^2\Big)\,, \end{aligned} \tag{4.1.3}$$

It is clear from eq.(4.1.1) that the quadratic terms of the form $\phi_H\phi_L$ vanish. The modes ϕ_H cannot be excited for processes at scales below or of order $b\Lambda$, so at first approximation they can be ignored, in which case we just recover the ϕ^4 theory S_L for the light mode only. Quantum mechanically, however, the modes ϕ_H contribute as virtual particles. Instead of neglecting them, we should more properly integrate them out, getting in this way an effective action for the light modes ϕ_L:

$$e^{-S_{eff}(\phi_L)} = e^{-S_L(\phi_L)}\int \mathcal{D}\phi_H(k) e^{-S_H^0(\phi_H) - S_{int}(\phi_L,\phi_H)}\,. \tag{4.1.4}$$

In eq.(4.1.4) the modes ϕ_L act like external fields. It is not difficult to see that the tree-level exchange of the ϕ_H modes generate effective ϕ_L couplings of the form $(\lambda^2/m^2)\phi_L^6$ (from two $\phi_H\phi_L^3$ vertices) plus an infinite set of higher derivative couplings involving six ϕ_L fields. Similar considerations can be made for all other couplings. At some order in perturbation theory all possible couplings will be generated. The effective action $S_{eff}(\phi_L)$ reads as

$$S_{eff}(\phi_L) = \int d^4x \Big(\frac{1}{2}(1+\delta Z)(\partial\phi_L)^2 + \frac{1}{2}(m^2+\delta m^2)\phi_L^2 + \frac{\lambda+\delta\lambda}{4!}\phi_L^4 + \delta Z_1(\partial\phi_L)^4 + \delta Z_2\phi_L^6 + \dots\Big) \tag{4.1.5}$$

where $\dots$ stands for higher dimensional operators. Recall that ϕ_L contains only momenta $k \leq b\Lambda$. The rescaled momentum $k' = bk$ satisfies the same constraint of the original system, $k' < \Lambda$. Correspondingly we redefine coordinates as $x' = x/b$. We also have to redefine the field ϕ so that it has a canonically normalized kinetic term:

$$\phi_L^{can} = b\sqrt{1+\delta Z}\,\phi_L\,. \tag{4.1.6}$$

[1] Strictly speaking, the actions (4.1.3) should be written in momentum space using eq.(4.1.1) and recalling the overall bound $|k| < \Lambda$. This results in unnecessary long expressions that are avoided in the rough, but more brief form (4.1.3).

The action S_{eff} reads now (redefining $x' \to x$ and $\phi_L^{can} \to \phi_L$)

$$S_{eff}(\phi_L) = \int d^4x \left(\frac{1}{2}(\partial\phi_L)^2 + \frac{1}{2}m^2(b)\phi_L^2 + \frac{\lambda(b)}{4!}\phi_L^4 + \delta Z_1(b)\phi_L^6 + \delta Z_2(b)(\partial\phi_L)^4 + \dots \right), \tag{4.1.7}$$

where[2]

$$m^2(b) = \frac{1}{b^2}\frac{m^2+\delta m^2}{1+\delta Z}\,, \qquad \lambda(b) = \frac{1}{b^0}\frac{\lambda+\delta\lambda}{(1+\delta Z)^2}\,,$$
$$\delta Z_1(b) = b^2\frac{\delta Z_1}{(1+\delta Z)^3}\,, \qquad \delta Z_2(b) = b^4\frac{\delta Z_2}{(1+\delta Z)^2}. \tag{4.1.8}$$

The process of integrating out heavy degrees of freedom and rescale the momentum is called "renormalization group" (RG).

The action (4.1.7) is the proper action for describing processes at scales $E \leq b\Lambda$. It is straightforward to see that a coupling $c_{\mathcal{O}}$ of a generic operator $\mathcal{O}$ of dimension Δ (in mass) scale as

$$c_{\mathcal{O}}(b) = b^{\Delta-4}c_{\mathcal{O}}\,. \tag{4.1.9}$$

At lower and lower energies (smaller and smaller b) among all infinite operators appearing in eq.(4.1.7), only the finite subset of those with $\Delta \leq 4$ do actually matter, all the others being "irrelevant". This is a very remarkable result, which dramatically simplifies the physical description of a system. We define as

- *Irrelevant* the operators with $\Delta > 4$
- *Relevant* the operators with $\Delta < 4$
- *Marginal* the operators with $\Delta = 4$

Correspondingly, we define as

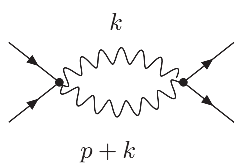
- *Irrelevant* the couplings with mass dimensions < 0
- *Relevant* the couplings with mass dimensions > 0
- *Marginal* the couplings with mass dimensions $= 0$

Relevant couplings grow in the IR and dominate the physics. In our ϕ^4 example, the only relevant coupling is the mass term, which is indeed the most important parameter governing the dynamics of a particle at low energies (in perturbation theory). When all

[2]The couplings δZ, δm^2, etc. appearing in eq.(4.1.5), after the redefinition $k \to bk$, can depend on b, as we will explicitly see in what follows. However, in order to avoid confusion with the couplings defined in eqs.(4.1.8), we have omitted to write this b-dependence.

relevant couplings vanish, the IR theory is controlled by marginal couplings only, in which case we say that the theory is (classically, see below) scale invariant, namely the form of the action does not change under the RG flow.

The fate of marginal operators under the renormalization group cannot be deduced from classical scaling and requires a quantum computation. In the ϕ^4 example, the only marginal operator is λ and its b-dependence is determined by that of δZ and $\delta\lambda$. It is very useful to determine $\lambda(b)$ at the lowest non-trivial order in perturbation theory, which is one-loop. At one-loop level the exchange of ϕ_H fields modify the ϕ_L^4 coupling by means of two $\phi_L^2\phi_H^2$ interactions. No correction arises in δZ at one-loop order, so that we can neglect it. By denoting the heavy fields with a wavy line, the relevant Feynman graph to consider is



$$+ \quad 2\,\text{perms.} \equiv -\delta_\lambda = \frac{3\lambda^2}{2}\int_{b\Lambda\leq k\leq\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(p+k)^2+m^2} \tag{4.1.10}$$

where p is the incoming external momentum and "perms." refer to two other diagrams obtained by permuting the external lines. When the external momentum and the mass m are much smaller than $b\Lambda$, δ_λ is easily computed:

$$\delta_\lambda = -\frac{3\lambda^2}{32\pi^4}\int_{b\Lambda\leq k\leq\Lambda}\frac{d^4k}{k^4} = -\frac{3\lambda^2}{16\pi^2}\log\frac{1}{b}\,. \tag{4.1.11}$$

The effective quartic interaction between the light modes is then

$$\lambda(b) = \lambda + \delta_\lambda = \lambda + \frac{3\lambda^2}{16\pi^2}\log b\,. \tag{4.1.12}$$

As we can seen from eq.(4.1.12), $\lambda(b)$ decreases at larger distances (smaller b), so we say that the coupling λ is marginally irrelevant, meaning in this way that it decreases in the IR, although not as quickly as a classically irrelevant operator. Similarly, we call marginally relevant a classically marginal coupling that increases at larger distance due to quantum effects. Classically marginal couplings that remain marginal at the quantum level are called exactly marginal. At the quantum level, as we have just seen, all the terms δZ, δm^2 etc., appearing in eqs.(4.1.7) and (4.1.8), can depend on b. We can correspondingly define a "quantum" scaling dimension of an operator that takes into account of this implicit b-dependence as follows:

$$\Delta_{\mathcal{O}} = \frac{b}{\mathcal{O}(b)}\frac{d}{db}\mathcal{O}(b)\,, \tag{4.1.13}$$

where $\mathcal{O}(b)$ is the operator we get from eq.(4.1.5) after the rescaling $x' = x/b$ but before the redefinition (4.1.6). If we apply the definition (4.1.13) to the elementary operator ϕ, we get

$$\Delta_\phi = \frac{b}{\phi_L^{can}}\frac{d}{db}\phi_L^{can} = 1+\gamma \tag{4.1.14}$$

where

$$\gamma = \frac{1}{2}b\frac{d\log(1+\delta Z(b))}{db} \tag{4.1.15}$$

is the quantum contribution to the scaling dimension of the field and is called the anomalous dimension of the field.

We started our analysis from the UV action (4.1.2) but it should be now clear that if we had started already at the UV with the most general (non-renormalizable) action of the form (4.1.5), under the renormalization group flow we would have always ended up to the usual ϕ^4 theory, plus an infinite number of irrelevant operators. From the Wilsonian RG point of view, then, renormalizable theories can be seen as critical surfaces, in parameter space, where a much larger class of theories flow to. It is important to stress here that our classification of the operators in relevant, irrelevant and marginal is based on perturbation theory. At strong coupling, it might happen that, say, an operator that is classically irrelevant becomes marginal due to non-perturbative effects.

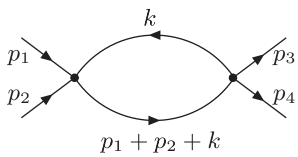
The Wilsonian picture of the RG flow is very intuitive and physical but, pragmatically speaking, it is not the best way to proceed in high energy physics. Distinguishing light and heavy modes in a single field can give rise to complicated expressions.[3] Cut-off regularization is often unavailable, like in gauge theories. Moreover, the Wilsonian RG flow requires, as a starting point, some UV-regulated Lagrangian, while in high energy physics we prefer to hide our ignorance about the UV physics in the renormalization of the parameters entering into the Lagrangian. For all these reasons, in the remaining of this chapter we will change perspective and consider the RG flow from an "high energy physics" point of view. As we will see, in a perturbative context, it essentially gives us a way of improving the perturbative expansion.

4.2 The Sliding Scale and the Summation of Leading Logs

The RG flow technique is very useful in standard four dimensional (weakly interacting) theories in high energy physics. In particular, it improves the perturbative expansion, allowing us to sum whole series of higher order effects. An example will clarify the problem and its resolution.

[3]On the contrary, when the field is sufficiently heavy, so that *all* its frequency modes can be integrated out, we typically have a great simplification of the physical system. The way of doing this is the subject of chapter 5.

Consider the 1PI 4-point function at one-loop level in our usual ϕ^4 theory in $d = 4$ space-time dimensions. Using a cut-off Λ in the momenta, we get



$$+ \quad \text{perms.} \quad = \quad \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\Big(\log\frac{\Lambda^2}{m^2 - sx(1-x)} - 1\Big)$$
$$+ \quad (s \to t) + (s \to u)\,, \tag{4.2.1}$$

where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ are the usual Mandelstam kinematical invariants. As we already discussed at length, we need a reormalization condition that fixes the counterterms needed to remove the logarithmic divergence in eq.(4.2.1). Let us define the coupling constant λ as the value of $\Gamma^{(4)}(p_i)$ at the symmetric point $s = t = u = 4m^2/3$:

$$\Gamma^{(4)}(s = t = u = 4m^2/3) \equiv -i\lambda\,. \tag{4.2.2}$$

In this way, the finite, renormalized 1PI four-point function reads

$$i\Gamma^{(4)}(s,t,u) = \lambda - \frac{\lambda^2}{32\pi^2}\int_0^1 dx \log\frac{m^2(1-4x(1-x)/3)}{m^2 - sx(1-x))} + (s \to t) + (s \to u)\,. \tag{4.2.3}$$

At high energies, when the absolute values of the Mandelstam variables is much greater than m, large logs $\sim \log m^2/E^2$ appear.[4] If E is sufficiently high, it can happen that the $\log m^2/E^2$ term is so large to compensate for the one-loop suppression given by $\sim \lambda/(16\pi^2)$, breaking down the perturbative expansion. Similarly, at two-loop level, terms of the form $\lambda^3 \log^2 m^2/E^2$ and $\lambda^3 \log m^2/E^2$ appear. In general, at l-loop level, all terms of the form $\lambda^{l+1}\log^l m^2/E^2$, $\lambda^{l+1}\log^{l-1} m^2/E^2$, $\ldots$, $\lambda^{l+1}\log m^2/E^2$ can appear. It is clear that if $\lambda \ll 1$, but $\lambda \log E^2/m^2 \sim 1$, the terms of the form $\lambda(\lambda^l \log^l m^2/E^2)$ are all of the same order. Logs of this form are called leading logs (LL) for obvious reasons. The terms of the form $\lambda^2(\lambda^l \log^l m^2/E^2)$ are denoted next-to-leading logs (NLL) and so on. There is actually a simple way to avoid the explicit appearance of large logs in $\Gamma^{(4)}$. The idea is based on the fact that we can choose the reormalization condition for the coupling as we wish. If we define the coupling constant at an energy scale $\mu \sim E$, replacing eq.(4.2.2) by

$$\Gamma^{(4)}(s = t = u = -\mu^2) \equiv -i\lambda(\mu)\,, \tag{4.2.4}$$

[4] Since $s+t+u = 4m^2$, the Mandelstam variables will necessarily have, at high energies, different signs, so that the argument of some of the logs in eq.(4.2.3) will be negative with branch cuts singularities. This is of course expected, since by the optical theorem, $i\Gamma^{(4)}$ should have an imaginary part. In order to avoid branch cut-singularities and imaginary amplitudes, that will not change the discussion that follows, we consider off-shell amplitudes at euclidean values of the momenta, where s, t and u are all negative.

the finite, 1PI four-point function (4.2.3) would now read

$$i\Gamma^{(4)}(s,t,u) = \lambda(\mu) - \frac{\lambda^2(\mu)}{32\pi^2}\int_0^1 dx \log\frac{m^2+\mu^2 x(1-x)}{m^2 - sx(1-x))} + (s\to t) + (s\to u)\,. \qquad (4.2.5)$$

and no large log term appears anymore. The arbitrary scale μ is denoted the sliding or renormalization scale. The coupling $\lambda(\mu)$ is determined by noting that the physics cannot depend on our arbitrary choice of scale μ. We must require that

$$\mu\frac{d\Gamma^{(4)}}{d\mu} = \left[\mu\frac{\partial}{\partial\mu} + \beta\Big(\lambda, \frac{m}{\mu}\Big)\frac{\partial}{\partial\lambda}\right]\Gamma^{(4)} = 0\,, \qquad (4.2.6)$$

where we have defined the function β as

$$\beta\Big(\lambda,\frac{m}{\mu}\Big) \equiv \mu\frac{d\lambda}{d\mu}\,. \qquad (4.2.7)$$

The behaviour of λ as a function of μ, as given by the first-order differential equation (4.2.7), is called the renormalizaion group (RG) flow of λ. It is straightforward to compute β given the explicit form (4.2.5) of $\Gamma^{(4)}$. We get

$$\beta\Big(\lambda,\frac{m}{\mu}\Big) = \frac{3\lambda^2}{16\pi^2}\int_0^1 dx\frac{\mu^2x(1-x)}{m^2+\mu^2x(1-x)} + \mathcal{O}(\lambda^3)\,. \qquad (4.2.8)$$

The coupling $\lambda(\mu)$ is determined by the first-order differential equation (4.2.7). In the extreme high (UV) and low (IR) energy regimes $\mu\gg m$, $\mu\ll m$, eq.(4.2.8) simplifies considerably, giving[5]

$$\begin{aligned}\beta_{UV} &\simeq \frac{3\lambda^2}{16\pi^2}\,,\\ \beta_{IR} &\simeq \frac{\lambda^2}{32\pi^2}\Big(\frac{\mu}{m}\Big)^2 \simeq 0,\end{aligned} \qquad (4.2.9)$$

whose solutions are simply

$$\lambda_{UV}(\mu) \simeq \frac{\lambda(\mu_0)}{1-\frac{3\lambda(\mu_0)}{16\pi^2}\log\frac{\mu}{\mu_0}}\,, \qquad \mu_0,\mu\gg m\,, \qquad (4.2.10)$$

$$\lambda_{IR}(\mu) \simeq \text{constant}\,, \qquad \mu\ll m\,. \qquad (4.2.11)$$

Strictly speaking, eqs.(4.2.10) and (4.2.11) are valid only in the asymptotic UV and IR regimes, but as a first crude approximation we can take $\mu_0 = m$ in eq.(4.2.10) and $\lambda(\mu)\simeq\lambda(m)$ for any $\mu\le m$ in eq.(4.2.11). We will perform a refined approximation in section 4.3, but for the moment this suffices to understand the main point of the RG analysis. Let

[5]Notice that β_{UV} in eq.(4.2.9) coincides with $bd\lambda(b)/db$ in eq.(4.1.12), as expected, being the same thing.

us compare the two equivalent 1PI 4-point functions (4.2.3) and (4.2.5) in the euclidean UV point $s = t = u \equiv -E^2 \gg m^2$:

$$i\Gamma^{(4)}(E) \simeq \lambda + \frac{3\lambda^2}{16\pi^2}\log\frac{E}{m} + \mathcal{O}(\lambda^2)\,, \qquad \text{(no RG)} \tag{4.2.12}$$

$$i\Gamma^{(4)}(E) \simeq \lambda(E) = \frac{\lambda}{1-\frac{3\lambda}{16\pi^2}\log\frac{E}{m}} + \mathcal{O}(\lambda^2)\,, \qquad \text{(RG)} \tag{4.2.13}$$

where $\lambda \simeq \lambda(m)$. Expanding in λ, we see that eq.(4.2.13) reproduces the one-loop result (4.2.12) but, in addition, automatically gives us all the LL logs of the form $\lambda\lambda^l \log^l E/m$. This is the key point of the RG evolution in perturbation theory: a powerful way to improve the perturbative expansion.

Various approximations have been made in comparing eqs.(4.2.3) and (4.2.5), such as $\lambda(m) \simeq \lambda(i2m/\sqrt{3})$ and $\mu_0 = m$. All these approximations change $\Gamma(E)$ at $\mathcal{O}(\lambda^2 \log^0 E/M)$ and are hidden in the $\mathcal{O}(\lambda^2)$ terms in eqs.(4.2.12) and (4.2.13). When $E \gg m$, these terms can consistently be neglected, being sub-leading with respect to the $\lambda^2 \log E/M$ terms. They are important if we want to go beyond the LL approximation, in which case they are the first terms in the $\lambda^2(\lambda^l \log^l E/M)$ (NLL) series. Resumming these logs require the knowledge of the $\lambda^3 \log E/M$ term, i.e. a two-loop perturbative computation.

The sliding scale μ and the RG evolution of the coupling are also useful in the IR. For instance, when $m = 0$, we immediately see that there is an IR singularity in the 1PI 4-point function (4.2.3), singularity which is avoided in eq.(4.2.5), which is well defined for $m = 0$.

Going back to eqs.(4.2.10) and (4.2.11), the RG evolution implies that the effective coupling constant in a QFT (i.e the one which does not give rise to large logs in amplitudes) depends on the energy scale, namely it is a "running" coupling constant. In particular eq.(4.2.10) implies that $\lambda(\mu_1) > \lambda(\mu_0)$ for $\mu_1 > \mu_0$ and predicts a pole at the scale

$$\mu_L = \mu_0\, e^{\frac{16\pi^2}{3\lambda(\mu_0)}} \tag{4.2.14}$$

where the coupling diverges (Landau pole).[6] The scale μ_L should not be taken too seriously, since at energies below μ_L, when the denominator in eq.(4.2.10) starts to significantly differ from one, perturbation theory breaks down and higher loop corrections are no longer negligible. However, eq.(4.2.14) indicates that, no matter how small $\lambda(\mu_0)$ is in the IR, at energies of order μ_L perturbation theory breaks down. On the other hand, perturbativity improves at low energies. When $m = 0$, and eq.(4.2.10) is valid for any range of μ_0 and μ, we see that $\lambda(\mu) \to 0$ as $\mu \to 0$, in agreement with the results derived using

[6] The scale dependence of μ_L on μ_0 is only apparent, as one can check by taking the derivative of μ_L with respect to μ_0.

the Wilsonian RG flow. In other words, provided the theory is in a perturbative regime at some scale μ_0, in the far IR it asymptotes a free field theory. Theories of this sort are denoted IR free. When $m \neq 0$, the running instead essentially stops for $\mu \lesssim m$. This is in agreement with the fact that at low energies no large logs to be resummed appear, see eq.(4.2.3). This is a manifestation of a much more general principle, actually a theorem (Appelquist-Carazzone), according to which the effects of massive particles at low energies should be negligible and go to zero when $m \to \infty$.[7] This is a key principle in QFT (and physics in general), allowing to reliably describe physical processes at some scale without necessarily knowing the "true" (if any) physical theory underlying all processes. In this sense any QFT should always be seen as an "effective" theory. We will come back to this point in the next chapter, where we will systematically study effective field theories.

4.3 Asymptotic Behaviours of β-Functions

The β-function is the crucial object to determine the evolution of a coupling constant. In general, in a theory with n couplings g_i, we have to solve a set of coupled differential equations of the kind

$$\mu \frac{dg_i}{d\mu} = \beta_i \,, i = 1, \ldots, n \,. \tag{4.3.1}$$

In general the β_i in eq.(4.3.1) depend on all the other couplings and masses of the theory. We can get rid of the masses by focusing only on the universal UV relevant coefficients, so that $\beta_i = \beta_i(g_j)$. It is not possible to describe the main properties of the solutions $g_i = g_i(\mu)$, because the system (4.3.1) is in general too complicated. We can simplify the situation by considering a single coupling g. In perturbation theory, $\beta(g)$ admits an expansion as follows:

$$\beta(g) = \beta_0 g^2 + \beta_1 g^3 + \mathcal{O}(g^4) \,. \tag{4.3.2}$$

Notice that it is always possible, by a proper coupling redefinition, to write the β-function expansion as in eq.(4.3.2). For instance, in QED $\beta(e)$ starts at cubic, rather than quadratic, order in the coupling, but it is enough to consider $\beta = \beta(e^2)$ rather than $\beta(e)$ to put the β in the form (4.3.2). At leading order, i.e. neglecting β_1, eq.(4.3.2) is solved by

$$g(\mu) = \frac{g_0}{1 - g_0 \beta_0 \log(\mu/\mu_0)} \,, \tag{4.3.3}$$

where $g_0 = g(\mu_0)$. The fate of $g(\mu)$ is entirely governed by the sign of β_0. When $\beta_0 > 0$, like in the ϕ^4 or QED cases, $g(\mu)$ is marginally irrelevant, it increases in the UV and at some high energy scale the theory is no longer perturbative.

[7] There are exceptions to this theorem that we will not consider here.

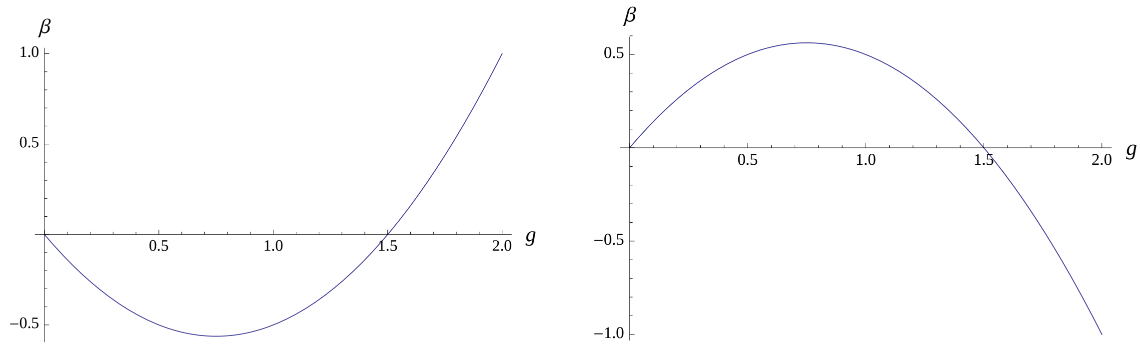


Figure 4.1: (Left panel) Schematic picture of the β-function of an UV stable fixed point (Right panel) Schematic picture of the β-function of an IR stable fixed point. In both cases we have taken $g^* = 3/2$. The units in both axes are arbitrary and irrelevant.

When $\beta_0 < 0$, on the other hand, the opposite happens. The coupling is marginally relevant and it decreases in the UV. In the limit of infinite energy, the coupling vanishes. Theories with (all) couplings of this kind are called asymptotically free. The low energy regime of these theories (including the spectrum of particles) is not perturbatively accessible. The coupling formally diverges at the scale

$$\Lambda = \mu e^{\frac{1}{g(\mu)\beta_0}} \,. \tag{4.3.4}$$

This scale, which is RG invariant, is said to be a dynamically generated scale, since there is no trace of it in the classical theory. It is a purely quantum effect. The most famous theory belonging to this class of theories is QCD.

It is useful to consider two other more exotic forms of β-function, that correspond to a perturbative expansion around a non-trivial value of the coupling (see figs.(4.1)). Let us assume that there exists a critical value of the coupling, g^*, such that $\beta(g^*) = 0$. If g is sufficiently close to g^*, we can expand β as follows:

$$\beta(g) = \beta(g^*) + \beta^*(g - g^*) + \mathcal{O}(g - g^*)^2 = \beta^*(g - g^*) + \mathcal{O}(g - g^*)^2 \,. \tag{4.3.5}$$

The solution of eq.(4.3.1), with β as in eq.(4.3.5), is

$$g(\mu) = g^* + (g(\mu_0) - g^*)\Big(\frac{\mu}{\mu_0}\Big)^{\beta^*} + \mathcal{O}(g - g^*)^2 \,. \tag{4.3.6}$$

Independently of $g(\mu_0)$, when $\beta^* < 0$ the coupling approaches g^* in the UV, while for $\beta^* > 0$ it approaches g^* in the IR. In the former case, g^* is called an ultraviolet stable fixed point of the RG flow, in the latter an infrared stable fixed point of the RG flow. When $g = g^*$ and β vanishes, no running occurs. In this case the theory is invariant under scaling transformations. More in general, it can be shown that it is invariant also under conformal transformations. A theory of this sort is called a Conformal Field Theory (CFT). Trivial CFT's are the free theories, for which $g = 0$ and β vanishes.

We now show that the coefficients β_0 and β_1 in eq.(4.3.2) do not depend on the renormalization scheme chosen, while higher order terms are scheme dependent. The coupling constants in two different schemes, call them schemes g and $\tilde{g}$, are equal at lowest order, but they start to differ at higher orders in the coupling (see e.g. eq.(4.5.11) for a concrete relation between coupling constants defined in different renormalization schemes). In general we have

$$\tilde{g}(g) = g + ag^2 + \mathcal{O}(g^3)\,, \tag{4.3.7}$$

where a is a constant. Their associated β-functions are related as

$$\tilde{\beta}(\tilde{g}) = \mu\frac{d\tilde{g}}{d\mu} = \mu\frac{dg}{d\mu}\frac{d\tilde{g}}{dg} = \beta(g)\frac{d\tilde{g}}{dg}\,. \tag{4.3.8}$$

Taking $\beta(g)$ as in eq.(4.3.2) we have

$$\begin{aligned}\tilde{\beta}(\tilde{g}) &= (\beta_0 g^2 + \beta_1 g^3 + \mathcal{O}(g^4))(1 + 2ag + \mathcal{O}(g^2)) \\ &= (\beta_0(\tilde{g} - a\tilde{g}^2)^2 + \beta_1\tilde{g}^3 + \mathcal{O}(\tilde{g}^4))(1 + 2a\tilde{g} + \mathcal{O}(\tilde{g}^2)) \\ &= \beta_0\tilde{g}^2 + \beta_1\tilde{g}^3 + \mathcal{O}(\tilde{g}^4)\,,\end{aligned} \tag{4.3.9}$$

where the $\mathcal{O}(\tilde{g}^4)$ terms are different from the $\mathcal{O}(g^4)$ terms in $\beta(g)$. Hence we have proved the scheme independence of the coefficients β_0 and β_1.

We will see in the following that even at one- (and two-) loop level the detailed form of a β-function in a theory depends on the renormalization scheme chosen, although its UV behaviour is universal and governed by the coefficients β_0 and β_1 above.

The generalization of eqs.(4.3.2) and (4.3.7) for multiple couplings are

$$\begin{aligned}\beta_i(g_j) &= \beta^{(0)}_{ijk} g_j g_k + \beta^{(1)}_{ijkl} g_j g_k g_l + \mathcal{O}(g_i^4)\,, \\ \tilde{g}_i(g_j) &= g_i + a_{ijk} g_j g_k + \mathcal{O}(g^3)\,.\end{aligned} \tag{4.3.10}$$

Proceeding as before, we get

$$\begin{aligned}\tilde{\beta}^{(0)}_{ijk} &= \beta^{(0)}_{ijk}\,, \\ \tilde{\beta}^{(1)}_{ijkl} &= \beta^{(1)}_{ijkl} + \frac{2}{3}\Big(a_{isl}\beta^{(0)}_{jks} - a_{skl}\beta^{(0)}_{ijs} + 2 \text{ perms. in } (j,k,l)\Big)\,.\end{aligned} \tag{4.3.11}$$

The two-loop coefficients of the β-function for multiple couplings are in general scheme-dependent.

4.4 The Callan-Symanzik RG equations

We have seen that at the quantum level we are necessarily led to introduce the sliding scale μ. It is often useful to define (renormalize) at the same scale μ not only the coupling constant but also the fields themselves. In other words, one substitues the usual physical condition of demanding unit residue of the two-point function at the pole mass, with a unit residue at $p^2 = -\mu^2$. Changing the definition of Z, now a function of μ, $Z = Z(\mu)$, will make external legs contribute to physical processes, because the Z entering in the LSZ reduction formulae is the physical μ-independent one. The independence of physics from μ translates now to a condition on the bare 1PI amplitudes. Taking again the $\lambda\phi^4$ theory as our working example, we have

$$\mu \frac{d\Big(Z^{-\frac{n}{2}}\Gamma^{(n)}\Big)}{d\mu} = Z^{-\frac{n}{2}} \left[\mu \frac{\partial}{\partial\mu} + \beta\Big(\lambda, \frac{m}{\mu}\Big)\frac{\partial}{\partial\lambda} - n\gamma\Big(\lambda, \frac{m}{\mu}\Big)\right]\Gamma^{(n)} = 0\,, \tag{4.4.1}$$

where

$$\gamma\Big(\lambda, \frac{m}{\mu}\Big) \equiv \frac{1}{2}\mu\frac{d\log Z}{d\mu} \tag{4.4.2}$$

is the anomalous dimension of the field ϕ, see eq. (4.1.13). Equations (4.4.1) are called the Callan-Symanzik (CS) equations. They can be solved as follows in the UV regime where we can neglect the m/μ dependence of β and γ. As a first step, we can get rid of the last term in eq.(4.4.1) by defining a new Green function $\hat{\Gamma}^{(n)}$:

$$\Gamma^{(n)}(p_i, \lambda, m, \mu) = e^{n\int_{\lambda_0}^{\lambda} d\lambda' \frac{\gamma(\lambda')}{\beta(\lambda')}}\hat{\Gamma}^{(n)}(p_i, \lambda, m, \mu) \tag{4.4.3}$$

such that

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right)\hat{\Gamma}^{(n)}(p_i, \lambda, m, \mu) = 0\,. \tag{4.4.4}$$

The above equation is solved by introducing an auxiliary variable t and t-dependent functions $\mu(t)$ and $\lambda(t)$ with

$$\mu(t) = e^t\mu, \qquad \lambda(0) = \lambda\,. \tag{4.4.5}$$

We then demand that $\lambda(t)$ is such that

$$\frac{d}{dt}\hat{\Gamma}^{(n)}(p_i, \lambda(t), m, \mu(t)) = 0\,. \tag{4.4.6}$$

When eq.(4.4.6) is satisfied, $\hat{\Gamma}^{(n)}(p_i, \lambda(t), m, \mu(t))$ is independent of t. Evaluating it at $t = 0$, we see that it coincides with our original green function $\hat{\Gamma}^{(n)}(p_i, \lambda, m, \mu)$. Using the chain rule of derivatives, eq.(4.4.6) equals

$$\left(\mu\frac{\partial}{\partial\mu} + \frac{d\lambda}{dt}\frac{\partial}{\partial\lambda}\right)\hat{\Gamma}^{(n)}(p_i, \lambda(t), m, \mu(t)) = 0\,. \tag{4.4.7}$$

Comparing eqs.(4.4.7) and (4.4.4), we see that the following equation should hold:

$$\frac{d\lambda(t)}{dt} = \beta(\lambda(t))\,, \tag{4.4.8}$$

For any t, provided that $\lambda(t)$ satisfies eq.(4.4.8), we have

$$\hat{\Gamma}^{(n)}(p_i, \lambda, m, \mu) = \hat{\Gamma}^{(n)}(p_i, \lambda(t), m, e^t\mu)\,. \tag{4.4.9}$$

In terms of the original Green functions $\Gamma^{(n)}$, we have

$$\Gamma^{(n)}(p_i, \lambda, m, \mu) = e^{n\int_0^t \gamma(t')dt'}\Gamma^{(n)}(p_i, \lambda(t), m, e^t\mu)\,. \tag{4.4.10}$$

Recall that the classical dimension of $\Gamma^{(n)}$ is $4-n$, so that

$$\Gamma^{(n)}(p_i, \lambda(t), m, e^t\mu) = e^{t(4-n)}\Gamma^{(n)}(e^{-t}p_i, \lambda(t), e^{-t}m, \mu)\,.t \tag{4.4.11}$$

Rescaling $p \to e^t p$, we finally get

$$\Gamma^{(n)}(e^t p_i, \lambda, m, \mu) = e^{t(4-n)+n\int_0^t \gamma(t')dt'}\Gamma^{(n)}(p_i, \lambda(t), e^{-t}m, \mu)\,. \tag{4.4.12}$$

Equation (4.4.12) tells us that a Green function at some energy scale $e^t p$, modulo an overall factor, equals the same Green function evaluated at the energy scale p, provided we replace the coupling λ with its running counterpart $\lambda(t)$, solution of eq.(4.4.8). In the high energy regime $t \to \infty$, mass corrections in the Green functions are negligible, in agreement with our expectations for an (IR) relevant coupling. The anomalous dimension γ affects the effective scaling dimension of the Green function, as expected.

The UV behaviour of $\Gamma^{(n)}$ is particularly simple for theories with an UV stable fixed point λ^*. In this case, $\int \gamma[\lambda(t')]dt' \simeq \int \gamma(\lambda^*)dt' = t\gamma^*$, with $\gamma^* \equiv \gamma(\lambda^*)$. The UV behaviour of $\Gamma^{(n)}$ reduces then to

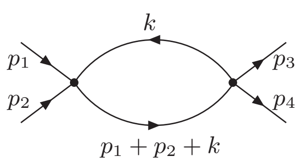
$$\Gamma^{(n)}(e^t p_i, \lambda, m, \mu) \simeq e^{4t-nt(1-\gamma^*)}\Gamma^{(n)}(p_i, \lambda^*, 0, \mu)\,. \tag{4.4.13}$$

Another interesting situation arises for UV free theories (like QCD) where the (one-loop) running of the coupling is given by eq.(4.3.3) with $\beta_0 < 0$. Assuming that $\gamma(t) = \gamma_0 g$, we have $\int \gamma[g(t')]dt' = \int \gamma(g)/\beta(g')dg' = \gamma_0/\beta_0 \log g(t)/g_0$ and thus for large t

$$\Gamma^{(n)}(e^t p_i, g, m, \mu) \simeq e^{t(4-n)}\Big(\frac{g(t)}{g_0}\Big)^{\frac{n\gamma_0}{\beta_0}}\Gamma^{(n)}(p_i, 0, \mu)\,. \tag{4.4.14}$$

4.5 Scheme dependence

The detailed form of the β-function of couplings and anomalous dimensions γ of fields has not per se an intrinsic physical meaning, since it depends on the renormalization scheme chosen. Only physical quantities are scheme-independent, and hence if distinct schemes give different expressions for physical amplitudes, necessarily the evolution of the couplings should compensate for the difference. This is evident if one considers that in certain schemes, typically associated to dimensional regularization such as minimal subtraction (MS) or modified minimal subtraction ($\overline{\rm MS}$), β and γ *do not* depend on the masses at all. For this reason, such schemes are denoted mass-indepedent, as opposed to momentum subtraction used in section 4.1 (hereafter denoted by MOM) and more in general to all schemes where β and γ depend on the masses of the particles. Is it then meaningful to talk about running coupling ? If so, how do we determine the "correct" running ? In order to answer to these questions, it is useful to focus on a concrete example and work it out in some detail. As usual, we take the ϕ^4 theory and compare the 1PI 4-point function (4.2.5), evaluated using MOM, with the expression one obtains in $\overline{\rm MS}$. The sliding scale in dimensional regularization arises as the scale relating the coupling constant in d dimensions (which acquires a dimensionality) with the physical one in 4 dimensions. By denoting $d = 4 - \epsilon$, in the ϕ^4 theory, one has $[\lambda_d] = 4 - d = \epsilon$ and hence $\lambda_d = \mu^\epsilon \lambda$:



$$= \frac{(-i\lambda\mu^\epsilon)^2}{2}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\frac{-i}{[k_E^2+m^2-sx(1-x)]^2}$$

$$= \frac{i\lambda^2\mu^{2\epsilon}}{2(4\pi)^{d/2}}\frac{\Gamma(\epsilon/2)}{[m^2-sx(1-x)]^{\epsilon/2}}\,. \qquad (4.5.1)$$

The $\overline{\rm MS}$ scheme is defined by adding the counterterm

$$\delta_\lambda = \frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma_E+\log 4\pi\right), \qquad (4.5.2)$$

where $\gamma_E \simeq 0.577$ is the Euler-Mascheroni constant. Summing over the t and u channel contributions, we get the finite, renormalized 4-point function

$$i\Gamma^{(4)}_{\overline{\rm MS}}(s,t,u) = \lambda(\mu) - \frac{\lambda^2(\mu)}{32\pi^2}\int_0^1 dx\log\frac{\mu^2}{m^2-sx(1-x))} + (s\to t) + (s\to u)\,. \qquad (4.5.3)$$

Notice how the numerators of the log term in eqs.(4.2.5) and (4.5.3) differ, the latter being simply μ^2. Demanding the μ-independence of the amplitude, as in section 4.1, we get

$$\beta_{\overline{\rm MS}} = \frac{3\lambda^2}{16\pi^2}\,, \qquad (4.5.4)$$

for *any* value of μ, independently of m. Thus, $\beta_{\overline{\rm MS}}$ in eq.(4.5.4) differs from $\beta_{\rm MOM}$ in eq.(4.2.8), although their UV behaviour is identical and given by eq.(4.2.10). This is a general result: the detailed form of β is scheme-dependent, its UV behaviour, given by mass-independent coefficients, is scheme-independent, as we have shown at the end of section 4.4. As far as the LL summation is concerned, both renormalization schemes are valid. At low energies, however, the two schemes lead to different behaviours: in MOM, λ essentially stops running below the scale m, in $\overline{\rm MS}$ the running never stops and would lead to a free theory when $\mu \to 0$! In the IR the physical picture is best given by MOM. This can be seen by noticing that eq.(4.2.5) is regular when $\mu \to 0$, while eq.(4.5.3) is IR divergent. This "fake" IR divergence and the corresponding large logs associated to it, are easily evaded by noticing that at arbitrarily low energies $s, t, u \ll m^2$, the large logs are avoided by taking $\mu^2 \simeq m^2$. In other words, in the $\overline{\rm MS}$, the necessity of avoiding spurious large logs forces us to never use the IR evolution of the coupling, being $\mu \simeq m$ the correct sliding scale in the IR. We thus conclude that the "correct" IR running behaviour is the one given by the MOM scheme. However, provided that one keeps in mind that the IR running in mass-independent schemes is fake, the latter can reliably be used. The best way to automatically get rid of this non-decoupling of heavy particles in mass-independent schemes is provided by using them in an effective field theory approach (more on effective field theories in chapter 5), where one integrates out the heavy particle so that, for $\mu \ll m$, the heavy state is no longer in the spectrum and do not contribute to the running anymore.

Amplitudes are easier to compute in mass-independent rather than mass-dependent schemes. However, physical couplings are typically defined by processes occurring at some energy scale and are directly related to the more physical mass-independent schemes. It is important to understand how to match couplings in mass-independent schemes with the physical couplings. Again, this is best illustrated with the specific instance of the ϕ^4 theory. First of all, let us find an approximation to the RG evolution of λ in MOM which is more refined than eqs.(4.2.10) and (4.2.11). We proceed as follows. The solution of eq.(4.2.7), for $\mu \gg m$, is given by

$$\lambda^{-1}(\mu) = -\frac{3}{16\pi^2}\log\mu + c\,, \tag{4.5.5}$$

where c is an integration constant. This is fixed by matching eq.(4.5.5) with the exact flow implied by eq.(4.2.7):

$$\lambda(\mu) = \lambda(\mu_0) + \int_{\mu_0}^{\mu} \frac{\beta}{\mu'} d\mu'\,. \tag{4.5.6}$$

Let us take $\mu_0 = 0$, $\lambda(0) \equiv \lambda$ and $\mu \gtrsim m$, so that in the range $[\mu_0, \mu]$, $\lambda(\mu) \simeq \lambda$, and eq.(4.5.6) is well approximated by

$$\lambda(\mu) \simeq \lambda + \frac{3\lambda^2}{32\pi^2}\int_0^1 dx \log\frac{\mu^2 x(1-x)}{m^2} = \lambda + \frac{3\lambda^2}{16\pi^2}\Big(\log\frac{\mu}{m}-1\Big)\,, \tag{4.5.7}$$

from which $c = \lambda^{-1} + 3/(16\pi^2)(1+\log m)$ and hence

$$\lambda^{-1}(\mu) = \lambda^{-1} - \frac{3}{16\pi^2}\Big(\log\frac{\mu}{m}-1\Big)\theta(\mu-m) \quad \text{(MOM)}\,. \tag{4.5.8}$$

Notice that $\lambda(\mu)$ in eq.(4.5.8) is discontinuous in m:

$$\lim_{\mu\to m^-}\lambda(\mu) = \lambda \neq \lim_{\mu\to m^+}\lambda(\mu) = \lambda + 3/(16\pi^2). \tag{4.5.9}$$

This discontinuity essentially takes into account the mass term disturbance to the UV running for $\mu \gtrsim m$. The constant $3/(16\pi^2)$ is often called mass threshold effect. It can be verified that eq.(4.5.8) is an excellent approximation to the exact one-loop running given by eq.(4.2.7) far away from the threshold region $\mu \simeq m$. The running coupling in the $\overline{\rm MS}$ scheme is simply given by

$$\lambda^{-1}(\mu) = \lambda^{-1} - \frac{3}{16\pi^2}\log\frac{\mu}{m}\,\theta(\mu-m) \quad (\overline{\rm MS})\,, \tag{4.5.10}$$

where the step function $\theta(\mu-m)$ is put by hand, for the reasons explained before. Comparing eqs.(4.5.8) with (4.5.10), we get, for $\mu > m$,

$$\lambda^{-1}_{\rm MOM}(\mu) = \lambda^{-1}_{\overline{\rm MS}}(\mu) + \frac{3}{16\pi^2}\,. \tag{4.5.11}$$

We can finally answer to the previous questions: is it meaningful to talk about running coupling ? If so, how do we determine the "correct" running ? There is no notion of "correct" coupling. The running given by any sensible scheme is meaningful, provided we consistently associate it to expressions computed in that scheme. The simplest mass-independent schemes and their associated simple running, can reliably be used in the UV, and then matched, by means of formulae analogous to eq.(4.5.11), to the physically defined coupling constants.

4.6 "Irrelevant" RG Flow of Dimensionful Couplings

The concept of sliding scale and running coupling can be extended to dimensionful (relevant or irrelevant) couplings. However, as should be clear from our overview on the general physical meaning of the RG flow in section 4.1, dimensionful couplings are governed by their classical dimensions and there is no real reason to study their quantum running.[8]

[8] This is of course only true at weak coupling in a perturbative context. At strong coupling everything can possibly happen, e.g. a classically irrelevant operator might turn into a marginal or even relevant operator at strong coupling.

Indeed, there is no analogue of the log resummation needed in treating marginal couplings and hence no need to improve the perturbative expansion.

In order to be concrete, let us consider the $\lambda\phi^4$ theory in five space-time dimensions and let us compute the one-loop RG evolution of λ. In five dimensions, the ϕ^4 theory is non-renormalizable and λ is an irrelevant coupling of mass dimension -1. We then introduce a mass scale M and write the interaction as $\lambda/M\phi^4$, with λ dimensionless. The scale M is the scale below which this theory makes sense as an effective field theory. We assume that the size of all dimensionful operators are governed by this scale, that is any dimensionful coupling of mass dimension $-n$ can be written as a dimensionless coupling of order one times $1/M^n$. Let us focus on the 1PI 4-point function. This is linearly divergent in a cut-off regularization. Let us first renormalize $\Gamma^{(4)}$ by momentum subtraction using eq.(4.2.4), with $\lambda \to \lambda/M$. The finite, 1PI four-point function, the 5d analogue of eq.(4.2.3), is

$$i\Gamma^{(4)}_{5d}(s,t,u) = \frac{\lambda(\mu)}{M} + \frac{\lambda^2(\mu)}{32\pi^2 M^2}\int_0^1 dx \left(\sqrt{m^2 - sx(1-x)} - \sqrt{m^2+\mu^2 x(1-x)}\right) \\ +(s\to t)+(s\to u)\,. \tag{4.6.1}$$

The logs are replaced by square roots in 5d. In order to see that the sliding scale is not of great help, fix, say, $\mu = 0$ and consider the high energy regime $-s \gg m^2$. Like the 4d case (4.2.4), the one-loop term grows and can in principle be comparable to the tree-level term. This will occur at energies $\lambda\sqrt{|s|}/(32\pi^2 M)\sim 1$, namely $|s| > M^2$. But this is beyond the regime in which the effective field theory makes sense, which is $|s| \ll M^2$! There is no energy regime in which the perturbative expansion can be improved by some RG resummation. The theory is simply weakly coupled for energies below M (and λ perturbative, of course) and becomes strongly coupled (unreliable) for energies above M. Nevertheless, there is nothing intrinsically wrong in introducing the sliding scale μ as we did. We can even compute the β-function for λ by using the CS equations (4.4.1). In this way we get the 5d analogue of eq.(4.2.8):

$$\beta_{5d} = \frac{3\lambda^2}{32\pi^2 M}\int_0^1 dx \frac{\mu^2 x(1-x)}{\sqrt{m^2+\mu^2 x(1-x)}} + \mathcal{O}(\lambda^3)\,. \tag{4.6.2}$$

For $\mu \gg m$ this simplifies to

$$\beta_{5d} \simeq \frac{3\lambda^2\sqrt{\mu^2}}{256\pi M}, \qquad \mu \gg m. \tag{4.6.3}$$

The approximate solution of the RG flow is

$$\lambda_{UV}(\mu) \simeq \frac{\lambda(\mu_0)}{1-\frac{3\lambda(\mu_0)}{256\pi M}(\mu-\mu_0)}\,, \qquad \mu_0,\mu \gg m\,, \tag{4.6.4}$$

$$\lambda_{IR}(\mu) \simeq \text{constant}\,, \qquad \mu \ll m\,. \tag{4.6.5}$$

Let us finally compare the $\Gamma^{(4)}$ one obtains with and without the use of the RG technique at high energies (the analogue of eqs.(4.2.12) and (4.2.13)) for $s = t = u \equiv -E^2 \gg m^2$:

$$i\Gamma^{(4)}_{5d}(E) \simeq \frac{\lambda}{M} + \frac{3\lambda^2 E}{256\pi M^2} + \mathcal{O}(\lambda^2)\,, \qquad \text{(no RG)} \tag{4.6.6}$$

$$i\Gamma^{(4)}_{5d}(E) \simeq \lambda(E) = \frac{\lambda}{1-\frac{3\lambda E}{256\pi M}} + \mathcal{O}(\lambda^2)\,. \qquad \text{(RG)} \tag{4.6.7}$$

Like in 4d, eq.(4.6.7) reproduces the one-loop result (4.6.6) and, in addition, encodes higher order terms. However, in contrast to the 4d case, the higher order terms are not "special" terms in the perturbative expansion, being of order $\lambda^{l+1}(E/M)^l$, which is the order expected for a generic l-loop computation. Hence eq.(4.6.7) should only be trusted at $\mathcal{O}(\lambda^2)$, in which case it merely reproduces the perturbative result obtained with no RG technique.

The RG evolution of non-marginal couplings is also scheme-dependent. This is best seen in our 5d ϕ^4 theory by noting that in DR regularization the $\Gamma^{(4)}$ is completely finite at one-loop level! Indeed, the Γ function is well-behaved for negative half-integers values of its argument and being the one-loop integral proportional to $\Gamma(2-d/2)$, no divergence occurs in this regularization. No sliding scale is needed and hence

$$\beta^{DR}_{5d} = 0\,, \tag{4.6.8}$$

in contrast to eq.(4.6.3).

It should be clear that the absence of logs in our computation is not general for any dimensionful coupling. It is enough to consider again $\Gamma^{(4)}$ in the $\lambda\phi^4$ theory in 6d to see logs appearing in the loop computation. In these cases, there is in principle a window of energies where the RG technique is useful, but in practice this is quite limited because log's grow slowly and the range of energies we can explore is limited by the range of validity of the non-renormalizable effective theory.

A similar analysis applies for relevant couplings, with the obvious crucial difference that their effect decreases, rather than increases, in the UV. Mass terms can also be seen as relevant couplings. As we are now going to explicitly show, their β-functions are scheme dependent. Consider once again the one-loop mass correction in the ϕ^4 theory, given by the tadpole graph in figure 4.2. In DR, the one-loop 1PI 2-point function reads

$$\Gamma^{(2)}(p) = \frac{-i\mu^\epsilon\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} - i\delta_m = -\frac{i\mu^\epsilon\lambda}{2(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(1-d/2)}m^{d-2} - i\delta_m\,. \tag{4.6.9}$$

In the $\overline{\text{MS}}$ scheme the counterterm is

$$\delta_m = \frac{m^2\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + \log 4\pi\right)\,, \tag{4.6.10}$$

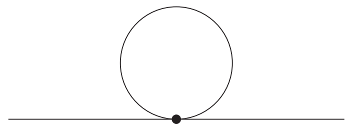


Figure 4.2: One-loop tadpole graph contributing to the renormalization of the mass in the ϕ^4 theory.

so that

$$\Gamma^{(2)}_{\overline{\rm MS}}(p) = \frac{im^2\lambda}{32\pi^2}\Big(1+\log\frac{\mu^2}{m^2}\Big)\,. \tag{4.6.11}$$

Using a cut-off regularization in the momentum integral would give

$$\Gamma^{(2)}(p) = -\frac{i\lambda}{16\pi^2}\Big(\Lambda^2+m^2\log\frac{m^2}{\Lambda^2+m^2}\Big) - i\delta_m\,. \tag{4.6.12}$$

The counterterm is now fixed by demanding m to be the physical mass, namely $\Gamma^{(2)}(p^2 = m^2) = 0$. We get

$$\delta_m = -\frac{\lambda}{16\pi^2}\Big(\Lambda^2+m^2\log\frac{m^2}{\Lambda^2+m^2}\Big) \tag{4.6.13}$$

and hence no finite one-loop correction left:

$$\Gamma^{(2)}(p) = 0. \tag{4.6.14}$$

By adding the tree-level term $i(p^2-m^2)$ and demanding the equality of the 1PI 2-point functions in the two schemes we get

$$m^2 = m^2_{\overline{\rm MS}}(\mu) - \frac{m^2\lambda}{32\pi^2}\Big(1+\log\frac{\mu^2}{m^2}\Big)\,, \tag{4.6.15}$$

where m in the left-hand side of eq.(4.6.15) is the physical, μ-independent, mass, whereas $m^2_{\overline{\rm MS}}(\mu)$ is the (unphysical) running mass in the $\overline{\rm MS}$ scheme. Thus

$$0 = \mu\frac{dm^2}{d\mu} = \beta_{m^2} - \frac{m^2\lambda}{16\pi^2} + \mathcal{O}(\lambda^2) \Longrightarrow \beta_{m^2} = \frac{m^2\lambda}{16\pi^2}\,, \qquad (\overline{\rm MS})\,. \tag{4.6.16}$$

Another possible definition of mass is provided by momentum subtraction. For instance, we might define $m^2(\mu)$ as

$$\Gamma^{(2)}(p^2=\mu^2) = -m^2(\mu)\,, \tag{4.6.17}$$

in which case one trivially gets the relation between the physical mass m^2 and the MOM mass defined in eq.(4.6.17):

$$m^2 = m^2_{\rm MOM}(\mu) + \mu^2\,, \tag{4.6.18}$$

and hence

$$\beta_{m^2} = -2\mu^2, \qquad ({\rm MOM})\,. \tag{4.6.19}$$

4.7 "Relevant" RG Flow of Dimensionful Couplings and Renormalization of Composite Operators

The β-function of a coupling depends in general on all the other coupling constants present in the theory. Thus, in a theory where marginal *and* (ir)relevant couplings are present together, the former can lead to large logs effects on the latter that should be summed. In other words, the quantum RG flow of (ir)relevant operators, induced by marginal couplings, can and should be taken into account. This is what we mean by "relevant" RG flow in the title of this section.

Before showing how this RG flow is computed, it is useful to introduce the concept of composite operator and its corresponding anomalous dimension. Composite operators are obtained by taking products of elementary fields (and their derivatives) at the same space-time point. Composite operators can be relevant, marginal or irrelevant: ϕ^2, ϕ^4, $(\partial\phi)^2$, ϕ^6, $(\partial\phi)^4$, etc. are all composite operators. Like the elementary fields, composite operators require a wave-function renormalization at a scale μ. If we denote by $\mathcal{O}$ a generic composite operator, in analogy to the elementary field case, we write the bare operator $\mathcal{O}^0 = \mathcal{O}(\mu)Z^{\mathcal{O}}(\mu)$.

Let $G^{(n)}$ be a generic n-point connected Green function of n operators $\mathcal{O}_i$:[9]

$$G_0^{(n)}(p_1,\ldots,p_n) = \langle \mathcal{O}_1^0(p_1)\ldots\mathcal{O}_n^0(p_n)\rangle = \prod_{i-1}^{n} Z^{\mathcal{O}_i}(\mu)G^{(n)}(p_1,\ldots,p_n)\,. \tag{4.7.1}$$

The straightforward generalization of the CS eqs.(4.4.1) – which are also valid for connected, rather than 1PI, Green functions – are

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_i(\lambda)\frac{\partial}{\partial\lambda_i} + \sum_{i=1}^{n}\gamma^{\mathcal{O}_i}(\lambda)\right)G^{(n)} = 0\,, \tag{4.7.2}$$

where

$$\gamma^{\mathcal{O}_i}(\lambda) = \mu\frac{d\log Z^{\mathcal{O}_i}}{d\mu} \tag{4.7.3}$$

is the anomalous dimension of the operator $\mathcal{O}_i$. In general, the situation is more complicated, because operators with the same quantum numbers typically mix under renormalization. The anomalous dimension becomes then a matrix of anomalous dimensions: $\mathcal{O}_i^0 = Z_{ij}^{\mathcal{O}}(\mu)\mathcal{O}_j(\mu)$ and

$$\gamma_{ij}(\lambda) = (Z_{ik}^{\mathcal{O}})^{-1}\mu\frac{dZ_{kj}^{\mathcal{O}}}{d\mu}\,. \tag{4.7.4}$$

Let us come back to our original problem of studying the RG evolution of (ir)relevant couplings, driven by marginal couplings. Depending on the energy scale, relevant and

[9]Composite or elementary. In the latter case $Z^{\phi}(\mu) = \sqrt{Z(\mu)}$.

irrelevant operators parametrically either dominate the physics or are negligible. For instance, in the IR relevant operators, such as masses, are the dominant effect, while in the UV irrelevant operators make a theory ill-defined. We focus here on the situation in which the (ir)relevant operators can be seen as a small deformation in the theory and can be studied at the linear level. For instance, the effect of a mass term in the UV or the insertion of an irrelevant operator in the IR. Linear level means that we neglect graphs obtained by more than one insertion of the (ir)relevant operator. Let us denote by c a generic coupling associated to the operator $\mathcal{O}_c$ and by λ_i the marginal couplings whose effect on c we want to sum by means of the RG flow. Let us denote by $G^{(n)}$ a generic n-point Green function obtained from a Lagrangian containing, among the other marginal interactions, the term $c\mathcal{O}_c$. We can bring down from the action the term $c\mathcal{O}_c$ so that (in momentum space)

$$
\begin{aligned}
G^{(n)}(c,p_1,\dots,p_n) = \langle\phi(p_1)\dots\phi(p_n)\rangle_c = & \sum_{k=0}^{\infty}\frac{c^k}{k!}\langle\phi(p_1)\dots\phi(p_n)\mathcal{O}_c^k(0)\rangle_0 \\
\equiv & \sum_{k=0}^{\infty}\frac{c^k}{k!}G^{(n,k)}(0,p_1,\dots,p_n)\,. \qquad (4.7.5)
\end{aligned}
$$

The CS eqs. satisfied by the $G^{(n,k)}$ are

$$
\left(\mu\frac{\partial}{\partial\mu}+\beta_i(\lambda)\frac{\partial}{\partial\lambda_i}+n\gamma+k\gamma^{\mathcal{O}_c}(\lambda)\right)G^{(n,k)}(0,p_1,\dots,p_n)=0\,. \qquad (4.7.6)
$$

We can multiply eq.(4.7.6) by $c^k/k!$ and sum over k to write

$$
\left(\mu\frac{\partial}{\partial\mu}+\beta_i(\lambda)\frac{\partial}{\partial\lambda_i}+n\gamma+c\gamma^{\mathcal{O}_c}(\lambda)\frac{\partial}{\partial c}\right)G^{(n)}(c,p_1,\dots,p_n)=0\,, \qquad (4.7.7)
$$

where we used that $c\partial G^{(n)}/\partial c = \sum_k kc^k/k!G^{(n,k)}$. The$\beta$-function for c equals $\beta_c = c\gamma^{\mathcal{O}}$. More in general, for a set of interactions $c_n\mathcal{O}_n$ that mix under renormalization, we have

$$
\beta_{c_n} = \gamma_{nm}(\lambda)c_m\,. \qquad (4.7.8)
$$

The RG behaviour of the c_n is governed by the anomalous dimensions of the operators $\mathcal{O}_n$, which are function of the marginal couplings λ_i. Notice that the linear dependence of β_{c_n} from the couplings c_m crucially arises from the assumption of neglecting interactions involving more than one coupling c_n.

Irrelevant operators can also come from an underlying renormalizable theory when some degrees of freedom are integrated out. We will study this topic in the chapter on Effective Field Theories, where we will also consider an explicit example of RG flow of an irrelevant coupling.

The mass term is a particular relevant operator. Although the naive definition of β_{m^2} given in section 4.6 is scheme-dependent, its definition at linear order in terms of eq.(4.7.8) is scheme-independent and allows to study the effect of marginal couplings in the evolution of the mass term, seen now as the coupling of the associated relevant operator. This is particularly useful in theories where the mass spectrum is not perturbatively accessible (like quarks in QCD) and the mass cannot be defined as the pole of the two-point function of elementary fields. In the ϕ^4 example, the composite operator associated to the mass is ϕ^2. The anomalous dimension γ^{ϕ^2} of ϕ^2 is derived by studying the Green function $G^{(2,1)} = \langle \phi\phi\phi^2 \rangle$. A simple computation in the MOM or $\overline{\rm MS}$ schemes is enough to show that the UV behaviour of γ^{ϕ^2} is universal. One finds

$$\gamma^{\phi^2} = \frac{\lambda}{16\pi^2}\,. \tag{4.7.9}$$

Using eq.(4.7.8), we conclude that $\beta_{m^2} = m^2\gamma^{\phi^2} = m^2\lambda/(16\pi^2)$, which coincide with the result found in eq.(4.6.16) in the $\overline{\rm MS}$ scheme.[10] There is nothing wrong in computing amplitudes in terms of unphysical masses (like the mass terms in the $\overline{\rm MS}$ scheme), provided one eventually relates these masses to the physical ones, using formulae like eq.(4.6.15). As a matter of fact, $\overline{\rm MS}$ masses are often used in the literature, being the $\overline{\rm MS}$ scheme one of the most popular scheme in perturbative computations. For particles, such as quarks, that do not appear in asymptotic states and for which the physical mass definition is unavailable, the story is a bit more complicated. The $\overline{\rm MS}$ quark mass, for example, can be defined to be $m^2_{\overline{\rm MS}}(M)$, evaluated at some indirectly derived "pole" mass M. For a heavy quark, for which $M \gg \Lambda_{QCD}$, the mass M can roughly be computed as the mass of the meson bound states $\bar{Q}Q$ divided by two. For light quarks, M can be computed using chiral perturbation theory, as explained in section 6.4.1. For $M \simeq \Lambda_{QCD}$, more complicated procedures are needed.

4.8 RG Improved Effective Potential

The RG technique is also useful in the context of effective potentials. The summation of LL leads to so called RG improved Effective Potentials. Our favorite ϕ^4 theory is particularly instructive in this case, since it shows how RG improved potentials help us in correcting fake perturbative results. We have seen in ??? that the CW potential for a massless ϕ^4

[10]This is not a coincidence. The schemes associated to DR, such as MS or $\overline{\rm MS}$, in which all power-like divergencies are set to zero, automatically select the universal contributions to the running parameters. This is another great virtue of mass-independent renormalization schemes.

theory reads (omitting the irrelevant constant term):

$$V_{eff}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2}{256\pi^2}\phi^4 \log \phi^2 \,. \tag{4.8.1}$$

Let us look for the extrema of ϕ:

$$0 = \frac{dV_{eff}(\phi)}{d\phi} = \frac{\phi^3}{384\pi^2}(64\pi^2 + 3\lambda + 6\lambda \log \phi^2) \to \phi = 0\,, \phi = \pm\bar{\phi} = \pm e^{-(\frac{1}{4}+\frac{16\pi^2}{3\lambda})} \,. \tag{4.8.2}$$

The classical minimum at $\phi = 0$ turns into a maximum and two new symmetric minima arise at $\pm\bar{\phi}$. The one-loop truncation of the potential is reliable provided the tree-level term is greater than the one-loop term, namely for field values such that

$$\frac{3\lambda}{16\pi^2}|\log \phi| \ll 1 \,. \tag{4.8.3}$$

The condition (4.8.3) is manifestly violated at the minimum $\bar{\phi}$, so we cannot trust the result we have just found for such small values of ϕ. This problem is easily solved by RG arguments. Let us define a running coupling $\lambda(\mu)$ by

$$V_{eff}(\phi = \mu) \equiv \frac{\lambda(\mu)}{4!}\mu^4 \,. \tag{4.8.4}$$

In terms of $\lambda(\mu)$ the potential reads

$$V_{eff}(\phi) = \frac{\lambda(\mu)}{4!}\phi^4 + \frac{\lambda^2(\mu)}{256\pi^2}\phi^4 \log \frac{\phi^2}{\mu^2} \,. \tag{4.8.5}$$

We already know how $\lambda(\mu)$ flows with the energy scale, but it is instructive to see how the β-function for λ can be computed by demanding the μ-invariance of V_{eff}. Recalling also that $\gamma_\phi = 0$ up to one-loop level, we get

$$0 = \mu\frac{dV_{eff}}{d\mu} = \beta(\lambda)\frac{\phi^4}{4!} - \frac{\lambda^2\phi^4}{128\pi^2} + \mathcal{O}(\lambda^3) \to \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \,. \tag{4.8.6}$$

By choosing $\mu = \phi$ in eq.(4.8.5), we can get rid of the log term and write

$$V_{eff}(\phi) = \frac{\lambda(\phi)}{4!}\phi^4 = \frac{\lambda_0}{1 - \frac{3\lambda_0}{16\pi^2}\log\frac{\phi}{\phi_0}}\frac{\phi}{4!} \,, \tag{4.8.7}$$

where ϕ_0 is an arbitrary scale. The minima at $\phi = \pm\bar{\phi}$ have disappeared in the potential (4.8.7), which manifestly increases monotonically when ϕ increases. The potential (4.8.7) is the RG improved version of the effective potential (4.8.1). Expanding the log term in eq.(4.8.7), we recover eq.(4.8.1) plus all the usual LL terms summed by the RG technique. The origin of the fake result (4.8.2) should now be clear. The minima at $\phi = \pm\bar{\phi}$ were obtained by forgetting the large log's that appear for so small values of ϕ (small energies). Being the ϕ^4 theory free in the IR, the effective coupling at such small energies becomes smaller and smaller and the actual minimum is in fact the tree-level one.

4.9 Anomalous Dimension of the Photon and QED β-function

We have seen in chapter ?? that the exact photon propagator has the following structure:

$$G_{\mu\nu}(p) = \frac{-i}{p^2}\left[\frac{\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{1-\Pi(p^2)} + \xi\frac{p_\mu p_\nu}{p^2}\right], \tag{4.9.1}$$

so that the radiative inverse propagator reads as

$$\Gamma_{\mu\nu}(p) = i(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\Pi(p^2)\,. \tag{4.9.2}$$

The physical wave function renormalization of the photon is $\Pi(0)=0$, implying that at the pole mass $p^2=0$, the residue of the transverse part of the propagator is $-i$. This condition is satisfied by appropriately fixing the counterterm Z_3 (see eq.???). Alternatively, we might demand that $\Pi(-\mu^2)=0$, in which case $Z_3 = Z_3(\mu^2)$. At one-loop level, Π is the sum of the counterterm and the purely one-loop graph contribution $I(p^2)$. In dimensional regularization (DR) we have

$$-\mu^\epsilon(Z_3-1) + I(-\mu^2) = 0\,, \tag{4.9.3}$$

where

$$I(p^2) = \mu^\epsilon\frac{8e^2\Gamma(\epsilon/2)}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{x(1-x)}{[m^2-p^2x(1-x)]^{\epsilon/2}}\,. \tag{4.9.4}$$

Expanding in ϵ, we get

$$Z_3 = 1 + \frac{c}{\epsilon} + \frac{e^2}{2\pi^2}\int_0^1 dx\,x(1-x)\log\left[m^2+\mu^2x(1-x)\right], \tag{4.9.5}$$

where c is an irrelevant μ-independent constant. At leading order in the coupling constant, eq.(4.4.2) gives

$$\gamma_A = \frac{1}{2}\mu\frac{d\log Z_3}{d\mu} = \frac{e^2}{2\pi^2}\int_0^1 dx\,\frac{x^2(1-x)^2\mu^2}{m^2+\mu^2x(1-x)}\,. \tag{4.9.6}$$

Interestingly enough, the QED WT allow to compute the QED β-function for the coupling without performing any further computation. Indeed, the μ-dependence of the running coupling $e(\mu)$ is easily extracted by noticing that $e_0 = Z_3^{-1/2}e$, where the bare charge e_0 manifestly does not depend on μ. Then

$$0 = \mu\frac{de_0}{d\mu} = Z_3^{-1/2}\Big(-e\gamma_A+\beta\Big) \Longrightarrow \beta = e\gamma_A = \frac{e^3}{2\pi^2}\int_0^1 dx\,\frac{x^2(1-x)^2\mu^2}{m^2+\mu^2x(1-x)}\,. \tag{4.9.7}$$

The UV and IR behaviours of $e(\mu)$ are similar to what found in the $\lambda\phi^4$ theory. For $\mu\gg m$, $\gamma_A\simeq e^2/(12\pi^2)$ and $\beta\simeq e^3/(12\pi^2)$. It is useful to write the RG behavior in terms of $\alpha\equiv e^2/(4\pi)$ and solve for its inverse. One gets, for $\mu_0,\mu\gg m$,

$$\alpha^{-1}(\mu) = \alpha^{-1}(\mu_0) - \frac{2}{3\pi}\log\frac{\mu}{\mu_0}\,. \tag{4.9.8}$$

Chapter 5

Effective Field Theories

We have so far mostly considered renormalizable quantum field theories, since these theories have simple renormalization properties. Once a finite number of observables are fixed (by experiment), these theories allow for very accurate predictions. It is however clear that renormalizable theories such as the SM cannot be the ultimate theory of nature, since gravity is not included. From this perspective, renormalizable theories are special because they allow us to hide in a redefinition of a finite number of parameters all our ignorance of the UV physics which is beyond the model in consideration. Aside from an improved calculability, there is no conceptual reason to focus only on such a restricted class of theories.

Unless one is so ambitious to try to construct the ultimate, possibly, finite theory of everything, any quantum field theory should be seen as an *effective* field theory, namely a theory that is reasonably accurate in a given energy regime and is replaced by some other more complete theory at a given UV scale M, beyond which it is incorporated in a larger theory. This happens all the time in physics and is the most efficient way to study a phenomenon keeping only the relevant degrees of freedom. Classical examples are the Fermi theory of electroweak interactions or the pion chiral Lagrangian. These theories are effective, being replaced, at sufficiently high energies, by the SM and QCD, respectively. It is obvious that if we are interested in processes happening at a given scale $E \ll M$ and involving light fields only, with masses much smaller than M, the heavy states with mass $\sim M$ cannot be produced, so that the latter can be integrated out. In a path integral approach, the complete 1PI generating functional of light fields is given by

$$e^{i\Gamma(\phi_l)} = \int_{1PI} \mathcal{D}\eta_l \, e^{iS_{IR}(\phi_l+\eta_l)} \,, \quad e^{iS_{IR}(\phi_l)} = \int \mathcal{D}\phi_h \, e^{iS_{UV}(\phi_l,\phi_h)} \,, \tag{5.0.1}$$

where ϕ_l and ϕ_h schematically denote light and heavy fields. Here S_{UV} and S_{IR} represent the underlying UV and IR effective actions, respectively. At tree-level, we simply have

$$S_{IR}^{(0)}(\phi_l) = S_{UV}(\phi_l, \phi_h(\phi_l))\,, \tag{5.0.2}$$

where $\phi_h(\phi_l)$ is the classical solution to the heavy field equations of motion. Being the integrated fields heavy, the action admits a well-defined momentum expansion in terms of local operators. We could proceed by computing the one-loop and higher loop effective actions $S_{IR}^{(l)}$ in a similar way. In this way, the whole effective action S_{IR} can be reconstructed. The key point of effective field theory is to *replace* this procedure by a simpler one, where we compute radiative effects involving the light fields starting from the action

$$S_{EFT}(\phi_l) = S_{IR}^{(0)}(\phi_l) + \Delta S(\phi_l)\,, \tag{5.0.3}$$

where ΔS encodes all higher dimensional local operators up to some order, depending on the accuracy we want to achieve. All the unknown couplings multiplying the higher dimensional local operators are then fixed by comparing given quantities as computed in the UV theory and in the EFT. In principle, one might want to directly compare S-matrix elements, but it is often easier to compare 1PI functions, generically off-shell.[1] This procedure is called "matching". Of course, matching requires to also perform a computation in the full theory but, as we will see, it is typically much easier to do computations in the EFT and then matching, rather than computing everything in the full theory.

5.1 Two Scalars

Consider

$$\mathcal{L}_{UV} = \frac{1}{2}(\partial H)^2 - \frac{1}{2}M^2 H^2 + \frac{1}{2}(\partial L)^2 - \frac{1}{2}m^2 L^2 - \frac{g}{2} H L^2\,, \tag{5.1.1}$$

where $m \ll M$ and the dimensionful coupling g is assumed to be of $\mathcal{O}(M)$.[2] The most general EFT for L reads as follows:

$$\mathcal{L}_{IR} = \frac{1}{2}Z_L(\partial L)^2 - \frac{1}{2}\tilde{m}^2 L^2 - \frac{\lambda}{4!}L^4 + \text{higher dimensional operators}\,. \tag{5.1.2}$$

[1] This might seem at the same time too conservative (physical observables correspond to on-shell S matrix elements and not to the whole Green functions) and improper (different off-shell Green functions can lead to the same on-shell results). We will nevertheless require the off-shell equality of 1PI functions in our examples, since typically one has to invoke field redefinitions to construct different off-shell Green functions leading to the same physics and we will never consider this possibility. Moreover, once the 1PI functions are compared at an unphysical point, by analyticity we are ensured that their on-shell properties are equal.

[2] This theory is not realistic, since the cubic potential HL^2 is not positive definite, but the considerations we want to make are insensitive to this stability problem.

The tree-level integration of H gives $H = -gL^2/(2M^2) + \mathcal{O}(p^2/M^4)$. Plugged back in eq.(5.1.1) gives

$$\mathcal{L}_{IR}^{(0)} = \frac{1}{2}(\partial L)^2 - \frac{1}{2}m^2 L^2 - \frac{\lambda}{4!}L^4 + \mathcal{O}(g^2 M^{-4}) \tag{5.1.3}$$

so that tree-level matching gives

$$Z_L = 1 + \mathcal{O}(g^2)\,, \quad \tilde{m}^2 = m^2 + \mathcal{O}(g^2)\,, \quad \lambda = -\frac{3g^2}{M^2} + \mathcal{O}(g^4)\,. \tag{5.1.4}$$

It is important to emphasize that the factors Z_L, $\tilde{m}^2$, etc. in eq.(5.1.4) are the analogues of $(1+\delta Z)$, $m^2+\delta m^2$, etc. defined in eq.(4.1.5) in the Wilsonian RG flow. They are finite terms, which at the quantum level indicate how the EFT fields, masses etc. differ from their UV counterparts to compensate for the different UV behaviour of the two theories, so that low-energy observables match in the two descriptions. They should not be confused with the usual counterterms needed to subtract divergencies, which we will never explicitly write, always fixing them by an $\overline{\text{MS}}$ subtraction scheme.

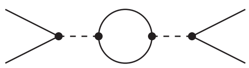
We can directly use $\mathcal{L}_{IR}^{(0)}$ as effective theory to compute Green functions for energies much smaller then M. For instance, the $\overline{\text{MS}}$ renormalized four-point 1PI function reads (see eq.(4.5.3)):

$$i\Gamma^{(4)}(s,t,u) = \lambda(\mu) + \frac{\lambda^2(\mu)}{32\pi^2}\int_0^1 dx \left[\log\frac{m^2 - sx(1-x)}{\mu^2} + s \to t + s \to u\right]. \tag{5.1.5}$$

By taking $s \simeq t \simeq u \simeq \mu$ we get $i\Gamma^{(4)}(\mu) \simeq \lambda(\mu)$, where

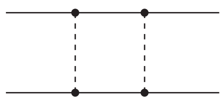
$$\lambda^{-1}(\mu) = \lambda^{-1}(\mu_0) - \frac{3}{16\pi^2}\log\left(\frac{\mu}{\mu_0}\right) \tag{5.1.6}$$

is the usual one-loop running in the ϕ^4 theory, assuming $\mu, \mu_0 > m$. The natural scale where the effective theory parameter λ should be matched, at one-loop level, with the underlying UV parameters g and M is at the scale $\mu_0 = M$, since this is the energy scale boundary between the two theories. This is understood by comparing the 1PI four point amplitude $\Gamma^{(4)}$. The following diagrams contribute to $\Gamma^{(4)}$ in the UV theory (continuous and dashed lines correspond to light and heavy fields, respectively). At leading order in an $1/M$ expansion and for zero external momentum, we get



$$+\text{ perms.} = \left(\frac{1}{2}\times 3\right)(-ig)^4\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left(\frac{i}{k^2-m^2}\right)^2\left(\frac{-i}{M^2}\right)^2 = -\frac{3ig^4}{32\pi^2M^4}\log\frac{m^2}{\mu^2}\,,$$

$$+\text{ perms.} = (6)(-ig)^4\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left(\frac{i}{k^2-m^2}\right)^2\frac{-i}{M^2}\frac{i}{k^2-M^2} = -\frac{3ig^4}{8\pi^2M^4}\left(1+\log\frac{m^2}{M^2}\right), \tag{5.1.7}$$



$$+\text{ perms.} = (6)(-ig)^4\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left(\frac{i}{k^2-m^2}\right)^2\left(\frac{i}{k^2-M^2}\right)^2 = -\frac{6ig^4}{8\pi^2M^4}\left(1+\frac{1}{2}\log\frac{m^2}{M^2}\right),$$

where in parenthesis we have denoted the geometrical factor (permutations included) of the graph. Including the tree-level term, we then get

$$i\Gamma^{(4)}(0) = -\frac{3g^2(\mu)^2}{M^2}+\frac{3g^4(\mu)}{8\pi^2M^4}\left(3+2\log\frac{m^2}{M^2}+\frac{1}{4}\log\frac{m^2}{\mu^2}\right). \tag{5.1.8}$$

Comparing eq.(5.1.5) (at zero momentum) with eq.(5.1.8) and using the tree-level relation (5.1.4) for λ, we get

$$\lambda(\mu) = -\frac{3g^2(\mu)}{M^2}+\frac{3g^4}{8\pi^2M^4}\left(3+2\log\frac{\mu^2}{M^2}\right). \tag{5.1.9}$$

Notice how all the (potentially large) logs involving the light mass m^2 cancelled from eq.(5.1.9). This is not a coincidence. All the IR properties of the UV theory (as well as large IR effects) are reliably described by the EFT by construction, so that the matching equations are always regular in the IR. It is quite clear from eq.(5.1.9) that the best scale to match the two theories is $\mu = M$, in which case we have

$$\lambda(M) = -\frac{3g^2(M)}{M^2}+\frac{9g^4}{8\pi^2M^4}\,. \tag{5.1.10}$$

Taking $\mu_0 = M$ in eq.(5.1.6) and using (5.1.10), will allow us to take into account of possible large logs of the form $\log(\mu/M)$. Notice that the g^4/M^4 term in eq.(5.1.10) can safely be neglected at one-loop level, starting to be relevant only at two-loop level, if NLL want to be resummed. In other words, no one-loop computation of the 1PI 4-point function would have been needed in matching the theory at $\mu \simeq M$ and we would have been able to resum the leading $\log(\mu/M)$ terms without performing any radiative computation in the UV theory! In general, an l-loop computation in the effective theory requires an $l-1$-loop matching computation in the UV theory. This example clearly shows the usefulness of effective theories even in such a simple set-up.

5.2 Yukawa Theory I: Heavy Scalar, Light Fermion

Consider a light fermion field coupled to a massive scalar by means of a Yukawa-like coupling. The UV and IR theory look then

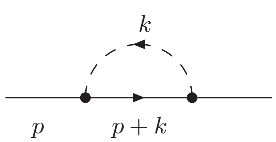
$$\mathcal{L}_{UV} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\psi \tag{5.2.1}$$

$$\mathcal{L}_{IR} = Z_\psi \bar{\psi} i\partial\!\!\!/ \psi - \tilde{m}\bar{\psi}\psi + \frac{\lambda}{2}\bar{\psi}\psi\bar{\psi}\psi + \text{h.d.o.} \tag{5.2.2}$$

The tree-level integration of ϕ gives $\phi = -g\bar{\psi}\psi/M^2 + \mathcal{O}(p^2/M^4)$. Plugging back in eq.(5.2.1) gives

$$Z_\psi = 1 + \delta Z_\psi, \quad \tilde{m} = m + \delta_m\,, \quad \lambda = \frac{g^2}{M^2}\,. \tag{5.2.3}$$

Let us compute the 1PI 2-point function $\Gamma^{(2)}(p)$ in both the UV and the IR EFT at one-loop level and let us see what we learn in comparing the two results. In the UV theory the relevant one-loop diagram is



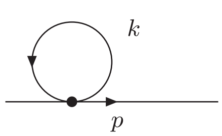
$$= (-ig)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - M^2}\frac{i(p\!\!\!/ + k\!\!\!/ + m)}{(k+p)^2 - m^2} \tag{5.2.4}$$

$$= \frac{ig^2}{16\pi^2}\int_0^1 dx\Big(p\!\!\!/(1-x) + m\Big)\log\left(\frac{\mu^2}{-p^2x(1-x) + m^2x + (1-x)M^2}\right).$$

Expanding for $M \gg m, p$, we get

$$\begin{aligned}\Gamma^{(2)}(p) &= p\!\!\!/ - m - \frac{g^2}{16\pi^2}\Bigg[p\!\!\!/\Big(\frac{1}{2}\log\frac{M^2}{\mu^2} - \frac{1}{4} + \frac{m^2}{2M^2} - \frac{p^2}{6M^2} + \mathcal{O}\Big(\frac{1}{M^4}\Big)\Big) \\ &+ m\Big(\log\frac{M^2}{\mu^2} - 1 - \frac{m^2}{M^2}\log\frac{m^2}{M^2} - \frac{p^2}{2M^2} + \mathcal{O}\Big(\frac{1}{M^4}\Big)\Big)\Bigg]\,.\end{aligned} \tag{5.2.5}$$

In the effective theory, the relevant one-loop diagram is



$$= (i\lambda\mu^\epsilon)\int\frac{d^dk}{(2\pi)^d}\frac{i(k\!\!\!/ + m)}{k^2 - m^2} = \frac{i\lambda m^3}{16\pi^2}\Big(1 + \log\frac{\mu^2}{m^2}\Big)\,, \tag{5.2.6}$$

and hence

$$\Gamma^{(2)}(p) = (1 + \delta Z_\psi)p\!\!\!/ - (m + \delta_m) + \frac{\lambda m^3}{16\pi^2}\Big(1 + \log\frac{\mu^2}{m^2}\Big)\,. \tag{5.2.7}$$

Matching eqs.(5.2.5) and (5.2.7) gives

$$\begin{aligned}
\delta Z_\psi(\mu) &= -\frac{g^2}{16\pi^2}\Big(\frac{1}{2}\log\frac{M^2}{\mu^2}-\frac{1}{4}+\frac{m^2}{2M^2}+\mathcal{O}\Big(\frac{1}{M^4}\Big)\Big)\,, \\
\delta_m(\mu) &= \frac{\lambda m^3}{16\pi^2}\Big(1+\log\frac{\mu^2}{m^2}\Big)+\frac{mg^2}{16\pi^2}\bigg(\log\frac{M^2}{\mu^2}-1-\frac{m^2}{M^2}\log\frac{m^2}{M^2}+\mathcal{O}\Big(\frac{1}{M^4}\Big)\bigg)\,. \\
&= -\frac{mg^2}{16\pi^2}\Big(1+\log\frac{\mu^2}{M^2}\Big)\Big(1+\frac{m^2}{M^2}\Big)\,,
\end{aligned} \tag{5.2.8}$$

where in the last line we have used the tree-level relation (5.2.3) for λ. Again, as expected, all logs involving the light mass m cancelled and it appears clear from eq.(5.2.8) that the matching is best performed at $\mu = M$. Matching the p^2/M^2 terms in eq.(5.2.7) would require the addition of the higher dimensional operators of the form $\bar\psi\partial\!\!\!/\Box\psi$ or $\bar\psi\Box\psi$. We neglect them, since their effect is small, suppressed by p^2/M^2 times a loop factor. The physical fermion mass is given, at one-loop level, by

$$m_{phys} = m(M) - m\frac{g^2}{16\pi^2}\Big(\frac{5}{4}+\frac{m^2}{2M^2}\Big)\,. \tag{5.2.9}$$

5.3 Yukawa Theory II: Heavy Fermion, Light Scalar

Consider now the opposite case with an heavy fermion field coupled to a light scalar:

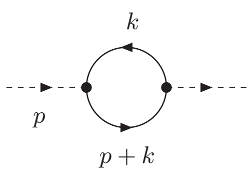
$$\mathcal{L}_{UV} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar\psi(i\partial\!\!\!/ - M)\psi - g\phi\bar\psi\psi \tag{5.3.1}$$

$$\mathcal{L}_{IR} = \frac{1}{2}Z_\phi(\partial\phi)^2 - \frac{1}{2}\tilde m^2\phi^2 + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 + \text{h.d.o.} \tag{5.3.2}$$

The tree-level integration of ψ gives $\psi = 0$, so that

$$Z_\phi = 1 + \delta Z_\phi, \quad \tilde m^2 = m^2 + \delta_{m^2}\,, \quad \lambda_3 = \mathcal{O}(g^3)\,, \quad \lambda_4 = \mathcal{O}(g^4)\,. \tag{5.3.3}$$

Let us again match the 1PI 2-point function in the UV and the IR theory. In the UV we have



$$= (-1)(-ig)^2\mu^\epsilon \text{Tr}\int\frac{d^dk}{(2\pi)^d}\frac{i(k\!\!\!/+M)}{k^2-M^2}\frac{i(p\!\!\!/+k\!\!\!/+M)}{(k+p)^2-M^2}$$

$$= -\frac{4ig^2}{16\pi^2}\int_0^1 dx\Big(M^2-p^2x(1-x)\Big)\bigg[3\log\Big(\frac{\mu^2}{-p^2x(1-x)+M^2}\Big)-5\bigg]\,, \tag{5.3.4}$$

taking $\text{Tr}\, I = 4$ (rather than d). Expanding for $M \gg m, p$, we get

$$\begin{aligned}
\Gamma^{(2)}(p) &= p^2 - m^2 - \frac{4g^2}{16\pi^2}\bigg[p^2\Big(-\frac{1}{2}\log\frac{M^2}{\mu^2}+\frac{4}{3}-\frac{p^2}{20M^2}+\mathcal{O}\Big(\frac{1}{M^4}\Big)\Big) \\
&+ M^2\Big(-3\log\frac{M^2}{\mu^2}-5+\mathcal{O}\Big(\frac{1}{M^4}\Big)\Big)\bigg]\,.
\end{aligned} \tag{5.3.5}$$

In the EFT we simply have

$$\Gamma^{(2)}(p) = p^2 - m^2 + p^2\delta Z_\phi - \delta_{m^2}\,. \tag{5.3.6}$$

Matching eqs.(5.3.5) and (5.3.6) gives

$$\begin{aligned}
\delta Z_\phi(\mu) &= -\frac{4g^2}{16\pi^2}\bigg(-\frac{1}{2}\log\frac{M^2}{\mu^2}+\frac{4}{3}\bigg)\,, \\
\delta_{m^2}(\mu) &= -\frac{4g^2M^2}{16\pi^2}\bigg(3\log\frac{M^2}{\mu^2}+5\bigg)\,.
\end{aligned} \tag{5.3.7}$$

The physical scalar mass is given by

$$m^2_{phys} = m^2(M) + \frac{4g^2}{16\pi^2}\Big(-5M^2+\frac{16}{3}m^2\Big)\,. \tag{5.3.8}$$

5.4 Naturalness and the Hierarchy Problem

The scalar mass correction in eq.(5.3.8) is of $\mathcal{O}(M^2g^2/(16\pi^2))$. For $M \gg m$, the ratio $(M/m)^2$ can overcome the one-loop suppression so that the radiative term is generically much bigger than the tree-level one. If we want to keep the physical scalar mass small, a fine-tuning between the $\overline{MS}$ mass term $m^2(M)$ and the one-loop correction is needed (of course this readjustment is needed at each order in perturbation theory). This is not a conceptual problem, and there is nothing wrong in principle to do that, but it is nevertheless unpleasant to have in the EFT a parameter that is so sensitive to the UV physics. Even a small change in the UV theory (say, $g \to g+\delta g$ in our toy example) will give rise to a large radiative correction to $m^2_{phys.}$ and to a new readjustment. This is the typical example of an "hierarchy problem" or "naturalness problem", namely the problem of explaining why a parameter in an EFT Lagrangian is much smaller than its expected value at the quantum level.[3] Natural parameters are often defined as follows:
dimensionless couplings should be of order one and dimensionful couplings should be of order of the largest mass scale in the theory, to the appropriate power. Exceptions arise if a symmetry is restored when a coupling (dimensionless or not) vanish, in which case it is natural to have that coupling arbitrarily small.

Let us check, using our results above, that this definition makes sense. In the scalar theory of section (5.1), when $g = \mathcal{O}(M)$ (natural value), we get that the dimensionless

[3] The hierarchy problem is often defined as due to quadratically (or higher) divergent graphs in regularizations with some cut-off Λ. Although this argument is essentially correct, it relies on the use of Λ and, taken as it is, does not apply to regularizations like DR, where no Λ is introduced and there are no real quadratic divergencies. It would then seem that the hiearchy problem is "scheme dependent", whereas of course it is not, as we have just shown.

coupling λ in the EFT is of $\mathcal{O}(M^2/M^2) = \mathcal{O}(1)$. Contrary to the scalar case, the fermion mass correction in eq.(5.2.9) is proportional to the mass itself. This implies that, for $m \to 0$, all loop corrections vanish as well. Hence, light fermion masses are natural. According to our general definition of natural parameters, some symmetry should be restored when $m \to 0$. This is indeed what happens, the symmetry being the chiral symmetry $\psi \to \gamma_5 \psi$ (combined with the discrete $\mathbf{Z}_2$ inversion symmetry $\phi \to -\phi$ in the UV theory). No symmetry is instead restored when the scalar mass term goes to zero, so that in this case we have a naturalness problem.

Despite hierarchy problems do not lead to real inconsistencies and we can live with them, they have been the main driving force in going beyond the SM, since the Higgs boson mass term is unnatural. Similarly, one of the main problems in theoretical physics today is provided by another hierarchy problem, which is why the vacuum energy we measure is so smaller than its natural value.

5.5 Non-Leptonic Decays

As a final example, we consider a commonly used EFT of the weak interactions, by means of four-fermion operators. The SM current is given by

$$\mathcal{L}_{SM} \supset \frac{g}{\sqrt{2}} W_\mu^+ \Big(V_{ud} \bar{u}_L \gamma^\mu d_L + V_{us} \bar{u}_L \gamma_\mu s_L) + h.c. \tag{5.5.1}$$

Integrating out the W gives, among others, the flavour changing four-Fermi interaction term

$$-\frac{4G_F}{\sqrt{2}} V_{us} V_{du}^\star (\bar{u}_L \gamma^\mu s_L)(\bar{d}_L \gamma_\mu u_L)\,, \tag{5.5.2}$$

with the matching relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}\,. \tag{5.5.3}$$

The most general EFT dimension six interaction terms that can contribute to K decays are

$$\mathcal{L}_{IR} = -c_1 O_1 - c_2 O_2 + h.c., \tag{5.5.4}$$

where

$$O_1 = (\bar{u}_L \gamma^\mu s_L)(\bar{d}_L \gamma_\mu u_L)\,, \qquad O_2 = (\bar{u}_L \gamma^\mu u_L)(\bar{d}_L \gamma_\mu s_L)\,. \tag{5.5.5}$$

Tree-level matching gives

$$c_1(m_W) = \frac{g^2}{2m_W^2} V_{us} V_{du}^\star (m_W)\,, \qquad c_2(m_W) = 0\,. \tag{5.5.6}$$

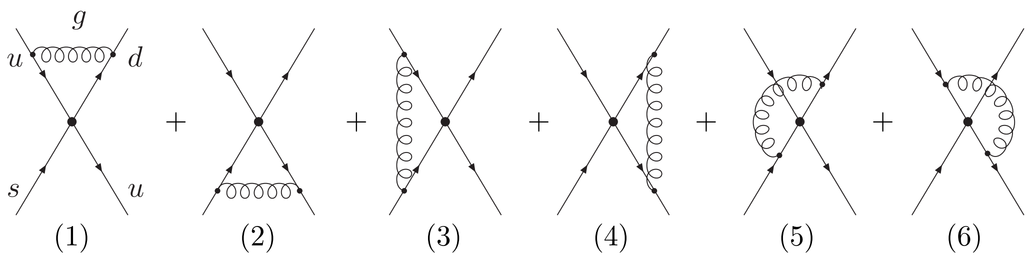


Figure 5.1: One-loop graphs contributing to the renormalization of the four-Fermi operator (5.5.2). All external momenta are vanishing.

Although at tree-level only O_1 appears, symmetry arguments do not forbid O_2, which is expected to arise at the quantum level, as we will explicitly see. Operators like (5.5.4) are relevant for the decays of the strange mesons, such as the K's. The natural scale where the operators $O_{1,2}$ should be renormalized is $\sim m_K$, being the scale of the decay process. It is then important to compute the RG flow of the couplings $c_i = c_i(\mu)$, $i = 1, 2$, and see their effect. The main effect is given by the leading QCD log corrections from m_W to m_K that should be resummed. This is then our aim and where EFT is useful for.

Recall that at the linear level, the RG evolution of the couplings $c_{1,2}$ is fixed by the anomalous dimensions of the corresponding operators (see eq.(4.7.8)):

$$\mu \frac{dc_i}{d\mu} = \beta_i = \gamma_{ij} c_j \,, \tag{5.5.7}$$

where γ_{ij} is the 2×2 matrix of anomalous dimensions of the composite operators O_1 and O_2 which, as we will see, mix under renormalization. We have then to determine γ_{ij}. By definition,

$$O_{i,0} = Z_{ij}(\mu) O_j(\mu) \,, \tag{5.5.8}$$

where $O_{i,0}$ are the bare composite operators. The relevant 1PI one-loop graphs renormalizing O_1 are depicted in figure (5.1). A similar set of graphs , with $d \leftrightarrow u$ in the external lines, renormalize O_2. Due to the conservation of the underlying electroweak currents, the diagrams (1) and (2) in figure (5.1) precisely compensate for the non 1PI one-loop graphs correcting the external quark lines,[4] so that we do not need to compute them. Neglecting

[4]Alternatively, one might define $O_{i,0} = Z_{ij}(\mu)/Z_q^2 O_j(\mu)$, taking care of the wave-function renormalization of the elementary quark fields, in which case only the 1PI graphs should be considered. In this case, Z_q^2 compensates for diagrams (1) and (2), leading to the same result.

all quark masses, diagram (3) gives, keeping only the finite μ-dependent terms,

$$
\begin{aligned}
(3) &= (ig_c)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\Big(\bar d_L\gamma^\mu \frac{i\not k}{k^2}\gamma^\nu t^a u_L\Big)\Big(\bar u_L\gamma_\mu\frac{-i\not k}{k^2}\gamma_\nu t^a s_L\Big)\frac{-i}{k^2} \\
&= -\frac{g_c^2\log\mu^2}{64\pi^2}\Big(\bar d_L\gamma^\mu\gamma^\rho\gamma^\nu t^a u_L\Big)\Big(\bar u_L\gamma_\mu\gamma_\rho\gamma_\nu t^a s_L\Big)\,. \qquad (5.5.9)
\end{aligned}
$$

Similarly, we get

$$
(4) = (3)\,,\qquad (5) = (6) = \frac{g_c^2\log\mu^2}{64\pi^2}\Big(\bar d_L\gamma^\mu\gamma^\rho\gamma^\nu t^a u_L\Big)\Big(\bar u_L\gamma_\nu\gamma_\rho\gamma_\mu t^a s_L\Big)\,. \qquad (5.5.10)
$$

Eqs.(5.5.9) and (5.5.10) are best written using Fierz identities in both the spinor and color indices. It is straightforward to verify that

$$
\begin{aligned}
t^a_{ij}t^a_{kl} &= \frac{1}{2}\Big(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\Big)\,, \\
\psi_{1L}\bar\psi_{2L} &= -\frac{1}{2}(\bar\psi_{2L}\gamma^\mu\psi_{1L})\gamma_\mu P_R \qquad (5.5.11)
\end{aligned}
$$

for two anti-commuting spinors. Using eq.(5.5.11), we have

$$
\begin{aligned}
\Big(\bar d_L\gamma^\mu\gamma^\rho\gamma^\nu t^a u_L\Big)\Big(\bar u_L\gamma_\mu\gamma_\rho\gamma_\nu t^a s_L\Big) &= -8\bar d_L\gamma^\mu s_L\bar u_L\gamma_\mu u_L + \frac{8}{3}\bar d_L\gamma^\mu u_L\bar u_L\gamma_\mu s_L\,, \\
\Big(\bar d_L\gamma^\mu\gamma^\rho\gamma^\nu t^a u_L\Big)\Big(\bar u_L\gamma_\nu\gamma_\rho\gamma_\mu t^a s_L\Big) &= -2\bar d_L\gamma^\mu s_L\bar u_L\gamma_\mu u_L + \frac{2}{3}\bar d_L\gamma^\mu u_L\bar u_L\gamma_\mu s_L\,. \qquad (5.5.12)
\end{aligned}
$$

Collecting all terms, we get ($\alpha_c = g_c^2/(4\pi)$)

$$
O_{1,0} = O_1(\mu) - \frac{\alpha_c}{4\pi}\log\mu^2\Big[O_1(\mu) - 3O_2(\mu)\Big]\,,\qquad O_{2,0} = O_2(\mu) - \frac{\alpha_c}{4\pi}\log\mu^2\Big[O_2(\mu) - 3O_1(\mu)\Big]\,, \qquad (5.5.13)
$$

from which the matrix of anomalous dimension is easily extracted:

$$
\gamma_{ij} = \frac{\alpha_c}{2\pi}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} + \mathcal{O}(\alpha_c^2)\,. \qquad (5.5.14)
$$

The diagonal basis for the operators O_1 and O_2 is $O_{1/2} = (O_1 - O_2)/2$, $O_{3/2} = (O_1 + O_2)/2$, the subscripts 1/2 and 3/2 referring to the isospin quantum numbers of the two operators, so that $c_iO_i = c_{1/2}O_{1/2} + c_{3/2}O_{3/2}$, with $c_{1/2} = c_1 - c_2$, $c_{3/2} = c_1 + c_2$. In the new basis, the system of linear differential equations (5.5.7) is diagonal:

$$
\mu\frac{dc_i}{d\mu} = \gamma_i c_i\,,\quad \gamma_{1/2} = \frac{\alpha_c}{\pi}\,,\quad \gamma_{3/2} = -\frac{2\alpha_c}{\pi}\,. \qquad (5.5.15)
$$

The equations are easily integrated by writing $\mu dc_i/d\mu = dc_i/d\alpha_c\beta_c$, where β_c is the (one-loop) QCD β function

$$
\beta_c = -\frac{\alpha_c^2}{2\pi}b_0 \qquad b_0 = 11 - \frac{2}{3}n_f\,. \qquad (5.5.16)
$$

We get

$$c_{1/2}(\mu) = c_{1/2}(\mu_0)\left(\frac{\alpha_c(\mu)}{\alpha_c(\mu_0)}\right)^{\frac{4}{b_0}}\,,\quad c_{3/2}(\mu) = c_{3/2}(\mu_0)\left(\frac{\alpha_c(\mu)}{\alpha_c(\mu_0)}\right)^{-\frac{2}{b_0}}\,. \tag{5.5.17}$$

The integration constants $c_i(\mu_0)$ are fixed by taking $\mu_0 = m_W$ and matching by using eq.(5.5.6): $c_+(m_W) = c_-(m_W) = g^2 V_{us}V^\star_{du}/(2m_W^2) \equiv c_0$. By taking $\mu \sim m_K$, we can estimate the effect of the QCD LL corrections. For $n_f = 4$,[5] one roughly gets $c_{1/2}(m_K) \simeq 2c_0$, $c_{3/2}(m_K) \simeq 0.7c_0$, with an enhancement by about a factor 3 of the isospin 1/2 operator with respect to the 3/2 one, showing that QCD corrections are far from being negligible.

5.6 (Ir)relevance of Higher Dimensional Operators

We have so far neglected all higher dimensional operators appearing in the EFT, assuming that their effect is small. This assumption is actually correct, and this is the main reason why we study EFT after all, but it is not as trivial as what naively expected. At the quantum level the insertion of higher dimensional operators in Feynman amplitudes lead to bad divergencies that can obscure the irrelevance of these operators at low energies. As a matter of fact, these operators are not always negligible at low-energies, their effect depending on the regularization and renormalization procedure used to deal with the divergencies. As an illustrative example, let us go back to the two scalar theory in section (5.1) and let us add the dimension six operator

$$\frac{\tilde\lambda}{M^2}L^3\Box L \tag{5.6.1}$$

to the Lagrangian (5.1.2). The operator (5.6.1) is indeed generated, at tree-level, when we integrate H out. Naively, we expect that (5.6.1) is suppressed at low-energies as E^2/M^2 with respect to the λ coupling in (5.1.2). However, at the quantum level eq.(5.6.1) lead to severe divergencies. For instance, at one-loop level the 1PI 4-point function with one λ and one $\tilde\lambda$ vertex is quadratically divergent, so that

$$\delta\Gamma^{(4)} \propto \lambda\frac{\tilde\lambda}{M^2}\int d^4k\frac{k^2}{(k^2+p^2)^2} \sim \lambda\frac{\tilde\lambda}{M^2}\Big(\Lambda^2 + p^2\log\frac{\Lambda^2}{p^2}+\dots\Big)\,, \tag{5.6.2}$$

where p generically denotes external momenta or the mass m. Since the cut-off Λ of the EFT is around M, we see that the divergence term can compensate for the manifest $1/M^2$ suppression to give a contribution to $\Gamma^{(4)}$ of the same order of the leading λ^2 term.

[5] This is, of course, a simplification, since we should take into account the b quark for energies above its mass and integrate out the c quark below its mass, by performing a more refined matching.

This potential problem is actually scheme-dependent and is manifestly absent in dimensional regularization with a mass-independent renormalization subtraction scheme, such as $\overline{\text{MS}}$. In DR, divergencies arise as poles in $1/\epsilon$ and, when subtracted, leave a renormalized amplitude where the sliding scale μ appears in logs only. Since in the EFT the only other mass scales present are the light masses and the low-energy momenta, by dimensional analysis all higher dimensional operators are manifestly suppressed. In the example above, for instance, we get, once the poles are subtracted,

$$\delta\Gamma^{(4)} \propto \lambda \frac{\tilde{\lambda}}{M^2} \int d^d k \frac{k^2}{(k^2+p^2)^2} \sim \lambda \frac{\tilde{\lambda}}{M^2} \left(p^2 \log \frac{\mu^2}{p^2} + \dots \right), \tag{5.6.3}$$

which is manifestly sub-leading with respect to the λ^2 term.

Of course by using any other regulator, after the divergence is subtracted, no depedence on the cut-off appears. However, in a generic regularization and renormalization prescription, *power-like* dependence on the sliding scale is induced, which can spoil the naive dimensional analysis. For instance, by renormalizing the operators L^4 and $L^3 \Box L$ at a scale μ, one would get

$$\delta\Gamma^{(4)} \sim \lambda \frac{\tilde{\lambda}}{M^2} \left(p^2 \log \frac{\mu^2}{p^2} + \mu^2 \log \frac{\mu^2}{p^2} + \dots \right). \tag{5.6.4}$$

If the subtraction point $\mu \ll M$, we can still neglect higher dimensional operators, but in a sufficiently complicated set-up powers of M might be induced by mixing with various operators. Moreover, there might be situations where it is convenient to choose μ close to M, in which case dimensional analysis manifestly break down. All these problems are always avoided by using DR with a mass independent subtraction. This is probably the most important welcome feature of this combined regularization/renormalization procedure.

Chapter 6

Spontaneously Broken Symmetries

Symmetries are a fundamental concept in QFT but they are often broken in Nature. If a physical system described by a Lagrangian $\mathcal{L}(\phi)$ is invariant under some symmetry, a possible breaking term is obtained by adding terms to $\mathcal{L}(\phi)$ which do not respect the symmetry (explicit breaking). If the corresponding breaking operators have dimensions less than four, the UV behaviour of the theory (and its divergences) will not be affected by the symmetry breaking terms. In this case we say that the symmetry is *softly* broken.

Another possibility occurs when $\mathcal{L}(\phi)$ is invariant under the symmetry but the ground state (the vacuum) is not. If G is the operator in the Hilbert space parametrizing the action of the symmetry group, $G|0\rangle \neq |0\rangle$. In this this case we have what is called a spontaneous symmetry breaking. Classical prototypical example is the breaking of the $SO(3)$ spatial rotations in a ferromagnet. The laws of Nature are all spatially symmetric, but the vacuum is not. The simplest example of a spontaneous symmetry breaking mechanism is provided by the $\lambda\phi^4$ theory, that enjoys a $\mathbf{Z}_2$ symmetry under which $\phi \to -\phi$, with $\langle\phi\rangle = \pm\phi_0 \neq 0$. Clearly, the $\mathbf{Z}_2$ symmetry exchanges the two degenerate vacua, $\mathbf{Z}_2| \pm \phi_0\rangle = | \mp \phi_0\rangle$. The two vacua are physically identical, so that we can choose any of the two and expand for small fluctuations around the selected one.

It is important to remark that we are tacitly assuming that the vacuum is one of the two vacua $| \pm \phi_0\rangle$, and *not* a linear combination of them. Indeed, if the vacuum would be $|\pm\rangle = 1/\sqrt{2}(|\phi_0\rangle \pm | - \phi_0\rangle)$, then the symmetry would be exact and not spontaneously broken (the vacua being mapped to themselves, up to a possible sign factor). In order to better appreciate this important point, let us consider the quantum mechanical analogue of the double well potential. There, in fact, the hamiltonian eigenvalues are given by $|\pm\rangle$, since tunneling effects induce non-trivial transitions of the form $\langle\pm\phi_0| \mp \phi_0\rangle$, and no spontaneous $\mathbf{Z}_2$ symmetry breaking occurs. On the contrary, in QFT we do have spontaneous breaking of symmetries, because such transitions are exponentially suppressed by the volume of space. When the latter is very large or infinite, such transitions do not occur and the two vacua $|\pm\phi_0\rangle$ are equivalent and completely disconnected from each other.

The discussion is easily generalized to continuous symmetries where we have an infinite family of degenerate vacua. Being all related by the (spontaneously broken) symmetry, it does not matter which one we choose. For continuous symmetries we also have an important theorem, the Goldstone theorem, relating the symmetry breaking pattern and the spectrum of a system.

6.1 The Goldstone Theorem

Let G be a group of continuous symmetry with generators t^α, $\alpha = 1, \ldots, \dim G$, acting on some system. By Noether's theorem, we have $\dim G$ conserved currents $J_\mu^\alpha(x)$: $\partial^\mu J_\mu^\alpha(x) = 0$ and the associated charges $Q^\alpha = \int d^3x J_0^\alpha(x)$. The group G is said to be spontaneously broken if, on the vacuum, the generators splits into two sets, labelled by a and i, such that

$$Q^a|0\rangle \neq 0\,, \qquad Q^i|0\rangle = 0\,, \tag{6.1.1}$$

with a non-empty set for a. The unbroken generators labelled by t^i form a subgroup of G, denoted by H, so that we have $\dim G - \dim H$ broken generators ($a = 1, \ldots, \dim G - \dim H$). Goldstone's theorem states that, independently of the specific pattern of breaking and physical system we are considering, in the spectrum there will appear one massless and spinless particle for each broken generator. The particle will be scalar or pseudoscalar, depending on the parity of the associated broken generator. These particles are called Goldstone or Nambu-Goldstone (NG) bosons. For simplicity, we will always refer to them in what follows as to the NG particles.

Proof of the theorem. Let ϕ_n be the set of fields responsible for the spontaneous symmetry breaking pattern $G \to H$. This implies that an infinitesimal action of the group G on the fields ϕ_n do not leave them invariant, namely $\langle \phi_n'(0)\rangle \neq \langle \phi_n(0)\rangle$, where ϕ' are the fields one gets after the infinitesimal action. In other words, we must have

$$\langle \delta\phi_n(0)\rangle = \epsilon^\alpha \langle [Q^\alpha, \phi_n(0)]\rangle = \epsilon^a \langle [Q^a, \phi_n(0)]\rangle \neq 0\,, \tag{6.1.2}$$

with $\langle [Q^a, \phi_n(0)]\rangle \equiv \delta\phi_n^a \neq 0$, for each a. Consider now the two-point function of a conserved current J_μ^a with the fields ϕ_n. Since $\partial^\mu J_\mu^a(x) = 0$, we have

$$\begin{aligned} \partial_x^\mu \langle 0|T[J_\mu^a(x)\phi_n(0)]|0\rangle &= \partial_x^\mu \Big(\theta(x^0)\langle J_\mu^a(x)\phi_n(0)\rangle + \theta(-x^0)\langle \phi_n(0) J_\mu^a(x)\rangle\Big) \\ &= \delta(x^0)\langle [J_0^a(x), \phi_n(0)]\rangle \end{aligned}\,. \tag{6.1.3}$$

Let us denote by $G^a_{\mu,n}$ the Fourier transform of the two-point function:

$$\langle 0|T[J^a_\mu(x)\phi_n(0)]|0\rangle = \int \frac{d^4p}{(2\pi)^4} G^a_{\mu,n}(p) e^{ip\cdot x}\,. \tag{6.1.4}$$

By Lorentz invariance, we have

$$G^a_{\mu,n}(p) = -ip_\mu H^a_n(p^2)\,. \tag{6.1.5}$$

Integrating (6.1.3) over space-time gives

$$\int d^4x\, \partial^\mu_x \langle 0|T[J^a_\mu(x)\phi_n(0)]|0\rangle = \int d^4p\, \delta(p) p^2 H^a_n(p^2) = \langle [Q^a, \phi_n(0)]\rangle \neq 0\,. \tag{6.1.6}$$

The integral of a divergence does not vanish when long-range particles appear, and these are precisely the NG bosons. More precisely, we must have

$$H^a_n(p^2) = \frac{\delta\phi^a_n}{p^2} + \dots \tag{6.1.7}$$

where $\dots$ represent terms that are regular when $p \to 0$. A pole in the two-point function of J^a_μ with ϕ_n is a signal of the presence of massless 1-particle states, one for each unbroken generator a. Since ϕ_n are spinless (otherwise in the vacuum we will break the Lorentz symmetries as well), such particles are necessarily of spin zero. Their intrinsic parity is then fixed by the parity of the associated current J^a. Q.e.d.

Denoting by NG_a the one-NG particle states, we have

$$\begin{aligned} \langle 0|J^a_\mu(x)|NG_b(p)\rangle &= \frac{ip_\mu F_{ab} e^{-ip\cdot x}}{\sqrt{2p_0(2\pi)^3}}\,, \\ \langle NG_b(p)|\phi_n(0)|0\rangle &= \frac{Z^b_n e^{-ip\cdot x}}{\sqrt{2p_0(2\pi)^3}}\,, \end{aligned} \tag{6.1.8}$$

where $\delta\phi^a_n = iF_{ab}Z^n_b$. We can also define the NG fields $\pi^a(x)$ with canonical 1-point function with the NG particles:

$$\langle NG_b(p)|\pi^a(x)|0\rangle = \frac{\delta_{ab} e^{ip\cdot x}}{\sqrt{2p_0(2\pi)^3}}\,. \tag{6.1.9}$$

Putting together (6.1.8) and (6.1.9), we see that

$$\phi_n(x) = Z^a_n \pi^a(x) + \dots = (iF)^{-1}_{ab}\delta\phi^b_n \pi^a(x) + \dots \tag{6.1.10}$$

where $\dots$ denote field components not associated to the NG bosons. For linearly realized symmetries, in a basis where all the field components ϕ_n are real and all the generators t^α are purely imaginary, we simply have

$$\delta\phi^a_n = it^a_{nm}\langle\phi_m\rangle\,. \tag{6.1.11}$$

The appearance of one NG particle per broken generator is physically explained by noticing that the potential for the ϕ_n is, by assumption, invariant under the symmetry. Hence, around any vacuum $\langle\phi_m\rangle$, the broken directions do not leave the minimum invariant, but necessarily shift it in a new minimum with exactly the same energy. Hence, we have a ($\dim G$-$\dim H$)-dimensional space of flat directions in the potential. It costs no energy to fluctuate around a flat direction, and the small fluctuation is identified with the NG boson.

The above proof of the Goldstone's theorem does not rely on perturbation theory, indicating that NG particles are expected to arise whenever a global symmetry group is spontaneously broken, no matter whether the theory undergoing the symmetry breaking is weakly or strongly coupled. The fields ϕ_n responsible for the symmetry breaking are also not necessarily elementary fields appearing directly in the Lagrangian at some energy scale, but might be composite fields built with different fields. The most relevant example of this kind is the spontaneous breaking of the $SU(2)$ chiral symmetry in QCD, induced by effective scalar fields ϕ_n constructed out of quark bilinears. In this case, the three NG bosons are spin zero mesons that appear as bound states of the original quarks, the pions π^0, $\pi^\pm$. We will come back to this relevant case in much more detail in the following.

We close this section by noticing that the Goldstone's theorem applies for internal symmetries only, namely for those symmetries whose generators commute with the ones of the Poincaré group. An example of non-internal symmetry is provided by the conformal group, that in four dimensions is given by $SO(4,2)$. A CFT is invariant under the conformal group. When the latter is broken spontaneously to the Poincaré group, one gets $15-10=5$ broken generators (special conformal transformations and dilatations), but actually only one massless NG boson, commonly denoted as the dilaton.

As we will shortly see, the Goldstone's theorem does not apply for local (i.e. space-time dependent) symmetries as well.

6.2 Vacuum Alignement and Pseudo-Goldstone Bosons

It is interesting to analyze theories where, in some approximation, a spontaneous symmetry breaking occurs and, in addition, sub-leading effects introduce explicit symmetry breaking terms. Relevant examples include the above mentioned case of QCD, where the up and down quark masses can be seen as the small explicit symmetry breaking terms of the $SU(2)$ chiral symmetry. We will consider in what follows linearly realized symmetries only, for which we know that quantum effects do not spoil the invariance of the action. By considering space-time independent field configurations ϕ_n, the whole effective action boils down to the effective potential $V(\phi)$. The latter, by definition, can be written as

$$V(\phi) = V_0(\phi) + V_1(\phi)\,, \tag{6.2.1}$$

where V_0 is the invariant term, and V_1 is the breaking one. In the field region of interest, by assumption, we have $|V_1| \ll |V_0|$. In the basis (6.1.11), we have

$$\delta\phi_n = i\epsilon^\alpha t^\alpha_{nm}\phi_m\,. \tag{6.2.2}$$

Invariance of V_0 implies

$$\frac{\partial V_0}{\partial\phi_n} t^\alpha_{nm}\phi_m = 0\,, \quad \forall\alpha\,. \tag{6.2.3}$$

Let ϕ_0 be the minimum of V_0 and $\phi = \phi_0 + \phi_1$ the minimum of the whole potential V:

$$\left.\frac{\partial V}{\partial\phi_n}\right|_{\phi=\phi_0+\phi_1} = 0\,. \tag{6.2.4}$$

Since $|V_1| \ll |V_0|$, we also have $\phi_1 \ll \phi_0$ and we can consistently expand (6.2.4) for small V_1 and ϕ_1. At first non-trivial order, we have

$$\left.\frac{\partial^2 V_0}{\partial\phi_n\partial\phi_m}\right|_{\phi_0}\phi_{1,m} + \left.\frac{\partial V_1}{\partial\phi_n}\right|_{\phi_0} = 0\,. \tag{6.2.5}$$

Taking a derivative with respect to ϕ of (6.2.3), and evaluating at ϕ_0, we have

$$\left.\frac{\partial^2 V_0}{\partial\phi_n\partial\phi_m}\right|_{\phi_0} t^\alpha_{mp}\phi_{0,p} = 0\,. \tag{6.2.6}$$

When the index α runs over the unbroken directions i, (6.2.6) trivially vanishes since $t^i_{mp}\phi_{0,p} = 0$. For $\alpha = a$, (6.2.6) is non-trivial and indicates that there are $\dim G$-$\dim H$ directions in field space with a massless eigenvector, that are the NG particles (this can be in fact seen as an alternative proof of the Goldstone's theorem). Multiplying (6.2.5) by $t^a_{np}\phi_{0,p}$ and using (6.2.6), we finally have

$$\left.\frac{\partial V_1}{\partial\phi_n}\right|_{\phi_0} t^\alpha_{np}\phi_{0,p} = 0\,. \tag{6.2.7}$$

In presence of a perturbation V_1, the vacuum ϕ_0 is no longer arbitrary among the would-be degenerate vacua (in absence of the perturbation) but is restricted to satisfy (6.2.7). Equation (6.2.7) is denoted vacuum alignment condition, because it shows that the symmetry breaking terms typically force the vacuum ϕ_0 to be parallel to the direction of the explicit breaking term. This is easily seen in the particularly simple example of an $SO(N) \to SO(N-1)$ breaking pattern. We choose as explicit breaking term $V_1 = u_n\phi_n$, where u_n is a fixed vector, explicitly breaking $SO(N)$ down to the $SO(N-1)$ subgroup that leaves it fixed. The condition (6.2.7) gives $u_n t^a_{np}\phi_{0,p} = 0$, namely $\phi_{0,n} \propto u_n$ (the matrices t^a being antisymmetric). The addition of the small perturbation to the invariant Lagrangian lifts the vacuum degeneracy and forces ϕ_0 to be parallel to u_n. The final unbroken group is then $SO(N-1)$ and not $SO(N-2)$, intersection of the two would-be misaligned $SO(N-1)$ subgroups.

The explicit term V_1 is generally responsible for another important effect, giving masses to the NG bosons, which are now denoted pseudo-NB bosons. Their masses can be extracted from the potential by using (6.1.10):

$$M^2_{ab} = \frac{\partial^2 V}{\partial \pi^a \partial \pi^b}\bigg|_{\phi} = F^{-1}_{ac}\delta\phi^c_n F^{-1}_{bd}\delta\phi^d_m \frac{\partial^2 V}{\partial\phi_n\partial\phi_m}\bigg|_{\phi} . \tag{6.2.8}$$

Expanding for small V_1 and ϕ_1, it is straightforward to see that the pseudo NG boson masses are linearly proportional to the explicit breaking term:

$$M^2_{ab} \propto V_1 . \tag{6.2.9}$$

6.3 Spontaneously Broken Gauge Symmetries: the Higgs Mechanism

The Goldstone's theorem does not apply in the case in which the broken symmetry is local.[1] It is impossible for gauge theories to keep at the same time Lorentz invariance and positivity of the Hilbert space, both conditions being important to establish the theorem. For local symmetries, no NG massless particles appear. More precisely, the would-be NG bosons are "eaten" by gauge fields that become massive.

The situation is easily illustrated by the abelian Higgs model (model that will be extensively analyzed in the final chapter in the special case $m = 0$), whose Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\Phi|^2 - V(\Phi), \quad V(\Phi) = -m^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 , \tag{6.3.1}$$

with $D_\mu\Phi = \partial_\mu\Phi - ieA_\mu\Phi$. For $m^2 > 0$, the minimum of the potential is at

$$|\Phi_0| = \sqrt{2m^2/\lambda} \equiv v . \tag{6.3.2}$$

It is useful to choose radial coordinates for the field Φ and write

$$\Phi = \frac{v+\rho}{\sqrt{2}} e^{i\theta/v} , \tag{6.3.3}$$

[1]This often used terminology is actually misleading. Gauge symmetries, being merely redundancies of the system, cannot be broken. A more proper term would be gauge theories in a non-linearly realized, or Higgs, phase.

where the factor v in the exponential has been introduced to get canonically normalized fields. Expanding around small fluctuations, we quickly get at quadratic order

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}(\partial_\mu\theta + evA_\mu)^2 - \frac{1}{2}(2m^2)\rho^2 + \dots \qquad (6.3.4)$$

The radial field is massive, with $m_\rho = \sqrt{2}m$, while the angular field θ is massless. Actually θ mixes at quadratic level with the gauge field A_μ and it is not an eigenstate of the Hamiltonian. We can easily get rid of the unwanted mixing term $eA^\mu\partial_\mu\theta$ by noticing that we have still to gauge-fix the $U(1)$ gauge symmetry. Under a $U(1)$ transformation with parameter $\alpha(x)$, $\Phi(x) \to \exp(i\alpha(x))\Phi(x)$. The radial field $\rho(x)$ is invariant, while the angular field $\theta(x)$ shifts as $\theta(x) \to \theta(x) + v\alpha(x)$. For any $v \neq 0$, we can take $\alpha = -\theta/v$ and set in this way $\theta(x) = 0$. In this gauge, denoted unitary gauge, the third term in (6.3.4) boils down to a mass term for the $U(1)$ gauge field, $m_A = ev$. We can say that the gauge field has "eaten" the field θ that becomes the longitudinal component of a massive gauge field. This mechanism is commonly denoted Higgs mechanism. The number of physical degrees of freedom (d.o.f.) in the process is unchanged. We started with 2 d.o.f. from the massless gauge field and 2 d.o.f. from the complex scalar field, for a total of 4, and ended up in 1 d.o.f. from the neutral field ρ and 3 d.o.f. from the massive gauge field, again for a total of 4.

Notice that in the limit $e \to 0$ the $U(1)$ symmetry becomes global and the gauge field decouples. In this limit the Goldstone's theorem applies and we recover the massless NG boson θ.

All the above analysis similarly applies to non-abelian symmetries. Let us consider the usual sets of real fields ϕ_n transforming as (6.2.2) under an infinitesimal local transformation of a group G. For simplicity, we assume the fields ϕ_n to be in some irreducible representation of G, although this is not strictly necessary. The Lagrangian of our system will include the terms

$$\mathcal{L} \supset \frac{1}{2}\Big(\partial_\mu\phi_n - igt^\alpha_{nm}A^\alpha_\mu\phi_m\Big)^2 - V(\phi)\,, \qquad (6.3.5)$$

where we assume that the potential $V(\phi)$ has minima for $\phi_m = v_m$ such that, in the "ungauged" limit $g \to 0$, the global group G is spontaneously broken to H. Expanding around the non-trivial vacuum, $\phi_m = v_m + \phi'_m$, the above covariant derivative gives rise to the following terms, up to quadratic order in the fluctuations,

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu\phi'_n)^2 - ig\partial_\mu\phi'_n t^\alpha_{nm}v_m A^\alpha_\mu + \frac{g^2}{2}(it^\alpha_{nm}v_m)(it^\beta_{np}v_p)A^\alpha_\mu A^{\mu,\beta}\,. \qquad (6.3.6)$$

When $\alpha,\beta = i$, the second and third terms in (6.3.6) vanish identically, since $t^i_{nm}v_m = 0$, while they are generically non-vanishing for $\alpha,\beta = a$. The second term in (6.3.6) is the generalization of the $\partial_\mu \theta A^\mu$ mixing term in the $U(1)$ case. Recall that the directions in field space given by $t^a_{nm} v_m$ correspond to the NG boson directions. As we will explicitly show in the next section, it is always possible to choose a gauge (denoted again unitary gauge) in which the fields ϕ_n do not contain NG boson fields, and hence

$$\phi'_n t^\alpha_{nm} v_m = 0\,. \tag{6.3.7}$$

The third term in (6.3.6) is a mass term for the gauge fields in the broken directions. The mass matrix

$$\mu^2_{\alpha\beta} = g^2 (i t^\alpha_{nm} v_m)(i t^\beta_{nl} v_l) \tag{6.3.8}$$

is symmetric, real and positive, showing that each gauge field A^a_μ along the broken directions get a non-vanishing mass, by eating the corresponding would-be NG boson. In the basis in which μ^2 is diagonalized, the propagators $G^a_{\mu\nu}$ for the massive gauge fields read

$$G^{a(UG)}_{\mu\nu}(p) = \frac{-i}{p^2-\mu_a^2}\Big(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{\mu_a^2}\Big)\,, \tag{6.3.9}$$

where UG stands for unitary gauge and μ_a^2 are the mass square eigenvalues. Notice that for large values of the momentum p, the gauge field propagator goes to a constant. The renormalizability properties of spontaneously broken gauge theories are correspondingly unclear. In order to fix this problem, it is sometimes more useful to use a more general class of gauge-fixing terms, denoted by ξ-gauges. The gauge fixing term $\mathcal{L}_{g.f.}$ for ξ-gauges is a generalization of the usual $1/(2\xi)(\partial_\mu A^\mu)^2$. It reads

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi} f_\alpha f_\alpha\,, \qquad f_\alpha = \partial_\mu A^\mu_\alpha + i\xi g \phi'_n t^\alpha_{nm} v_m\,. \tag{6.3.10}$$

The associated ghost Lagrangian is obtained by taking an infinitesimal gauge variation of f_α, with parameter ω^α, $\mathcal{L}_{gh.} = -g\omega^\star_\alpha \delta_\omega f_\alpha$. We get[2]

$$\mathcal{L}_{gh.} = (\partial^\mu \omega^\star_\alpha)(D_\mu \omega_\alpha) - \xi g^2 (i t^\alpha_{nm} v_m)(i t^\beta_{nl}\phi_l)\omega^\star_\alpha \omega_\beta\,. \tag{6.3.11}$$

From (6.3.11) we see that he ghost fields along the broken directions have a mass square that equal the gauge boson mass matrix times ξ. For any ξ, the mixing term in (6.3.11), upon an integration by parts, cancels the second unwanted term in (6.3.6). The unitary gauge (6.3.7) corresponds to the limit $\xi \to \infty$ of the above class of gauges. In this limit, the term proportional to ξ in $\mathcal{L}_{g.f.}$ oscillates very rapidly and averages to zero any field configuration for which this term is non-zero. We then effectively recover the unitary gauge (6.3.7). In this gauge we see from (6.3.11) that the ghost fields become infinitely massive and can be neglected.

[2] We take $\delta v_m = 0$, so that $\delta\phi'_m = \omega^\alpha t^\alpha_{mn}\phi_n$.

Let us derive the propagators for gauge fields, ghosts and scalar (NG and not) fields in an arbitrary ξ-gauge. In momentum space, setting μ^2 to diagonal form, the quadratic terms of the gauge bosons in the Lagrangian add to

$$\mathcal{L}_{quad.}(A) = \frac{1}{2} A_\mu^\alpha(-p) A_\nu^\beta(p) \delta_{\alpha\beta} \Big(- p^2 \eta^{\mu\nu} + p^\mu p^\nu \Big(1 - \frac{1}{\xi}\Big) + \eta^{\mu\nu} \mu_\alpha^2 \Big) , \tag{6.3.12}$$

valid for both broken and unbroken generators (in the latter case with $\mu_i^2 = 0$, of course). By inverting the above quadratic terms, we easily get

$$G_{\mu\nu}^{\alpha\beta(\xi)}(p) = \frac{-i\delta_{\alpha\beta}}{p^2 - \mu_a^2} \Big(\eta_{\mu\nu} - \frac{(1-\xi) p_\mu p_\nu}{p^2 - \xi \mu_\alpha^2} \Big) . \tag{6.3.13}$$

The ghost propagator $G_{\alpha\beta}^{\omega(\xi)}(p)$ is trivially obtained from (6.3.11):

$$G_{\alpha\beta}^{\omega(\xi)}(p) = \frac{i\delta_{\alpha\beta}}{p^2 - \xi \mu_\alpha^2} . \tag{6.3.14}$$

The scalar propagators are different for NG and non-NG bosons. The total mass matrix in the Lagrangian reads

$$M_{mn}^2 = \frac{\partial^2 V}{\partial \phi_m \partial \phi_n} \Big|_{\phi = v} + \xi g^2 (i t_{mp}^a v_p)(i t_{nq}^a v_q) . \tag{6.3.15}$$

The mass matrix (6.3.15), acting on the NG boson directions $(t_{ns}^b v_s)$, gives

$$M_{mn}^2 (t_{ns}^b v_s) = \xi g^2 (i t_{mp}^a v_p)(i t_{nq}^a v_q)(t_{ns}^b v_s) = \xi \delta_{ab} \mu_a^2 (t_{mp}^a v_p) , \tag{6.3.16}$$

where the first term in (6.3.15) vanishes thanks to the relation (6.2.6). In a generic ξ-gauge, the would-be NG bosons have a mass which is $\sqrt{\xi}$ times the gauge boson mass. On the other hand, for the non-NG boson directions, defined by (6.3.7), it is the second term in (6.3.15) that vanishes, giving as masses the eigenvalues m_i^2 of $\partial^2 V / \partial \phi_m \phi_n$. We then get as scalar propagators

$$\begin{aligned} G_{ij}^S(p) &= \frac{i\delta_{ij}}{p^2 - m_i^2} , \quad \text{(no NG bosons)} \\ G_{ab}^{S(\xi)}(p) &= \frac{i\xi\delta_{ab}}{p^2 - \xi \mu_a^2} , \quad \text{(NG bosons)} \end{aligned} \tag{6.3.17}$$

where the index i runs over all, but the NG boson, scalar directions.

In the unitary gauge $\xi \to \infty$, the NG bosons decouple and can be seen as eaten by the gauge fields. In any other gauge, they should be kept. Notice how the gauge boson propagator (6.3.13) goes like $1/p^2$ for large momenta for any finite values of ξ and it is only in the unitary gauge that it behaves as p^0. This makes any finite ξ-gauge more suitable than the unitary gauge to address renormalizability properties of spontaneously broken gauge theories. Commom choices of ξ-gauges are the Landau gauge $\xi = 0$, in which ghosts and NG bosons are massless, and the Feynman gauge $\xi = 1$, in which the gauge propagator simplifies considerably.

6.4 Effective Field Theories for General Broken Symmetries*

We have already remarked that the appearance of massless NG bosons is a generic prediction of the Goldstone's theorem, no matter whether the underlying theory is weakly or strongly coupled. We will see in this section that, surprisingly enough, a lot can be said about the interactions of the NG bosons with themselves and with possible extra fields at low energies, without even knowing the underlying UV Lagrangian (both particles and interactions) responsible of the spontaneous symmetry breaking pattern. All we need to know is the symmetry breaking pattern $G \to H$, namely the starting and final global groups. The key point is that the effective low energy theory is not merely invariant under the subgroup H, but it "remembers" its origin from an underlying bigger group of symmetry G. More precisely, the effective theory turns out to be invariant under the whole group G. The distinction between H and G is only on the way the symmetries are realized: linearly for H and non-linearly for G/H. We will here consider the general construction of the effective field theory of NG bosons, discussed for the first time by Callan, Coleman, Wess and Zumino [4].

Let us start by writing the general commutation relations among the generators $t_i \in \mathcal{H}$ and $t_a \in \mathcal{G}/\mathcal{H}$, $\mathcal{G}$ and $\mathcal{H}$ being the Lie algebras of the corresponding groups:

$$[t_i, t_j] = i f_{ijk} t_k \,, \quad [t_i, t_a] = i f_{iab} t_b \,, \quad [t_a, t_b] = i f_{abc} t_c + i f_{abi} t_i \,. \tag{6.4.1}$$

We take here the structure constants completely antisymmetric in its indices. Since H is a subgroup of G, $f_{ija} = -f_{iaj} = 0$. Let ϕ_m be the fields responsible for the $G \to H$ spontaneous symmetry breaking pattern (again in a basis where all the fields are real). Omitting for simplicity the matrix indices m, n, we have $t_i v = 0$ and $t_a v \neq 0$. Let us define fields $\tilde{\phi}$ that do not contain NG boson field directions:

$$\phi(x) = \gamma(x) \tilde{\phi}(x), \quad \gamma(x) \in G \,, \tag{6.4.2}$$

such that

$$\tilde{\phi}^t(x) t_\alpha v = 0, \quad \forall \alpha \,. \tag{6.4.3}$$

In this basis the NG bosons all sit in the matrix field $\gamma(x)$. The definition of $\tilde{\phi}$ is not unique. If $\tilde{\phi}$ satisfies (6.4.3), so does $\tilde{\phi}' = \tilde{\phi} + c_i t_i \tilde{\phi}$, for arbitrary coefficients c_i. Indeed,

$$\begin{aligned} \tilde{\phi}'^t t_\alpha v &= (\tilde{\phi}^t + c_i \tilde{\phi}^t t_i^t) t_\alpha v = (\tilde{\phi}^t - c_i \tilde{\phi}^t t_i) t_\alpha v \\ &= -c_i \tilde{\phi}^t t_i t_a v = -c_i \tilde{\phi}^t [t_i, t_a] v = -i c_i f_{iab} \tilde{\phi}^t t_b v = 0 \,. \end{aligned} \tag{6.4.4}$$

The field matrix $\gamma(x)$ is defined only modulo x-dependent transformations under H: $\gamma(x) \sim \gamma(x) h(x)$, with $h(x) \in H$. Given this equivalence class, we can always choose as representative a $\gamma(x)$ of the form

$$\gamma(x) = e^{i\xi_a(x)t_a}\,, \tag{6.4.5}$$

where ξ^a are essentially the NGB's fields. Under a global G transformation, we have $\phi \to \phi' = g\phi = g\gamma(x)\tilde{\phi}(x)$. The field ϕ' can also be decomposed as (6.4.2). In general, we will have

$$\phi'(x) = \gamma(\xi'_a(x))h(\xi_a(x), g)\tilde{\phi}(x)\,, \tag{6.4.6}$$

for some $h(\pi_a(x), g) \in H$, so that

$$\begin{aligned} \phi'(x) =& \gamma(\xi'_a(x))\tilde{\phi}'(x)\,, \quad \tilde{\phi}'(x) = h(\xi_a(x), g)\tilde{\phi}(x)\,, \\ \gamma(\xi'_a(x)) =& g\gamma(\xi_a(x))h^{-1}(\xi_a(x), g) \end{aligned} \tag{6.4.7}$$

We notice that the field $\tilde{\phi}(x)$, under global G transformations, transforms as *local* transformations under H. The transformations of ξ_a and $\tilde{\phi}$ simplify considerably when $g = h \in H$. The second commutation relations in (6.4.1) shows that the broken generators t_a transform linearly under transformations of h, namely

$$h t_a h^{-1} = t_b R_{ba}(h)\,, \tag{6.4.8}$$

where R is some representation (in general reducible) of H. The very same commutation relations shows that when $g = h$, the factor $h(\xi(x), g)$ defined in (6.4.6) reduces to the global x-independent element h: $h(\xi(x), g) = h$. We then have, using also (6.4.8) (we often omit from now on, for simplicity, the partial or total dependence on x, $\xi_a(x)$, etc. of the various fields)

$$\gamma' \equiv \gamma(\xi') = h\gamma(\xi)h^{-1} = \gamma(R(h)\xi)\,. \tag{6.4.9}$$

Under H, then, the transformations of both ξ and $\tilde{\phi}$ are simple and linear:

$$\begin{aligned} \xi'_a(x) &= R_{ab}(h)\,\xi_b(x)\,, \\ \tilde{\phi}'(x) &= h\,\tilde{\phi}(x)\,. \end{aligned} \tag{6.4.10}$$

Let us now study the field combination $\gamma^{-1}\partial_\mu\gamma$. By recalling the formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \ldots \tag{6.4.11}$$

valid for arbitrary matrices X and Y, we see that the combination $\gamma^{-1}\partial_\mu\gamma$ is a field defined in the algebra $\mathcal{G}$ of the group G. As such, it can be decomposed as

$$\gamma^{-1}\partial_\mu\gamma = it_a D^a_\mu(x) + it_i E^i_\mu(x)\,, \tag{6.4.12}$$

where necessarily $D^a_\mu(x) = D^{ab}(\xi(x))\partial_\mu \xi_b(x)$, $E^i_\mu(x) = E^{ia}(\xi(x))\partial_\mu \xi_a(x)$, for some fields D^{ab} and E^{ia}. Under a global transformation $g \in G$ we clearly have

$$(g\gamma)^{-1}\partial_\mu(g\gamma) = \gamma^{-1}\partial_\mu\gamma\,. \tag{6.4.13}$$

On the other hand $g\gamma = \gamma' h$ and hence, after multiplication by h and h^{-1} to the left and right, respectively, we get

$$\gamma^{-1\prime}\partial_\mu\gamma' = h(\gamma^{-1}\partial_\mu\gamma)h^{-1} - (\partial_\mu h)h^{-1}\,. \tag{6.4.14}$$

Using the decomposition (6.4.12) for γ' and identifying both sides, we get the transformations of D^a_μ and E^i_μ under global G transformations:

$$\begin{aligned} t_a D^a_\mu(\xi') &= h t_a h^{-1} D^a_\mu(\xi) = t_a R_{ab}(h) D^b_\mu(\xi)\,, \\ t_i E^i_\mu(\xi') &= h t_i h^{-1} E^i_\mu(\xi) + i(\partial_\mu h)h^{-1}\,. \end{aligned} \tag{6.4.15}$$

Since H is a subgroup of G, we have

$$h t_i h^{-1} = R_{ij}(h) t_j\,, \qquad (\partial_\mu h)h^{-1} = i t_i R_{ia}(h)\partial_\mu \xi_a(x)\,, \tag{6.4.16}$$

and thus

$$\begin{aligned} D^a_\mu(\xi') &= R_{ab}(h) D^b_\mu(\xi)\,, \\ E^i_\mu(\xi') &= R_{ij}(h) E^j_\mu(\xi) - R_{ia}(h)\partial_\mu\xi_a(x)\,. \end{aligned} \tag{6.4.17}$$

Under global G transformations, the factors D^a_μ transform linearly, while the factors E^i_μ transform non-linearly like a gauge field. We can in fact use E^i_μ to build covariant derivatives for $\tilde\phi$. From (6.4.7), we get that under G transformations

$$\partial_\mu\tilde\phi' = (\partial_\mu h)\tilde\phi + h(\partial_\mu\tilde\phi) = h\Big(h^{-1}(\partial_\mu h)\tilde\phi + \partial_\mu\tilde\phi\Big)\,. \tag{6.4.18}$$

Defining

$$D_\mu\tilde\phi \equiv (\partial_\mu\tilde\phi + i t_i E^i_\mu)\tilde\phi\,, \tag{6.4.19}$$

we have

$$D_\mu\tilde\phi' = h D_\mu\tilde\phi\,. \tag{6.4.20}$$

The symmetries constrain the NG bosons to appear in the low-energy Lagrangian only through the combinations D^a_μ and E^i_μ in derivative interactions. We conclude that any Lagrangian invariant under H transformations and constructed with D^a_μ, $\tilde\phi$ and their covariant derivatives, will automatically be invariant under the whole group G. Invariant terms can also be constructed using the "field strength"

$$F_{\mu\nu}(E) = \partial_\mu E_\nu - \partial_\nu E_\mu + [E_\mu, E_\nu] \tag{6.4.21}$$

and derivatives thereof, keeping in mind that not all invariants constructed in this way are independent of each other.

The transformation of the fields ξ_a and $\tilde{\phi}$ is linear when $g = h \in H$ (see (6.4.10)) and non-linear for $g \in G/H$. In particular, it is important to notice that the NG fields ξ_a transform as a shift for infinitesimal transformations along G/H, at leading order in the field fluctuations. For $g = 1 + i\epsilon_a t_a$, indeed, we have

$$\phi' = g\phi = (1 + i\epsilon_a t_a)\phi = \phi + i\epsilon_a t_a(1 + i\xi_b t_b + \mathcal{O}(\xi^2))\tilde{\phi}\,. \tag{6.4.22}$$

On the other hand $\delta\phi = i\delta\xi_a t_a\tilde{\phi} + \delta\tilde{\phi} + \ldots$ and hence

$$\delta\xi_a = \epsilon_a + \mathcal{O}(\pi^2, \epsilon\xi)\,, \quad \delta\tilde{\phi} = 0 + \mathcal{O}(\epsilon\xi)\,. \tag{6.4.23}$$

The leading order term encoding the NGB kinetic terms is

$$\mathcal{L} \supset \frac{1}{2}F^2_{ab}D^a_\mu D^b_\mu\,, \tag{6.4.24}$$

where $D^a_\mu = \partial_\mu\xi^a + \ldots$. The linearly realized symmetry H constrains the coefficients F^2_{ab}, in general different, so that the above term is H-invariant. These coefficients are closely related to the F_{ab} entering in (6.1.8). In order to see that, let us compute the explicit form of the broken currents J^a_μ from the the Lagrangian (6.4.33). Using the usual trick of promoting a global symmetry to a local one, so that $\delta_\epsilon\mathcal{L} = -\partial_\mu\epsilon J_\mu$, we have,

$$\delta_{\epsilon^a}\mathcal{L} = -F^2_{ab}D^b_\mu\partial_\mu\epsilon^a + \ldots = -\partial_\mu\epsilon^a F^2_{ab}\partial_\mu\xi^b + \ldots \Rightarrow J^a_\mu = F^2_{ab}\partial_\mu\xi^b + \ldots\,, \tag{6.4.25}$$

where we have used the leading order transformation (6.4.23) for the ξ's. Comparing with (6.1.8), we notice that the F_{ab} in (6.4.33) is identified with that in (6.1.8). Moreover we see that

$$\xi^a(x) = F^{-1}_{ab}\pi^b(x)\,, \tag{6.4.26}$$

in terms of the canonically normalized NGB's.

In general, we might have that a subgroup H_g of G, distinct from H, is gauged. In this case, the above formalism can easily be extended. It is convenient, for this purpose, to pretend that the whole group G is gauged and only at the end recover the original theory by setting to zero the gauge fields not in H_g. Notice that gauging a subgroup of H_g explicitly breaks the global symmetry G, since it determines a specific direction in field space. Correspondingly, the uneaten NG bosons become pseudo NG bosons and get a mass proportional to the gauge coupling constant associated to the gauge group H_g. Having said that, one can check that all our previous results, until (6.4.11) included, are formally still valid, provided the obvious understanding that now the G transformations are x-dependent. The G invariant field combination to consider now, analogue of $\gamma^{-1}\partial_\mu\gamma$, is[3]

$$\gamma^{-1}D_\mu\gamma \equiv \gamma^{-1}(\partial_\mu - it^\alpha A^\alpha_\mu)\gamma\,, \tag{6.4.27}$$

so that the inhomogeneous term in the gauge field transformation cancels the analogue term coming from $\partial_\mu g$. The quantity $\gamma^{-1}D_\mu\gamma$ is also defined in the algebra $\mathcal{G}$, so we have

$$\gamma^{-1}D_\mu\gamma \equiv it_a\hat{D}^a_\mu(x) + it_i\hat{E}^i_\mu(x)\,, \tag{6.4.28}$$

where $\hat{D}^a_\mu$ and $\hat{E}^i_\mu$ are the gauged versions of the fields D^a_μ and E^i_μ defined in (6.4.12), which now include the gauge fields A^α_μ. Following the same steps as above, we immediately get

$$(\gamma^{-1}D_\mu\gamma)' = h(x)(\gamma^{-1}D_\mu\gamma)h^{-1}(x) - \partial_\mu h(x)h^{-1}(x)\,. \tag{6.4.29}$$

Correspondingly, the relations (6.4.15) are valid for $\hat{D}^a_\mu$ and $\hat{E}^i_\mu$ as well. As before, any Lagrangian invariant under H transformations and constructed with $\hat{D}^a_\mu$, $\tilde{\phi}$ and their covariant derivatives, will automatically be invariant under the whole gauge group G. In order to also construct H invariant operators involving the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, that transforms linearly under G as $F_{\mu\nu} \to gF_{\mu\nu}g^{-1}$, we can define NG boson-dependent field strengths

$$f_{\mu\nu} = \gamma^{-1}F_{\mu\nu}\gamma\,. \tag{6.4.30}$$

The latter transform as $f_{\mu\nu} \to h\gamma^{-1}g^{-1}gF_{\mu\nu}g^{-1}g\gamma h^{-1} = hf_{\mu\nu}h^{-1}$ under G and can be used together with $\hat{D}^a_\mu$, $\tilde{\phi}$ and their associated operators constructed acting with covariant derivatives. We can now set to zero the "spurionic" gauge field components that do not belong to H_g. In order to distinguish H_g from G, we denote by $t^{\hat{\alpha}}$ the generators of G that are gauged. In general we have

$$t^{\hat{\alpha}} = g_{\hat{\alpha}\alpha}t^\alpha \tag{6.4.31}$$

where $g_{\hat{\alpha}\alpha}$ are the gauge coupling constants associated to the gauging, all different in the most general case, and set

$$A^\alpha_\mu = g_{\hat{\alpha}\alpha}A^{\hat{\alpha}}_\mu\,, \tag{6.4.32}$$

where $A^{\hat{\alpha}}_\mu$ are the actual dynamical gauge fields. The leading order term (6.4.33) in the low-energy effective Lagrangian becomes

$$\mathcal{L} \supset \frac{1}{2}F^2_{ab}\hat{D}^a_\mu\hat{D}^b_\mu\,, \tag{6.4.33}$$

[3]In order not to confuse the group element g with the gauge coupling constant g, we adopt here a non-canonical normalization for the gauge fields A^α_μ.

where $\hat{D}^a_\mu = \partial_\mu \xi^a - g_{\hat{\alpha}a} A^{\hat{\alpha}}_\mu$. As expected, the NGB's along the gauged directions are eaten by the gauge fields. Their mass is given by

$$\mu^2_{\hat{\alpha}\hat{\beta}} = F^2_{ab}\, g_{\hat{\alpha}a}\, g_{\hat{\beta}b}\,. \tag{6.4.34}$$

This relation is the generalization of eq.(6.3.8). The latter only applies for weakly coupled descriptions of the spontaneous symmetry breaking in terms of free fields.

The simplest instance of symmetry breaking pattern is $G = U(1)$, $H = \emptyset$ and ϕ a complex scalar field, i..e. the global, ungauged, version of the set-up we considered at the beginning of section (6.3) for the Higgs mechanism. The field decomposition (6.4.6) is in this case easily obtained by going to radial coordinates (compare with (6.3.3)), so we identify

$$\xi(x) = \frac{\theta(x)}{v}, \qquad \tilde{\phi}(x) = \frac{v + \rho(x)}{\sqrt{2}}\,. \tag{6.4.35}$$

When the group G is gauged, the field decomposition (6.4.6) can be interpreted as a gauge transformation. Correspondingly, we can gauge away $\gamma(x)$. The gauge in which $\gamma(x) = 1$ is nothing else than the unitary gauge defined before.

When the structure constants f_{abc}, introduced in the third commutator in (6.4.1), vanish (the coset G/H in this case is called symmetric), the Lie algebra defined by the relations (6.4.1) is invariant under the $\mathbf{Z}_2$ symmetry R under which the broken generators change sign:

$$R(t_i) = t_i\,, \qquad R(t_a) = -t_a\,. \tag{6.4.36}$$

We clearly have $R(\gamma) = \gamma^{-1}$. Under G transformations

$$\gamma' = g\gamma h^{-1}\,. \tag{6.4.37}$$

On the other hand

$$\begin{aligned} \gamma'^{-1} =& R(\gamma') = R(g)R(\gamma)R(h^{-1}) = R(g)\gamma^{-1}h^{-1} \\ & \Longrightarrow \gamma' = h\gamma R(g)^{-1}\,. \end{aligned} \tag{6.4.38}$$

Using both (6.4.37) and (6.4.38), we see that

$$U' \equiv \gamma'^2 = g\gamma h^{-1} h\gamma R(g)^{-1} = g\, U R(g)^{-1}\,. \tag{6.4.39}$$

For coset spaces, the square of the matrix γ transform linearly under transformations of G. This is a very useful result, because it can considerably simplify the construction of the low-energy effective Lagrangians, as we will see in the next two subsections. One can also obtain simple explicit expressions for $\hat{D}^a_\mu$ and $\hat{E}^i_\mu$, that will not be written here. It is perhaps time to explicitly see all this formalism at work in some examples.

6.4.1 Relevant Example I: $SU(3)_V \times SU(3)_A \to SU(3)_V$*

Consider QCD with three active quarks (say, the up, down and strange quark) in the limit in which they are all massless:

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \overline{Q}\, i\not{D} Q\,, \tag{6.4.40}$$

where $Q = (u\ d\ s)^t$. In addition to the obvious $SU(3)_c$ local (color) symmetry, the Lagrangian (6.4.40) is invariant under an additional $SU(3)_V \times SU(3)_A$ global symmety acting on Q as

$$Q \to \exp(i\theta^V_a \lambda_a + i\theta^A_a \lambda_a \gamma_5) Q\,, \tag{6.4.41}$$

where λ_a are the $SU(3)$ Gell-Mann matrices, normalized such that $\mathrm{tr}\,\lambda_a\lambda_b = \delta_{ab}/2$ in the fundamental representation. At sufficiently low energies, when the QCD coupling constant becomes strong, a quark bilinear gets a non-vanishing vacuum expectation value:

$$\langle \overline{Q}_i Q_j \rangle = \hat{\Lambda}^3 \delta_{ij}\,, \tag{6.4.42}$$

that breaks $G = SU(3)_V \times SU(3)_A$ down to $H = SU(3)_V$. Here $\hat{\Lambda}$ is the scale where the chiral symmetry breakdown occurs. The eight NG bosons are encoded in the matrix field

$$\gamma(x) = e^{i\lambda^a \xi^a(x)}\,. \tag{6.4.43}$$

The $SU(3)_V$ unbroken symmetry fixes the parameters F_{ab} to be proportional to the identity. We define $F_{ab} \equiv \delta_{ab} f_\pi$, and write the 8 NG bosons as

$$\lambda_a \xi_a = \frac{1}{\sqrt{2} f_\pi} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi_0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi_0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ \bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{pmatrix}, \tag{6.4.44}$$

where f_π is the pion decay constant: $f_\pi \simeq 92$ MeV. With this normalization the 8 spin zero mesons $\pi^\pm$, π^0, etc. are all canonically normalized.

The commutator of two vector or two axial transformations is a vector transformation, while the commutator of a vector and axial transformation is an axial transformation. The schematic commutation relation for $SU(3)_V \times SU(3)_A$ is then $[V,V] = V$, $[V,A] = A$, $[A,A] = V$, where V and A schematically represent the $SU(3)_V$ and $SU(3)_A$ generators. This is a symmetric coset space, being this algebra invariant under the automorphism action $V \to V$, $A \to -A$. We can also construct L and R transformations defined as $V = (L+R)/2$, $A = (L-R)/2$, under which the above automorphism exchanges L and R:

$L \leftrightarrow R$. According to the general results of section (6.4), the matrix $U = \gamma^2$ transforms homogeneously under g. We have[4]

$$U \to RUL^\dagger \,, \tag{6.4.45}$$

with obvious notation. In other words, U transforms as a $(\mathbf{3}, \bar{\mathbf{3}})$ representation of $SU(3)_R \times SU(3)_L$. Thanks to (6.4.45), we can easily write the leading, two-derivative, term involving the NG bosons and invariant under the symmetry (6.4.45):

$$\mathcal{L}_{Kin} = \frac{f_\pi^2}{4} \mathrm{tr} \left(\partial_\mu U \partial^\mu U^\dagger \right) . \tag{6.4.46}$$

All the interactions among the 8 mesons π, K and η are collected in the single term (6.4.46). This is of course the first term of an infinite series of higher derivative operators involving the matrices U. By expanding U in terms of field fluctuations, we notice that the effective coupling constant of all the interaction terms that can arise from (6.4.46) is E^2/f_π^2. For sufficiently low-energies $E \ll f_\pi$, the above term is the most important one, all the higher derivative interactions being suppressed by additional powers of E/f_π. The effective theory defined by (6.4.46) becomes unreliable when $E \sim f_\pi$, in which case an alternative description is needed. More precisely, the energy scale where we expect our effective theory to break down is $E = \Lambda \simeq 4\pi f_\pi \sim 1$ GeV, including the 4π factor coming from loops (analogue of the effective QED expansion parameter $e^2/(16\pi^2)$). It is natural to assume that $\Lambda \simeq \hat{\Lambda}$, the scale where chiral symmetry breaking occurs. This scale is related to the scale Λ_{QCD} where quarks confine (confinement scale), but it is *not* the same scale. The two scales are indeed slightly different, with the chiral symmetry breaking scale slightly higher than the confinement one. They are also conceptually different. In fact, there exist vacua in gauge (supersymmetric) theories where it is believed that quarks confine but no chiral symmetry breaking occurs.

In the real world, of course, the up, down and strange quarks are not massless and the spin zero mesons π, K and η are not exact NG bosons. The two things are related. The actual up, down and strange masses explicitly break the $SU(3)_A$ symmetry, but are small enough, compared to Λ, to be considered as a perturbation of the QCD Lagrangian (6.4.40) (this would not be the case for the charm, bottom and top, that are all heavier than Λ). We can get the relation between the quark and meson masses by using the

[4] There is actually no way to distinguish an $SU(3)_L$ from an $SU(3)_R$ transformation when acting on a 3×3 matrix such as U. Strictly speaking, we should write U as a 6×6 matrix, containing U and $U^\dagger$, to properly distinguish the two $SU(3)$'s. In doing that, one gets eq.(6.4.45).

following trick. We formally promote the quark mass term

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{6.4.47}$$

to be an external source that transforms under $SU(3)_L \times SU(3)_R$ as $M \to LMR^\dagger$, so that the term $\bar{Q}_L M Q_R + h.c.$ is made invariant. Of course, this is just a trick, because the true mass term, with M as in (6.4.47), is not invariant, but it is useful to implement the actual form (6.4.47) only at the end of our considerations. At low-energies, now, we can construct $SU(3)_L \times SU(3)_R$ invariant terms using both U and M. If M is small, we can consider terms linear in M only. We simply have

$$\mathcal{L}_m = c f_\pi^3 \mathrm{tr}\, UM + h.c. \,, \tag{6.4.48}$$

where c is an undetermined dimensionless coefficient. By plugging in (6.4.48) the actual form of the NG fields (6.4.44) and of the mass term (6.4.47) we get, after some simple algebra:

$$\begin{aligned} m_{\pi^\pm}^2 &= C(m_u + m_d)\,, \\ m_{K^\pm}^2 &= C(m_u + m_s)\,, \\ m_{K^0}^2 &= C(m_d + m_s)\,, \end{aligned} \tag{6.4.49}$$

where $C = 2cf_\pi$. The π^0 and η^0 mix with each other. Expanding for $m_{u,d} \ll m_s$ we get

$$\begin{aligned} m_{\pi^0}^2 &\simeq C(m_u + m_d)\,, \\ m_\eta^2 &\simeq C\frac{4m_s + m_u + m_d}{3}\,. \end{aligned} \tag{6.4.50}$$

It is surprising that using symmetries only we can fix the meson masses in terms of the quark masses and one unknown coefficient C. The above relations are a particular example of (6.2.9), showing that in general the masses of the pseudo NG bosons are linearly dependent on the source of the symmetry breaking violating term in the Lagrangian. In deriving (6.4.49) and (6.4.50) we have actually neglected another relevant source of explicit breaking of the $SU(3)_L \times SU(3)_R$ symmetry, given by the electric charge. As explained in section (6.4), the latter is introduced by gauging a $U(1)$ subgroup of $SU(3)_L \times SU(3)_R$. Given the form of Q and the charges of the up, down and strange quarks, the $U(1)$ generator of QED is given by

$$Q_{el} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{6.4.51}$$

We do not need to go through the full formalism of section (6.4) to get the NG boson interactions with the photon. Thanks to the linear transformation (6.4.45), it is enough to replace the ordinary derivatives in (6.4.46) with covariant ones:

$$\mathcal{L}_{Kin+\gamma} = \frac{f_\pi^2}{4}\mathrm{tr}\left(D_\mu U D^\mu U^\dagger\right), \tag{6.4.52}$$

where

$$D_\mu U = \partial_\mu U + ieQ_{el}U - ieUQ_{el}\,. \tag{6.4.53}$$

It is straightforward to check that (6.4.52) contains the expected interactions of the QED form for the charged mesons $\pi^\pm$ and $K^\pm$. These interactions, at the quantum level, contribute to the mass of the charged mesons by an amount Δ_{el}, which is the same, at leading order, for $\pi^\pm$ and $K^\pm$, thanks to the $SU(3)_V$ global symmetry. Eventually, the effect of the QED interactions is to shift the charged meson masses in (6.4.50) by this amount Δ_{el}:

$$\begin{aligned} m_{\pi^\pm}^2 &\to m_{\pi^\pm}^2 + \Delta_{el}, \\ m_{K^\pm}^2 &\to m_{K^\pm}^2 + \Delta_{el}\,. \end{aligned} \tag{6.4.54}$$

The neutral meson masses, at this order, are clearly unaffected by QED interactions. There are various ways in which the mass formulas (6.4.49), (6.4.50) and (6.4.54) can be used. For instance, the following relation holds, independently of C, Δ_{el} and the quark masses:

$$3m_\eta^2 + 2m_{\pi^\pm}^2 - m_{\pi^0}^2 = 2m_{K^\pm}^2 + 2m_{K^0}^2 \tag{6.4.55}$$

and it is experimentally very well reproduced. Alternatively, we might use the meson masses as experimental input parameters to compute the quark masses or, better, their ratios. One has

$$\begin{aligned} \frac{m_u}{m_s} &= \frac{2m_{\pi^0}^2 - m_{\pi^\pm}^2 + m_{K^\pm}^2 - m_{K^0}^2}{m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^\pm}^2} \simeq 0.027\,, \\ \frac{m_d}{m_s} &= \frac{m_{\pi^\pm}^2 + m_{K^0}^2 - m_{K^\pm}^2}{m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^\pm}^2} \simeq 0.050\,, \end{aligned} \tag{6.4.56}$$

where we have inserted in the last relation the approximate masses for the spin zero mesons.

The actual global symmetry of the QCD Lagrangian (6.4.40) is $U(3)_V \times U(3)_A$, rather than $SU(3)_V \times SU(3)_A$, since the $U(1)_V \times U(1)_A$ transformations

$$Q \to \exp(i\theta^V + i\theta^A\gamma_5)Q \tag{6.4.57}$$

are a symmetry of (6.4.40). The $U(1)_V$ symmetry is nothing else than the $U(1)$ baryon number and is unbroken, while $U(1)_A$ is manifestly broken by the quark condensate (6.4.42). Correspondingly, we should have a ninth pseudo NG boson, called η'. It can be shown, by adding the η' to the matrix field (6.4.43), that its mass $m_{\eta'}$ is bounded to be $m_{\eta'} \leq \sqrt{3} m_\pi$. But no such particle has been observed within this range of energies. This puzzle, denoted the $U(1)_A$ problem, has been solved by noticing that the $U(1)_A$ symmetry is explicitly broken – by an amount larger than the explicit breaking of the quark masses – by instantons, non-perturbative effects that are outside the content of these lectures. The actual η' has indeed a much larger mass, $m_{\eta'} \simeq 960$ MeV.

6.4.2 Relevant Example II: $SO(5) \to SO(4)^\star$

The example that follows is a bit more exotic and refers to the electroweak sector of the Standard Model (SM). We have briefly seen in section 5.4 that scalar masses are unnatural, in the sense that they are quadratically sensitive to UV physics that push their values to that scale. This problem clearly applies to *elementary* scalars, including the Higgs boson in the SM. But what if the Higgs is not elementary, but rather a composite particle, very much like the pion ? We do not have a hierarchy problem for the pion mass, since we know that this particle is composite and at some energies of order Λ_{QCD}, "diffuse" in its quark constituents. Moreover, pions are actually naturally lighter than Λ_{QCD} itself because, as we have extensively seen in the previous subsection, are pseudo NG bosons of an approximate global symmetry.

Before developing on this idea of considering the Higgs particle as a pseudo NG boson of some spontaneously broken global symmetry, it is useful to recall the symmetries of the standard, elementary Higgs Lagrangian, in the SM. In the global limit in which we switch off the $SU(2)_L \times U(1)_Y$ gauge couplings, the SM Higgs sector is

$$\mathcal{L}_H^{SM} = (\partial_\mu H)^\dagger (\partial^\mu H) + m^2 H^\dagger H - \frac{\lambda}{2} (H^\dagger H)^2 \,, \tag{6.4.58}$$

where H is an $SU(2)_L$ doublet of the form

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} h_2 + i h_1 \\ h_4 - i h_3 \end{pmatrix} . \tag{6.4.59}$$

A closer look at (6.4.58) reveals that the Higgs Lagrangian is invariant under an $SO(4)$, rather than $SU(2)_L$, global symmetry. This is manifest if we recast the four real Higgs components h_i $(i = 1, 2, 3, 4)$ in a 4-plet and notice that (6.4.58) is invariant under the transformations $h_i \to O_{ij} h_j$, with $O \in SO(4)$. It is actually more convenient to exploit the local isomorphism of $SO(4)$ with $SU(2) \times SU(2)$. The fundamental **4** of $SO(4)$ becomes a bidoublet $(\mathbf{2},\mathbf{2})$ of $SU(2)\times SU(2)$. One of the two $SU(2)$ is identified with the original $SU(2)_L$, and we denote by $SU(2)_R$ the other one. We can easily write the $SO(4)$ generators in the $SU(2)_L\times SU2)_R$ basis. The 6 anti-symmetric hermitian generators of $SO(4)$ are proportional to

$$t^{ab}_{ij} = -t^{ba}_{ij} = \delta^a_i\delta^b_j - \delta^b_i\delta^a_j\,, \tag{6.4.60}$$

where $a,b=1,\dots,4$ label the generators and i,j their matrix components. A simple check of the algebra reveals that the combinations

$$\begin{aligned} t^1_L &= -\frac{i}{2}(t^{23}+t^{14}), \quad t^2_L = -\frac{i}{2}(t^{31}+t^{24}), \quad t^3_L = -\frac{i}{2}(t^{12}+t^{34}), \\ t^1_R &= -\frac{i}{2}(t^{23}-t^{14}), \quad t^2_R = -\frac{i}{2}(t^{31}-t^{24}), \quad t^3_R = -\frac{i}{2}(t^{12}-t^{34}), \end{aligned} \tag{6.4.61}$$

satisfy the commutation relations of the $SU(2)_L$ and $SU(2)_R$ algebras, respectively. The action of $SU(2)_L\times SU(2)_R$ is best seen by writing explicitly the Higgs field as a bidoublet using the 2×2 matrices $\sigma^\mu = (1, i\sigma_i)$ $(i=1,2,3)$:

$$H_{BD} = \frac{h_4 1_2 + i h_i\sigma_i}{2} = \frac{1}{2}\begin{pmatrix} h_4 + ih_3 & -h_2 - ih_1 \\ -h_2 + ih_1 & h_4 - ih_3 \end{pmatrix}. \tag{6.4.62}$$

Under $SU(2)_L\times SU(2)_R$, the bidoublet (6.4.62) transforms as

$$H_{BD} \to g_L H_{BD} g_R^\dagger\,, \tag{6.4.63}$$

with $g_L\in SU(2)_L$, $g_R\in SU(2)_R$. It is manifest from (6.4.63) that when the Higgs field develops a non-vanishing VEV, say $\langle h_4\rangle \equiv v \neq 0$, i.e. $H_{BD}\propto 1_2$, the $SU(2)_L\times SU(2)_R$ global symmetry is broken to the diagonal $SU(2)_c$ symmetry, with $g_L=g_R$. This unbroken $SU(2)_c$ global symmetry is called custodial symmetry and plays an important role in establishing that the W and Z boson mass ratio at tree-level is

$$\frac{m_Z^2}{m_W^2} = \frac{g^2+g'^2}{g^2}\,, \tag{6.4.64}$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge coupling constants. In this formalism, the gauge fields are introduced by gauging the whole $SU(2)_L$ global group and a subgroup of $SU(3)_R$, along the σ_3 generator, which is identified with the $U(1)_Y$ symmetry. The full, gauged, Higgs Lagrangian reads

$$\mathcal{L}_H^{SM} = \mathrm{tr}\,(D_\mu H_{BD})^\dagger(D^\mu H_{BD}) + m^2 \mathrm{tr}\, H_{BD}^\dagger H_{BD} - \frac{\lambda}{2}\mathrm{tr}\,(H_{BD}^\dagger H_{BD})^2\,, \tag{6.4.65}$$

where

$$D_\mu H_{BD} = \partial_\mu H_{BD} - igW_\mu^L H_{BD} + ig' H_{BD} W_\mu^R, \tag{6.4.66}$$

and

$$W^L_\mu = \frac{1}{2}\sigma_i W^i_\mu\,, \qquad W^R_\mu = \frac{1}{2}\sigma_3 B_\mu\,. \tag{6.4.67}$$

The custodial $SU(2)_c$ symmetry is broken by the hypercharge coupling g' only.[5] If we set $g'=0$, the Lagrangian (6.4.65) is invariant under $SU(2)_L\times SU(2)_R$ global transformations, provided we rotate $W^L_\mu \to g_L W^L_\mu g_L^\dagger$. The 3 would-be NG bosons associated to the $SU(2)_L\times SU(2)_R \to SU(2)_c$ breaking are all eaten by the W^L's, and $SU(2)_c$ ensures that the 3 vector bosons all have equal masses $m_W = gv/2$. When $g'\neq 0$, the custodial symmetry is broken and the 3 massive gauge fields no longer have equal masses. However, the breaking pattern fixes their tree-level masses. From (6.4.66) we immediately see that the gauge bosons along the off-diagonal components of $SU(2)_L$ are unaffected by g' and retain their masses $m_W = gv/2$. On the other hand, along the $U(1)_L\times U(1)_R$ gauged subgroups in the σ_3 directions, the gauge field along the unbroken symmetry remains massless (the photon) and the orthogonal direction (the Z) gets a mass equal to $m_Z = \sqrt{g^2+g'^2}v/2$. The importance of emphasizing the symmetries of the Higgs Lagrangian becomes clear when we replace the Higgs field by an unspecified sector responsible for the electroweak symmetry breaking. In this more general context, we would conclude that the W and Z boson masses are given by eq.(6.4.34), with $\hat\alpha = 1L, 2L, 3L, 3R$. More explicitly, we have

$$g_{aLb} = g\delta_{ab}\,, \qquad g_{aRb} = g'\delta_{ab}\delta_{a3}\,, \tag{6.4.68}$$

where $a,b=1,2,3$ run over the broken $SU(2)$ generators. The crucial point is played by the custodial $SU(2)_c$ symmetry, that forces the F^2_{ab} terms to be proportional to the identity (like $SU(3)_V$ in the QCD example before): $F^2_{ab} = \delta_{ab}F^2$. Putting all together, the gauge boson mass matrix is of the form

$$\mu^2 = F^2 \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & gg' \\ 0 & 0 & gg' & g'^2 \end{pmatrix}. \tag{6.4.69}$$

We automatically recover the SM gauge boson masses, in particular the relation (6.4.64), with the identification $F = 2v$. We conclude by stressing that any sector replacing the usual Higgs in the SM Lagrangian will give the correct leading order electroweak gauge boson masses, provided it includes an $SU(2)_c$ global symmetry.

After this long, but necessary, digression on the SM, let us come back to our idea of the Higgs field as a pseudo NG boson. Assuming this idea, the Higgs should be a

[5] The fermion Yukawa couplings also break $SU(2)_c$, but here for simplicity we are focusing on the bosonic sector of the SM, neglecting fermions altogether.

composite of certain constituents, analogues of quarks and gluons in QCD, that enter at strong coupling at some energy scale. Being the analogue of the pion in QCD, the Higgs particle is expected to be the lightest resonance of the strongly coupled theory, that might include additional heavier resonances (in analogy with the hadron spectroscopy in QCD). Of course, in contrast to the QCD case, we do not know here what is the UV theory that becomes strongly coupled. But we have by now learned that a lot can be said about the dynamics of pseudo NG bosons, by only specifying the group theoretical structure of the spontaneous symmetry breaking pattern. As far as we are concerned, we have to assume that the UV theory, no matter what it is, has an approximate global symmetry group G, spontaneously broken to H, such that the NG bosons along the G/H directions have the quantum numbers of the SM Higgs. Let us denote by f_H the scale where this breaking occurs. The unbroken group H should be large enough to accommodate an $SU(2)_L \times U(1)_Y$ subgroup, that we will gauge and identity with the SM electroweak gauge group. Finally, we might also demand that H includes the custodial symmetry $SU(2)_c$ that ensures the correct tree-level mass ratio (6.4.64) between the SM gauge bosons. The minimal groups that give rise to the 4 Higgs NG bosons and nothing else are

$$
\begin{aligned}
&G = SU(3) \to H = SU(2) \times U(1)\,, \\
&G = SO(5) \to H = SO(4) \cong SU(2)_L \times SU(2)_R\,.
\end{aligned}
\tag{6.4.70}
$$

This is understood by looking at how the adjoint representations of $SU(3)$ and $SO(5)$ decompose under the breaking pattern above. We have

$$
\begin{aligned}
&\mathbf{8} \to \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_{1/2} \oplus \bar{\mathbf{2}}_{-1/2}\,, \\
&\mathbf{10} \to \mathbf{6} \oplus \mathbf{4} = (\mathbf{1},\mathbf{3}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2})\,.
\end{aligned}
\tag{6.4.71}
$$

The subscript in the first line of (6.4.71) refers to the $U(1)$ charge and in the second line we have reported the decomposition in terms of $SO(4)$ and $SU(2)_L \times SU(2)_R$ representations, respectively. The broken generators in the $SU(3)$ case transforms as the last two terms in the first line of (6.4.71), and the corresponding NG bosons form a complex doublet of $SU(2)$. The broken generators in the $SO(5)$ case transforms as the last term in the second line of (6.4.71), and the corresponding NG bosons form a 4-plet of $SO(4)$ or a bidoublet of $SU(2)_L \times SU(2)_R$. Both NG bosons have the quantum numbers of the ordinary Higgs field. However, the $SU(3) \to SU(2)$ case does not include the $SU(2)_c$ custodial symmetry, since the only unbroken $SU(2)$ must be the $SU(2)_L$ group. We then focus on the $SO(5) \to SO(4)$ case in the following.

The $SO(5)$ generators are as in (6.4.60), with the indices a, b, i, j running now from 1 to 5, rather than from 1 to 4. The $SO(4)$ subgroup can be taken to be the one generated by the matrices (6.4.61), with the understanding that they are now 5×5 matrices with an additional row and column of zeros. The remaining $SO(5)/SO(4)$ broken generators are given by

$$t^a = -\frac{i}{\sqrt{2}} t^{a5}, \quad a = 1,2,3,4\,. \tag{6.4.72}$$

The 4 NG bosons h_a are encoded as usual in the matrix

$$\gamma = e^{i\sqrt{2}\frac{h_a t_a}{f_H}}\,, \tag{6.4.73}$$

where the $\sqrt{2}$ factor arises from our choice of normalizations of the $SO(5)$ generators. It is straightforward to verify that G/H is a symmetric coset space, invariant under the automorphism (6.4.36), with $t_i \in SO(4)$ ($i = 1,\ldots,6$) and $t_a \in SO(5)/SO(4)$ ($a = 1,\ldots,4$). The matrix $U = \gamma^2$ transforms homogeneously under $SO(5)$ transformations, as in (6.4.39). For a generic gauging, the covariant derivative for U reads (for non-canonically normalized gauge fields A_μ)

$$D_\mu U = \partial_\mu U + i A_\mu U - i U A_\mu^R\,, \tag{6.4.74}$$

where, for $A_\mu = A_\mu^i t_i + A_\mu^a t_a$, we have $A_\mu^R = A_\mu^i t_i - A_\mu^a t_a$. In our case, the gauging is all within H, so that A_μ^a all vanish and $A_R = A$. The covariant derivative (6.4.74) reduces to (now for canonically normalized gauge fields, including gauge coupling constant factors)

$$D_\mu U = \partial_\mu U + i(g W_\mu^L + g' W_\mu^R)U - iU(g W_\mu^L + g' W_\mu^R)\,, \tag{6.4.75}$$

with

$$W_\mu^L = \sum_{\alpha=1}^{3} t_L^\alpha W_\mu^\alpha\,, \qquad W_\mu^R = t_R^3 B_\mu\,. \tag{6.4.76}$$

The Higgs Lagrangian associated to this "composite Higgs" scenario is

$$\mathcal{L}_H^{CH} = \frac{f_H^2}{4} \mathrm{tr}(D^\mu U)^\dagger D_\mu U\,. \tag{6.4.77}$$

The NG nature of the Higgs forbids a potential for H, in the limit in which the $SO(5)$ global symmetry is exact and only spontaneously broken to $SO(4)$. But like $SU(3)_V$ in the QCD case is broken by $U(1)_{EM}$, the $SO(5)$ symmetry is explicitly broken by the $SU(2)_L \times U(1)_Y$ gauging[6]. This implies that a potential for the Higgs field, even if not included at tree-level, will be generated by radiative effects. We will not elaborate more on this idea, but if suffices to say here that when fermions are also included and coupled to the Higgs matrix field U, electroweak symmetry breaking can in fact be induced at the

[6] $SO(5)$ might also be broken by additional terms, such as the analogues of the quarks masses in QCD. Without knowing the UV theory, we might for simplicity assume that there are no such terms.

quantum level. When $h_4 \neq 0$, a straightforward computation shows that the Lagrangian (6.4.77) gives rise to the following mass terms for the SM gauge fields:

$$m_W^2 = \frac{1}{4} g^2 f_H^2 \sin^2\left(\frac{\langle h_4 \rangle}{f_H}\right), \qquad m_Z^2 = \frac{1}{4}(g^2 + g'^2) f_H^2 \sin^2\left(\frac{\langle h_4 \rangle}{f_H}\right), \quad m_\gamma^2 = 0\,. \qquad (6.4.78)$$

As expected, thanks to the $SU(2)_c$ custodial symmetry, the mass ratio (6.4.64) is reproduced. We see that for $\langle h_4 \rangle / f_H \to 0$ we can expand the sin factor and recover the usual SM formula for the W's and Z, where we identify

$$\langle h_4 \rangle = v \equiv \sqrt{\frac{\sqrt{2}}{2G_F}} \simeq 246 \ \text{GeV}\,. \qquad (6.4.79)$$

This is the limit in which we push to very high energies the $SO(5) \to SO(4)$ breaking pattern, but in so doing we recover the hierarchy problem. On the other hand, various phenomenological bounds constrain the more natural limit $\langle h_4 \rangle \simeq f_H$, so that a little tuning is needed to achieve a mild separation between $\langle h_4 \rangle$ and f_H. Notice that away from the SM limit $f_H \to \infty$, $\langle h_4 \rangle$ does not coincide with v, as defined in (6.4.79). From (6.4.78), for finite f_H, we get

$$v = f_H \sin\left(\frac{\langle h_4 \rangle}{f_H}\right). \qquad (6.4.80)$$

Chapter 7

Anomalies*

There are different ways of regularizing a QFT. The best choice of regulator is the one which keeps the maximum number of symmetries of the classical action unbroken. Cut-off regularization, for instance, breaks gauge invariance and that's why we prefer to work in the somewhat more exotic dimensional regularization, where instead gauge invariance is always manifestly unbroken. It is also possible that there exists *no* regulator that preserves a given classical symmetry. In this case we say that the symmetry is anomalous, namely the quantum theory necessarily breaks it, independently of the choice of regulator.

Roughly speaking, anomalies can affect global or local symmetries. The latter case is particularly important, because local symmetries are needed to decouple unphysical fields. Indeed, anomalies in linearly realized local gauge symmetries lead to inconsistent theories. Theories with anomalous global symmetries are instead consistent, yet the effect of the anomaly can have important effects on the theories. We have already seen an example of anomaly. In classically scale invariant theories, a scale dependence is typically generated by quantum effects by means of a non-vanishing β-function. In this sense, we might say that the massless version of the $\lambda\phi^4$ theory is anomalous, since $\beta_\lambda \neq 0$. Most of the time, however, the word "anomaly" is used for global anomalies associated to chiral currents and their related anomalies in local symmetries. Historically, the first anomaly, discovered by Adler, Bell and Jackiw, was associated to the non-conservation of the axial current in QCD. Among other things, the axial anomaly resolved a puzzle related to the $\pi^0 \to 2\gamma$ decay rate, predicted by effective Lagrangian considerations to be about three orders of magnitude smaller than the observed one.

In the next section we will study the basic anomaly associated to a global $U(1)$ chiral transformation (using the so-called Fujikawa approach) and then consider anomalies in local symmetries as derived by a perturbative one-loop diagram computation.

7.1 Path integral derivation of the chiral anomaly*

In the path-integral formulation of field theory, anomalies arise from the transformation of the measure used to define the fermion path integral. We will consider in what follows path-integral in euclidean space-time.

Let $\psi_A(x)$ be a massless Dirac fermion defined on R^4, in an arbitrary representation $\mathcal{R}$ of a gauge group G ($A = 1, \ldots, \dim \mathcal{R}$). The minimal coupling of the fermion to the gauge fields is described by the Lagrangian

$$\mathcal{L} = \bar{\psi}(x)_A i\gamma^\mu D_{\mu\,B}^{A}\,\psi^B(x)\,. \tag{7.1.1}$$

The covariant derivative is given by

$$D_{\mu\,B}^{A} = \partial_\mu \delta^A_B + A^\alpha_\mu\, T^{A}_{\alpha\,B}\,, \tag{7.1.2}$$

in terms of the gauge connection A^α_μ and the anti-Hermitian generators $T^{A}_{\alpha\,B}$ of the group G in the representation $\mathcal{R}$ ($\alpha = 1, \ldots, \dim G$), with γ^μ satisfying the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ ($\mu, \nu = 1, 2, 3, 4$).

The classical Lagrangian (7.1.1) is invariant under the global chiral transformation

$$\psi \to e^{i\alpha\gamma_5}\psi, \tag{7.1.3}$$

where $\gamma_5 = \prod_{\mu=1}^4 \gamma_\mu$, normalized so that $\gamma_5^2 = I$, and α is a constant parameter. The chiral current

$$J^\mu_A = \bar{\psi}_A \gamma^\mu \gamma_5 \psi^A \tag{7.1.4}$$

is classically conserved. At the quantum level, however, this conservation law can be violated and turned into an anomalous Ward identity. To derive it, we consider the quantum effective action Γ defined by

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int d^4x\,\mathcal{L}}\,, \tag{7.1.5}$$

and study its behavior under an infinitesimal chiral transformation of the fermions, with a space-time-dependent parameter $\alpha(x)$, given by[1]

$$\delta_\alpha \psi = i\alpha\gamma_5\psi\,, \quad \delta_\alpha\bar{\psi} = i\alpha\bar{\psi}\gamma_5\,. \tag{7.1.6}$$

Since the external gauge fields A are inert, the transformation (7.1.6) represents a redefinition of dummy integration variables, and should not affect the effective action: $\delta_\alpha\Gamma = 0$.

[1] For simplicity of the notation, we omit the gauge index A in the following equations. It will be reintroduced later on in this section.

This statement carries however a non-trivial piece of information, since neither the classical action nor the integration measure is invariant under (7.1.6). The variation of the classical action under (7.1.6) is non-vanishing only for non-constant α, and has the form $\delta_\alpha \int \mathcal{L} = \int J_5^\mu \partial_\mu \alpha$. The variation of the measure is instead always non-vanishing, because the transformation (7.1.6) leads to a non-trivial Jacobian factor, which has the form $\delta_\alpha[\mathcal{D}\psi\mathcal{D}\bar{\psi}] = \exp\{i\int \alpha\mathcal{A}\}$, as we will see below. In total, the effective action therefore transforms as

$$\delta_\alpha \Gamma = \int d^4x\, \alpha(x) \Big[i\mathcal{A}(x) + \langle \partial_\mu J_5^\mu(x) \rangle \Big] \,. \tag{7.1.7}$$

The condition $\delta_\alpha \Gamma = 0$ then implies the anomalous Ward identity:

$$\langle \partial_\mu J_5^\mu \rangle = -i\mathcal{A} \,. \tag{7.1.8}$$

In order to compute the anomaly $\mathcal{A}$, we need to define the integration measure more precisely. This is best done by considering the eigenfunctions of the Dirac operator $i\not{D} \equiv i\gamma^\mu D_\mu$. Since the latter is Hermitian, the set of its eigenfunctions $\psi_k(x)$ with eigenvalues λ_k, defined by $i\not{D}\psi_n = \lambda_n \psi_n$, form an orthonormal and complete basis of spinor modes:

$$\int d^4x\, \psi_k^\dagger(x)\psi_l(x) = \delta_{k,l} \,, \quad \sum_k \psi_k^\dagger(x)\psi_k(y) = \delta^{(4)}(x-y) \,. \tag{7.1.9}$$

The fermion fields ψ and $\bar{\psi}$, which are independent from each other in Euclidean space, can be decomposed as

$$\psi = \sum_k a_k \psi_k \,, \quad \bar{\psi} = \sum_k \bar{b}_k \psi_k^\dagger \,, \tag{7.1.10}$$

so that the measure becomes

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_{k,l} da_k d\bar{b}_l \,. \tag{7.1.11}$$

Under the chiral transformation (7.1.6), we have

$$\delta_\alpha a_k = i\int d^4x \sum_l \psi_k^\dagger \alpha\gamma_5 \psi_l a_l \,, \quad \delta_\alpha \bar{b}_k = i\int d^4x \sum_l \bar{b}_l \psi_l^\dagger \alpha\gamma_5 \psi_k \,, \tag{7.1.12}$$

and the measure (7.1.11) transforms as

$$\begin{aligned} \delta_\alpha \Big[\mathcal{D}\psi\mathcal{D}\bar{\psi}\Big] &= \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \det(\delta_{kl} + i\psi_k^\dagger \alpha\gamma_5\psi_l)^{-2} = \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \exp\Big[-2\mathrm{tr}\log(\delta_{kl} + i\psi_k^\dagger\alpha\gamma_5\psi_l)\Big] \\ &= \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\Big\{-2i\sum_k \int d^4x\, \psi_k^\dagger\alpha\gamma_5\psi_k + \mathcal{O}(\alpha^2)\Big\} \,. \end{aligned} \tag{7.1.13}$$

For simplicity we can take α to be constant. This formal expression is ill-defined as it stands, since it decomposes into a vanishing trace over spinor indices ($\mathrm{tr}\,\gamma_5 = 0$) times an infinite sum over the modes ($\sum_k 1 = \infty$). A convenient way of regularizing it is to introduce a gauge-invariant Gaussian cut-off. The anomaly $\mathcal{A}$ can then be defined as

$$\mathcal{A} = -2 \lim_{\beta\to 0} \sum_k \psi_k^\dagger \gamma_5 \psi_k e^{-\beta\lambda_k^2} = -2 \lim_{\beta\to 0} \mathrm{Tr}\left[\gamma_5 e^{-\beta(i\not{D})^2}\right], \tag{7.1.14}$$

where the trace has to be taken over the mode and the spinor indices, as well as over the gauge indices. Using the completeness relation (7.1.9), we can write

$$\mathcal{A} = -2 \lim_{\beta\to 0} \lim_{y\to x} \mathrm{Tr}\Big[\gamma_5 e^{-\beta(i\not{D})^2}\Big]\delta^{(4)}(x-y) = -2 \lim_{\beta\to 0} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\Big[\gamma_5 e^{-\beta(\not{k}+i\not{D})^2}\Big]. \tag{7.1.15}$$

By using the commutator properties of the γ matrices, we can rewrite

$$(\not{k} + i\not{D})^2 = (k+iD)^2 - \frac{1}{4} F^{\mu\nu}[\gamma_\mu, \gamma_\nu]. \tag{7.1.16}$$

Rescaling the momentum $k \to k/\sqrt{\beta}$, we get

$$\mathcal{A} = -2 \lim_{\beta\to 0} \frac{1}{\beta^2} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\, \gamma_5 e^{-(k^2 + 2i\sqrt{\beta} D\cdot k - \beta D^2 - \beta F)}, \tag{7.1.17}$$

where $F \equiv F^{\mu\nu}[\gamma_\mu, \gamma_\nu]/4$. The trace over spinor indices is vanishing unless at least two factors of F (i.e. 4 γ's) are included in the trace. In this way, we get two powers of β that compensate for the overall factor $1/\beta^2$ in eq.(7.1.17). Hence, in the limit $\beta \to 0$, we can safely neglect the terms proportional to $D \cdot k$ and D^2 in the exponential. In this way, we finally get

$$\mathcal{A} = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F^\alpha_{\mu\nu} F^\beta_{\rho\sigma} \mathrm{tr}(T_\alpha T_\beta). \tag{7.1.18}$$

In Minkowski space, with $\partial_4 = -i\partial_0$, and in terms of hermitian (rather than anti-hermitian) generators and canonically normalized gauge fields ($AT \to -igAT$) the non-conservation of the axial current takes the form

$$\partial_\mu J_5^\mu = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F^\alpha_{\mu\nu} F^\beta_{\rho\sigma} \mathrm{tr}(T_\alpha T_\beta). \tag{7.1.19}$$

Anomalies in the chiral transformation (7.1.3) occur in any number of even space-time dimensions (in odd-dimensional space-times, chirality is not defined, since there is no analogue of the matrix γ_5). The derivation of the anomaly reviewed here, due to Fujikawa, is easily generalized to any number of dimensions. In $2n$ dimensions, we define the chiral matrix γ_{2n+1} as $\gamma_{2n+1} = i^n \prod_{\mu=1}^{2n} \gamma_\mu$ so that $\gamma_{2n+1}^2 = I$ for any n. All the steps are essentially identical to the 4d case and we get the generalization of eq.(7.1.17):

$$\mathcal{A} = 2 \lim_{\beta\to 0} \frac{1}{\beta^n} \int \frac{d^{2n}k}{(2\pi)^{2n}} \mathrm{Tr}\, \gamma_{2n+1} e^{-(k^2 + 2i\sqrt{\beta} D\cdot k - \beta D^2 - i\beta F)}. \tag{7.1.20}$$

In $2n$ dimensions, spinors have 2^n components, so we get for the anomaly

$$\mathcal{A} = -\frac{2}{(4\pi)^n n!}\epsilon^{\mu_1\cdots\mu_{2n}} F^{\alpha_1}_{\mu_1\mu_2}\ldots F^{\alpha_n}_{\mu_{2n-1}\mu_{2n}} \mathrm{tr}(T_{\alpha_1}\ldots T_{\alpha_n})\,. \tag{7.1.21}$$

The anomaly (7.1.21) can more elegantly be rewritten in a compact form in terms of differential two-forms. Let us introduce auxiliary anticommuting variables $\phi^1,\ldots,\phi^{2n}$ and define

$$F = \frac{1}{2}F_{\mu\nu}\phi^\mu \wedge \phi^\nu\,. \tag{7.1.22}$$

Then eq.(7.1.21) becomes simply

$$\mathcal{A} = -2\int d^{2n}\phi\,\mathrm{tr}\,e^{F/(2\pi)}\,, \tag{7.1.23}$$

where the integration over the auxiliary fermion variables ϕ^μ automatically selects the correct number of field strengths F. In evaluating the integrated form of the anomaly, $\int d^{2n}x\mathcal{A}$, we can replace the auxiliary fermion variables ϕ by the differentials dx^μ, so that we have

$$\int d^{2n}x\,\mathcal{A} = -2\int d^{2n}x\,\mathrm{ch}(F)\,, \tag{7.1.24}$$

where

$$\mathrm{ch}(F) = \mathrm{tr}\,e^{F/(2\pi)} \tag{7.1.25}$$

is the so-called Chern character of the gauge connection. There is a deep connection between anomalies and certain mathematical results that will not be discussed here.

7.2 Anomalies from one-loop graphs*

Anomalies were originally discovered by evaluating three-point functions between external currents at one-loop level. It is clear from the previous path integral derivation that anomalies are expected from loops of internal fermion lines from current correlations that involve the chiral matrix γ_5. We are then led to compute the three point functions between two vector and one axial current:

$$\Gamma_{\mu\nu\rho}(k_1,k_2) \equiv \langle J^A_\mu(-k_1-k_2)J_\nu(k_1)J_\rho(k_2)\rangle \tag{7.2.1}$$

where $J^A_\mu = \bar\psi\gamma_\mu\gamma_5\psi$, $J_\nu = \bar\psi\gamma_\nu\psi$, $J_\rho = \bar\psi\gamma_\rho\psi$. For simplicity, we will consider massless fermions. In this case, the classical conservation of the axial and vector currents would imply

$$(k_1+k_2)^\mu\Gamma_{\mu\nu\rho} = k_1^\nu\Gamma_{\mu\nu\rho} = k_2^\rho\Gamma_{\mu\nu\rho} = 0\,. \tag{7.2.2}$$

As we will see, due to the anomaly, it turns out to be impossible to impose all three conditions (7.2.2) simultaneously.

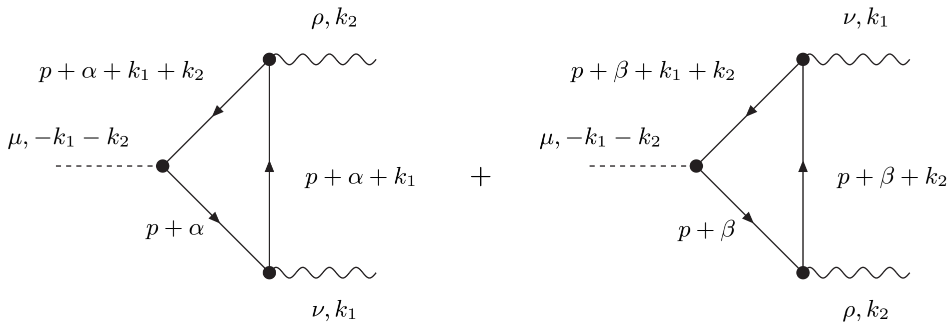


Figure 7.1: One-loop graphs contributing to the anomaly. All external momenta are incoming. The wavy and dashed lines represent the vector and axial currents, respectively.

The two diagrams in fig.7.1 contribute at one-loop level to $\Gamma_{\mu\nu\rho}$. They give

$$\Gamma_{\mu\nu\rho}(k_1,k_2) = -\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\Big(\gamma_\mu\gamma_5 S(p+\alpha+k)\gamma_\rho S(p+\alpha+k_1)\gamma_\nu S(p+\alpha) + \gamma_\mu\gamma_5 S(p+\beta+k)\gamma_\nu S(p+\beta+k_2)\gamma_\rho S(p+\beta)\Big)\,, \quad (7.2.3)$$

where $k = k_1 + k_2$ and $S(p) = \not{p}/p^2$ is the fermion propagator. In eq.(7.2.3) α and β are arbitrary constants, whose relevance will be clear in the following. Let us first compute the divergence of the axial current, $k^\mu\Gamma_{\mu\nu\rho}$. It is convenient to write

$$S(p+\alpha)\not{k}\,S(p+\alpha+k) = S(p+\alpha)(\not{p}+\not{\alpha}+\not{k}-\not{p}-\not{\alpha})S(p+\alpha+k) = S(p+\alpha)-S(p+\alpha+k)\,, \quad (7.2.4)$$

so that

$$k^\mu\Gamma_{\mu\nu\rho}(k_1,k_2) = \int\frac{d^4p}{(2\pi)^4}\Big[f_{\rho\nu}(p,\beta+k_2,\beta+k) - f_{\rho\nu}(p,\alpha,\alpha+k_1) + (\rho\leftrightarrow\nu,\alpha\leftrightarrow\beta,k_1\leftrightarrow k_2)\Big] \quad (7.2.5)$$

where

$$f_{\rho\nu}(p,a,b) \equiv \mathrm{tr}\Big(S(p+a)\gamma_\rho S(p+b)\gamma_\nu\gamma_5\Big) = 4i\epsilon_{\gamma\rho\omega\nu}\frac{p_\omega(a-b)_\gamma + a_\gamma b_\omega}{(p+a)^2(p+b)^2}\,. \quad (7.2.6)$$

Noticing that $f_{\rho\nu}(p+c,a,b) = f_{\rho\nu}(p,a+c,b+c)$, the second term in eq.(7.2.5) becomes identical to the first one by shifting the virtual momentum $p \to p+\beta-\alpha+k_2$, giving naively $k^\mu\Gamma_{\mu\nu\rho} = 0$. However, these expressions are divergent and momentum shift is not allowed. Indeed, for a generic function $f(x)$, we have

$$\int_{-\infty}^{\infty}dx\Big[f(x+a)-f(x)\Big] = \int_{-\infty}^{\infty}dx\Big[af'(x)+\dots\Big] = a\Big[f(\infty)-f(-\infty)\Big]+\dots \quad (7.2.7)$$

Equation (7.2.7) vanishes if $f(\pm\infty) = f'(\pm\infty) = \ldots = 0$, conditions automatically satisfied for convergent integrals, but not for divergent ones. In the case at hand, the analogue of f is $p^3 f_{\rho\nu}$ and $f_{\rho\nu} \sim 1/p^3$, so $f(\pm\infty) \neq 0$. The integral is governed by the asymptotic behaviour of f and thus it is easy to evaluate. Rotating the virtual momentum p to euclidean values and applying Stokes theorem, we have

$$
\begin{aligned}
i\int\frac{d^4p_E}{(2\pi)^4}\Big[f_{\rho\nu}(p_E+k,a,b) - f_{\rho\nu}(p_E,a,b)\Big] &= \frac{i\Omega_4}{(2\pi)^4}\lim_{p_E\to\infty} k^\mu p_{E,\mu} p_E^2 f_{\rho\nu}(p_E) \\
&= \frac{1}{8\pi^2}\epsilon_{\gamma\rho\omega\nu}k^\omega(a-b)^\gamma \qquad (7.2.8)
\end{aligned}
$$

where $\Omega_4 = 2\pi^2$ is the volume of the unit four-sphere and we have used the $SO(4)$ invariance to replace $p_{E,\mu}p_{E,\omega}$ by $-\delta_{\mu\omega}p_E^2/4$. Using eq.(7.2.8), the divergence of the axial current is easily computed:

$$
\begin{aligned}
k^\mu\Gamma_{\mu\nu\rho}(k_1,k_2) &= \frac{1}{8\pi^2}\epsilon_{\gamma\rho\omega\nu}\Big[(k_2+\beta-\alpha)^\omega(-k_1)^\gamma + (k_1+\alpha-\beta)^\omega k_2^\gamma\Big] \\
&= \frac{1}{8\pi^2}\epsilon_{\gamma\rho\omega\nu}(k_1-k_2+\Delta)^\omega(k_1+k_2)^\gamma\,, \qquad (7.2.9)
\end{aligned}
$$

with $\Delta \equiv \alpha - \beta$. It is now clear why we have introduced the otherwise redundant parameters α and β. In a convergent expression, they would trivially be reabsorbed in a shift of the virtual momentum p, but in the case at hand the final expression turns out to depend on their difference Δ. Being Δ arbitrary, one could choose $\Delta = k_2 - k_1$, so that the "anomalous" term (7.2.9) would cancel. However, care has to be paid on the divergence of the two vector currents J_ν and J_ρ. Proceeding exactly as before, we can compute

$$
\begin{aligned}
k_1^\nu\Gamma_{\mu\nu\rho}(k_1,k_2) &= \frac{1}{8\pi^2}\epsilon_{\gamma\mu\omega\rho}\Big[(k_1+k_2)^\gamma(\beta-\alpha)^\omega - (\beta-\alpha-k_1)^\omega k_2^\gamma\Big] \\
&= \frac{1}{8\pi^2}\epsilon_{\gamma\mu\omega\rho}k_1^\omega(\Delta+k_2)^\gamma\,, \qquad (7.2.10) \\
k_2^\rho\Gamma_{\mu\nu\rho}(k_1,k_2) &= \frac{1}{8\pi^2}\epsilon_{\gamma\mu\omega\nu}\Big[(k_1+k_2)^\gamma(\alpha-\beta)^\omega - (\alpha-\beta-k_2)^\omega k_1^\gamma\Big] \\
&= \frac{1}{8\pi^2}\epsilon_{\gamma\mu\omega\nu}k_2^\gamma(\Delta-k_1)^\omega\,. \qquad (7.2.11)
\end{aligned}
$$

The choice $\Delta = k_2 - k_1$ would then lead to the non-conservation of the two vector currents J_ν and J_ρ. Since three-point functions between 3 vector currents do not lead to any anomaly, we insist in having vanishing divergence for the vector currents. This uniquely fixes $\Delta = k_1 - k_2$, opposite to the choice leading to the conservation of the axial current. Plugging this value in eq.(7.2.9), we get

$$
k^\mu\Gamma_{\mu\nu\rho}(k_1,k_2) = \frac{1}{2\pi^2}\epsilon_{\gamma\rho\omega\nu}k_1^\omega k_2^\gamma\,. \qquad (7.2.12)
$$

We see that by changing Δ we can shift the anomalous term from one current to another, but what is important is that there is *no* choice of Δ for which all three currents are conserved.

Notice that no regulator has been chosen to evaluate the one-loop graphs 7.1, contrary to the usual prescription of first regularizing and then renormalizing the amplitudes. Despite the one-loop graphs, being divergent, require a regulator, their divergence (i.e. the anomaly) is well defined and finite, as we have just seen. In any given regularization, where all amplitudes are made finite and the shift in the virtual momentum allowed, the anomaly would appear differently. For instance, in dimensional regularization, the anomaly arises from subtleties related to the definition of γ_5 in $d \neq 4$ dimensions (see i.e. [1] for a computation of the anomaly in dimensional regularization). In Pauli-Villars regularization, where a heavy (PV) fermion is added, the anomaly is related to the explicit breaking of the axial symmetry given by the PV fermion mass term. There is no regulator that preserves at the same time the vector and axial symmetries and as a result an anomaly always appears.

It is not difficult to see that eq.(7.1.19) is consistent with eq.(7.2.12) for a single fermion ($T_\alpha = 1$). Taking the Fourier transform of eq.(7.2.12), we have

$$\begin{aligned} & \frac{\delta^2}{g\delta A^\nu(-k_1) g\delta A^\rho(-k_2)} \langle i k^\mu J_\mu^A \rangle = i k^\mu \langle J_\mu^A(-k_1-k_2) J_\nu(k_1) J_\rho(k_2) \rangle = i k^\mu \Gamma_{\mu\nu\rho}(k_1,k_2) \\ & = -\frac{1}{16\pi^2} \epsilon_{\alpha\beta\gamma\delta}(-4) \int d^4p \frac{\delta^2}{\delta A^\nu(-k_1)\delta A^\rho(-k_2)} \Big[p^\alpha A^\beta(p)(k-p)^\gamma A^\delta(k-p) \Big] \\ & = \frac{1}{2\pi^2} \epsilon_{\gamma\rho\omega\nu} k_1^\omega k_2^\gamma \, , \end{aligned} \tag{7.2.13}$$

where we have omitted the delta function $\delta(k-k_1-k_2)$ indicating the energy-momentum conservation.

It can be shown that in $2n$ space-time dimensions the anomaly (7.1.21) is reproduced by one-loop diagrams involving $n+1$ currents with internal fermion lines running in the loop.

7.3 Gauge Anomalies*

As we have already mentioned, anomalies can also affect local symmetries, in which case we refer to them as "gauge anomalies". The latter can arise in chiral, parity non-invariant, gauge theories, where left and right-moving fermions transform in representations of the gauge group that are not complex conjugate with each other. It is convenient to consider all fermions as left-handed by defining, for each right-handed component ψ_R its left-handed counterpart

$$\psi_L^c = C\psi_R^\star\,, \tag{7.3.1}$$

C being the matrix of complex conjugation in spinor space. If ψ_R transforms under a representation r_R under a (generically non-abelian) gauge group G, ψ_L^c will transform as $r_R^\star$. Let n be the total number of left-handed fermions in the theory, sum of the ones obtained by means of eq.(7.3.1) with the already existing left-handed fermions. We collect all such n fermions in a single left-handed fermion ψ^A, with $A=(r_1,\dots,r_n)$ and r_i the representation under the gauge group G of the various components. As in section 7.2, we evaluate the 3-point function at one-loop level between three currents:

$$\Gamma^{abc}_{\mu\nu\rho}(k_1,k_2)\equiv\langle J^a_\mu(-k_1-k_2)J^b_\nu(k_1)J^c_\rho(k_2)\rangle \tag{7.3.2}$$

where, in a four-component notation,

$$J^a_\mu=\bar\psi\gamma_\mu P_L T^a\psi, \tag{7.3.3}$$

with $P_L=(1+\gamma_5)/2$. Similarly to eq.(7.2.3), we have

$$\begin{aligned}\Gamma^{abc}_{\mu\nu\rho}(k_1,k_2) = -\int\frac{d^4p}{(2\pi)^4}\Bigg[&\mathrm{tr}(T^aT^cT^b)\mathrm{tr}\Big(\gamma_\mu P_L S(p+\alpha+k)\gamma_\rho P_L S(p+\alpha+k_1)\gamma_\nu P_L S(p+\alpha)\\ &+\mathrm{tr}(T^aT^bT^c)\gamma_\mu P_L S(p+\beta+k)\gamma_\nu P_L S(p+\beta+k_2)\gamma_\rho P_L S(p+\beta)\Big)\Bigg].\end{aligned} \tag{7.3.4}$$

Since the currents transform under a symmetry transformation, their classical conservation would imply the identities

$$\begin{aligned}(k_1+k_2)^\mu\Gamma^{abc}_{\mu\nu\rho} &= Nf^{abc}\Big(\Gamma_{\nu\rho}(k_1)-\Gamma_{\nu\rho}(-k_2)\Big)\,,\\ k_1^\nu\Gamma^{abc}_{\mu\nu\rho} &= Nf^{abc}\Big(\Gamma_{\rho\mu}(-k_2)-\Gamma_{\rho\mu}(-k)\Big)\,,\\ k_2^\rho\Gamma_{\mu\nu\rho} &= Nf^{abc}\Big(\Gamma_{\mu\nu}(k)-\Gamma_{\mu\nu}(k_1)\Big)\,,\end{aligned} \tag{7.3.5}$$

where

$$\langle J^a_\mu(k)J^b_\nu(-k)\rangle = N\delta^{ab}\Gamma_{\mu\nu}(k)\,. \tag{7.3.6}$$

It is convenient to write

$$\mathrm{tr}(T^aT^bT^c)=\frac{1}{2}\mathrm{tr}(\{T^a,T^b\}T^c)+\frac{1}{2}\mathrm{tr}([T^a,T^b]T^c)=D^{abc}+\frac{i}{2}f^{abc}N \tag{7.3.7}$$

with $D^{abc}=\mathrm{tr}(\{T^a,T^b\}T^c)/2$ and $\mathrm{tr}(T^cT^d)=N\delta^{cd}$ with N a constant depending on the number and kind of fermion representations. It is possible to show that the terms proportional to the structure constants f in eq.(7.3.4) do not lead to any anomaly and just reproduce the terms in the right-hand side of eq.(7.3.5). Similarly, it is not difficult to show that there is a unique choice of β as a function of α (or viceversa, of course) such that all the terms in D^{abc} that do not involve γ_5 vanish for all the three current conservations. The anomaly is finally found in the terms proportional to D^{abc} involving γ_5. These are eventually identical to the ones we already computed:

$$\Gamma^{abc}_{\mu\nu\rho}(k_1,k_2)|_{ano.} = \frac{1}{2} D^{abc}\Gamma_{\mu\nu\rho}(k_1,k_2)\,. \tag{7.3.8}$$

The divergencies of the gauge currents is then related to the divergencies of the associated abelian global currents defined in the last section. Without further computations, we know there is no choice of $\Delta(\alpha)$ such that all three currents are conserved, unless the coefficients D^{abc} vanish. An anomaly in a gauge current is deadly for a theory, because it does not allow anymore the decoupling of unphysical states. This is best understood considering abelian gauge theories, where $f_{abc}=0$ and thus the currents J are exactly conserved. Photon scattering amplitudes will receive from the triangular fermion loops a contribution proportional to $\epsilon^\mu(k)\epsilon^\nu(k_1)\epsilon^\rho(k_2)\Gamma_{\mu\nu\rho}(k_1,k_2)$, where ϵ are the polarizations of the external photons[2]. The computation of the divergence of one of the external current is equivalent to replace the photon polarization vector by its momentum, i.e. considering longitudinal photons. In presence of an anomaly, the gauge current is no longer conserved, the Ward-Identities are not conserved and hence unphysical, longitudinal photons, cannot be decoupled. The theory is not unitary (at any scale) and thus inconsistent. Similar considerations are valid for non-abelian gauge theories.

If one of the external currents, say J^a_μ, is the current of a global symmetry, we choose $\Delta = k_2 - k_1$ so that the remaining gauge currents J^b_ν and J^c_ρ are conserved, Proceeding as in eq.(7.2.13), we get

$$\langle \partial_\mu J^\mu_a \rangle_{ano.} = -\frac{g^2}{8\pi^2} D_{abc}\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^b_\nu \partial_\rho A^c_\sigma\,. \tag{7.3.9}$$

In the abelian case, with $T^a=1$, eq.(7.3.9) reduces to eq.(7.1.19), with an additional 1/2 factor. In the non-abelian case, additional one-loop (square and pentagon) diagrams contribute to the anomaly and should be considered. When they are summed to eq.(7.3.9), the whole non-abelian form of the field strength is reconstructed and we get the final result (for $T^a=1$)

$$\langle \partial_\mu J^\mu_a \rangle = -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma} \mathrm{tr}(T^b T^c)\,. \tag{7.3.10}$$

If all the external currents are gauge currents, we should choose Δ so that the anomaly is symmetric under the exchange of any of the two currents, as required by Bose symmetry.

[2]Strictly speaking, when the photons are outcoming, ϵ should be replaced by its complex conjugate, but this is irrelevant for our considerations.

This is achieved by taking

$$\Delta = -\frac{1}{3}(k_1 - k_2)\,, \tag{7.3.11}$$

which gives an anomaly one-third smaller than the anomaly (7.3.9) since it is equally distributed among the three external currents. Including anomalous square and pentagon graphs, one gets

$$\langle D_\mu J_a^\mu \rangle = -\frac{g^2}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\, T_a\Big[\partial_\mu A_\nu \partial_\rho A_\sigma - \frac{ig}{2}\big(\partial_\mu A_\nu A_\rho A_\sigma - A_\mu \partial_\nu A_\rho A_\sigma + A_\mu A_\nu \partial_\rho A_\sigma\big)\Big]\,. \tag{7.3.12}$$

Notice that the non-abelian anomaly (7.3.12) cannot be written in a covariant way in terms of field strengths only. By a suitable redefinition of the gauge current J_μ, we might actually rewrite the anomaly in terms of F's only ("covariant anomaly"), but the resulting anomaly would not satisfy an important consistency condition we will briefly discuss later. For this reason the non-covariant anomaly (7.3.12) is sometimes denoted "consistent anomaly". It has been shown by Adler and Bardeen that no additional contributions to the anomaly arise from higher loops to all orders of perturbation theory.

So far, we have only considered massless fermions. Fermion mass terms explicitly break the axial symmetry (7.1.3), but it can easily be shown that they do not change the form of the axial anomaly (7.1.21). In other words, any fermion in a theory, massless or massive, equally contribute to the axial anomaly. The situation is drastically different for gauge anomalies. The latter can receive a non-vanishing contribution only from massless fermions. In our basis of left-handed fields, this implies that any pair of fermions that can mix through a mass term give equal and opposite contribution to the anomaly. This is proved as follows. In terms of two-components spinors χ_i, a generic mass term reads

$$\mathcal{L}_m = \chi_i^t \sigma^2 \hat{M}_{ij}\chi_j + h.c. \tag{7.3.13}$$

where i run over all the fermions in the theory and $\hat{M}$ is a symmetric mass matrix. Let us focus on a subset of terms in eq.(7.3.13) coupling two left-handed fermion multiplets χ_1 and χ_2 in irreducible representations r_1 and r_2 of the group G, with $\mathrm{dim} r_1 = \mathrm{dim} r_2$:

$$\mathcal{L}_m \supset \chi_1^t \sigma^2 M \chi_2 + h.c. \tag{7.3.14}$$

where M is a non-singular mass matrix. The term (7.3.14) is gauge-invariant if

$$-\,T_1^t M = M T_2\,, \tag{7.3.15}$$

where T_1 and T_2 are the generators in the representations r_1 and r_2. Eq.(7.3.15) implies that $-T_1^t$ and T_2 are related by a similarity transformation, since $-T_1^t = M T_2 M^{-1}$. Then

$$D^1_{abc} = \frac{1}{2}\mathrm{tr}\{T_{1a}, T_{1b}\}T_{1c} = (-1)^3\frac{1}{2}\mathrm{tr}\{T^t_{2a}, T^t_{2b}\}T^t_{2c} = -D^2_{abc}\,. \tag{7.3.16}$$

The contribution to the gauge anomaly given by χ_1 is exactly cancelled by that of χ_2. Let us clarify this result with a couple of simple examples. Consider a $U(1)$ gauge theory with a Dirac fermion with charge q. In terms of left-handed fields, the Dirac fermion consists of one left-handed fermion χ_1 with charge q and its conjugate χ_2 with charge $-q$ and obviously admit the mass term $m\chi_1\sigma^2\chi_2 + h.c.$. The total gauge anomaly is proportional to

$$+\,q^3 + (-q)^3 = 0\,. \tag{7.3.17}$$

Similarly, Dirac fermions in any representation T of a gauge group consists of one left-handed fermion multiplet in the representation T and its conjugate in the complex conjugate representation $-T^\star = -T^t$ and thus do not lead to anomalies. In these simple cases of manifestly parity-invariant theories the absence of any anomaly is pretty obvious since all currents are manifestly vector-like (i.e no γ_5 appears in a 4-component Dirac notation). Non-trivial anomalies can only arise in non-parity-invariant theories, namely in so called chiral gauge theories, where a fermion and its complex conjugate transform in representations of the gauge group that are not complex conjugate between each other. The absence of anomalies in chiral gauge theories require a non-trivial cancellation between different fermion multiplets. The most important example of a theory of this sort is the SM, where gauge anomalies cancel between quarks and leptons, as we will see in the next section.

7.4 A Relevant Example: Cancellation of Gauge Anomalies in the SM*

The SM gauge group is $G_{SM} = SU(3)\times SU(2)\times U(1)$. Its fermion content, in terms of left-handed fields, is composed by three copies (generations) of the following representations:

$$(\mathbf{3},\mathbf{2})_{\frac{1}{6}} + (\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} + (\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} + (\mathbf{1},\mathbf{2})_{-\frac{1}{2}} + (\mathbf{1},\mathbf{1})_1 \tag{7.4.1}$$

corresponding to the quark doublet, up quark singlet, down quark singlet, lepton doublet and charged lepton singlet, respectively. In principle there can be ten possible kinds of gauge anomalies, associated to all possible combinations of $SU(3)$ $SU(2)$ and $U(1)$ currents in the trangular graph. Five of them, where a non-abelian group factor ($SU(3)$ or $SU(2)$) appears only once, $SU(3)^2\times SU(2)$, $SU(3)\times SU(2)^2$, $SU(3)\times SU(2)\times U(1)$, $SU(3)\times U(1)^2$ and $SU(2)\times U(1)^2$ are trivially vanishing, since for $SU(n)$ groups the generators are traceless: $\mathrm{tr}\,T = 0$. The $SU(2)^3$ anomaly is also manifestly vanishing because for $SU(2)$ the symmetric factor D^{abc} vanishes. In the case at hand, with doublets only, this is easily seen: $D^{abc} = 1/2\mathrm{tr}\{t^a,t^b\}t^c = \delta^{ab}\mathrm{tr}\,t^c = 0$. In the general case, D^{abc} vanishes because all $SU(2)$ representations are equivalent to their complex conjugates, namely there exists a matrix A such that $T^t = T^\star = -ATA^{-1}$. Using this relation, one immediately sees that $D^{abc} = 0$ for any representation.

The remaining combinations $SU(3)^3$, $SU(3)^2 \times U(1)$, $SU(2)^2 \times U(1)$ and $U(1)^3$ have to be checked. Let us then compute the values of the symmetric coefficients D^{abc} in each of the above 4 cases and show that they always vanish. In order to distinguish the different coefficients, we denote by a subscript c, w and Y the $SU(3)$, $SU(2)$ and $U(1)$ factors, respectively. It is enough to consider a single generation of quarks and doublets, because the cancellation occurs generation per generation. Let us start with the $SU(3)^3$ anomaly. Only quarks contribute to it. We get

$$D^{abc}_{ccc} = 2D^{abc}_{\mathbf{3}} + D^{abc}_{\bar{\mathbf{3}}} + D^{abc}_{\bar{\mathbf{3}}} = 2D^{abc}_{\mathbf{3}} - D^{abc}_{\mathbf{3}} - D^{abc}_{\mathbf{3}} = 0\,, \tag{7.4.2}$$

using that $D^{abc}_{\bar{\mathbf{3}}} = -D^{abc}_{\mathbf{3}}$. For $SU(3)^2 \times U(1)$ we have

$$D^{ab}_{ccY} = 2\mathrm{tr}_{\mathbf{3}}\, t^a t^b \times \frac{1}{6} + \mathrm{tr}_{\bar{\mathbf{3}}}\, t^a t^b \times \Big(-\frac{2}{3}\Big) + \mathrm{tr}_{\bar{\mathbf{3}}}\, t^a t^b \times \frac{1}{3} = \mathrm{tr}_{\mathbf{3}}\, t^a t^b \Big(\frac{1}{3} - \frac{2}{3} + \frac{1}{3}\Big) = 0\,, \tag{7.4.3}$$

with $\mathrm{tr}_{\mathbf{3}}\, t^a t^b = \mathrm{tr}_{\bar{\mathbf{3}}}\, t^a t^b$. For $SU(2)^2 \times U(1)$, only doublets contribute. We get

$$D^{ab}_{wwY} = 3\mathrm{tr}_{\mathbf{2}}\, t^a t^b \times \frac{1}{6} + \mathrm{tr}_{\mathbf{2}}\, t^a t^b \times \Big(-\frac{1}{2}\Big) = \mathrm{tr}_{\mathbf{2}}\, t^a t^b \Big(\frac{1}{2} - \frac{1}{2}\Big) = 0\,. \tag{7.4.4}$$

For $U(1)^3$ anomalies all quarks and leptons contribute and one gets

$$D_{YYY} = 3 \times 2 \times \Big(\frac{1}{6}\Big)^3 + 3 \times \Big(-\frac{2}{3}\Big)^3 + 3 \times \Big(\frac{1}{3}\Big)^3 + 2 \times \Big(-\frac{1}{2}\Big)^3 + (1)^3 = 0\,. \tag{7.4.5}$$

There is actually a fifth non-trivial anomaly to check, that arises when we couple the SM to gravity. It is an anomaly involving a $U(1)$ current and two energy-momentum tensors, and it is called a mixed $U(1)$-gravitational anomaly.[3] This anomaly is proportional to $\sum_n q_n$, where n runs over all fermions with charges q_n. In the SM, the mixed $U(1)$-gravitational anomaly is proportional to

$$3 \times 2 \times \frac{1}{6} + 3 \times \Big(-\frac{2}{3}\Big) + 3 \times \frac{1}{3} + 2 \times \Big(-\frac{1}{2}\Big) + 1 = 0\,. \tag{7.4.6}$$

Notice how the fermion charges nicely combine to give a vanishing result for all the anomalies, in particular the $U(1)^3$ and the mixed $U(1)$-gravitational ones.

The SM is then a chiral, anomaly-free gauge theory. More precisely, we have shown that the SM is anomaly-free in the unbroken phase where all the gauge group is linearly realized (i.e. no Higgs mechanism is at work). Since the anomaly does not depend on the Higgs field, our results automatically imply that the SM is anomaly-free in the broken phase as well. In particular, the unbroken gauge group $SU(3) \times U(1)_{EM}$ is manifestly anomaly-free being all currents vector-like.

[3] There also exist pure gravitational anomalies. These are vanishing in 4 space-time dimensions, but can occur in $4n + 2$ dimensions (n a non-negative integer). We will not discuss these anomalies, that are beyond our course.

7.5 The Wess-Zumino Consistency Conditions*

In this section we will consider some more formal developments about anomalies. It is convenient in this context to consider the effective action $\Gamma(A)$ arising upon integration of the fermions:

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S(\psi,\bar{\psi},A)}\,. \tag{7.5.1}$$

In presence of an anomaly, $\Gamma(A)$ is not gauge invariant. Indeed, under an infinitesimal gauge transformation $A \to A - D\epsilon_1$

$$\delta_{\epsilon_1}\Gamma(A) = -\int d^4x\,\frac{\delta\Gamma(A)}{\delta A_\mu^a}D_\mu\epsilon_1^a = \int d^4x\,D_\mu J_a^\mu\epsilon_1^a = \int d^4x\,\mathcal{A}_a\epsilon_1^a = \int d^4x\,\epsilon_1^a\mathcal{F}_a\Gamma(A)\,, \tag{7.5.2}$$

where the anomaly $\mathcal{A}_a$ is the right-hand side of eq.(7.3.12) and we have defined the functional operator

$$\mathcal{F}_a = \partial_\mu\frac{\delta}{\delta A_\mu^a} + f_{abc}A_\mu^b\frac{\delta}{\delta A_\mu^c}\,. \tag{7.5.3}$$

Under a further infinitesimal transformations, we have

$$\delta_{\epsilon_2}\delta_{\epsilon_1}\Gamma(A) = \int d^4x\int d^4y\,\epsilon_2^b(y)\epsilon_1^a(x)\mathcal{F}_b(y)\mathcal{F}_a(x)\Gamma(A)\,. \tag{7.5.4}$$

Due to the group structure, performing the commutators of the two infinitesimal transformations parametrized by ϵ_1 and ϵ_2 should be equivalent to perform a single infinitesimal transformation with parameter $[\epsilon_1,\epsilon_2]$, where $\epsilon_{1,2} = \epsilon_{1,2}^aT_a$. In other words, we should have

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\Gamma(A) = \delta_{[\epsilon_1,\epsilon_2]}\Gamma(A)\,, \tag{7.5.5}$$

identity that can easily be verified to hold in general. The relation (7.5.5) implies a non-trivial condition for the anomaly, known as the Wess-Zumino consistency conditions:

$$\mathcal{F}_a(x)\mathcal{A}_b(y) - \mathcal{F}_b(y)\mathcal{A}_a(x) = f_{abc}\mathcal{A}_c(x)\delta(x-y)\,. \tag{7.5.6}$$

These conditions can be more conveniently expressed in terms of BRST transformations by defining

$$G(\omega,A) \equiv \int d^4x\,\omega_a(x)\mathcal{A}_a(x)\,. \tag{7.5.7}$$

Equations (7.5.6) imply that G is BRST-invariant, namely

$$sG(\omega,A) = 0\,. \tag{7.5.8}$$

This is easily shown by recalling the BRST transformations (**??**) of ω and A, according to which

$$\begin{aligned} sG(\omega,A) = & \int d^4x \Big(-\frac{1}{2} f_{abc}\omega_b(x)\omega_c(x)\mathcal{A}_a(x) \\ & -\omega_a(x)\int d^4y \frac{\delta \mathcal{A}_a(x)}{\delta A^b_\mu(y)}(\partial_\mu\omega_b(y)+f_{bcd}A^c_\mu(y)\omega_d(y)\Big) \\ = & \int d^4x d^4y \Big(-\frac{1}{2}\omega_a(x)\omega_b(y)\Big)\Big[\delta(x-y)f_{abc}\mathcal{A}_c(x)-\mathcal{F}_a(x)\mathcal{A}_b(y)+\mathcal{F}_b(x)\mathcal{A}_a(y)\Big], \end{aligned} \tag{7.5.9}$$

which vanishes if eq.(7.5.6) is satisfied.

We have extensively seen in the previous sections that there is some arbitrariness in computing anomalies, related to the fact that we can shift the latter from one current to another. From an effective action point of view, this shift corresponds to the possibility of adding local, non-gauge invariant, counterterms to $\Gamma(A)$ that change $\delta_\epsilon\Gamma(A)$ and hence the anomaly. The impossibility of keeping all currents conserved corresponds to the impossibility of finding a local[4] counterterm such that $\delta_\epsilon\Gamma(A)=0$. We can now be more precise exploiting eq.(7.5.8). If there would exist a local functional of the gauge fields $F(A)$, such that $G(\omega,A)=sF(A)$, namely suppose that $G(\omega,A)$ would be BRST-exact, then the anomaly would be cancelled by adding to $\Gamma(A)$ the counterterm $-F(A)$. Anomalies then form an equivalence class. Two anomalies related by the addition of a local functional $F(A)$ are equivalent and belong to the same cohomology of the BRST operator s. Thanks to the Wess-Zumino consistency conditions, it is possible to reconstruct the whole form of the non-abelian anomaly (7.3.12) by only knowing the terms quadratic in the gauge fields, i.e. by computing triangular graphs only. The procedure does not uniquely fixes the anomaly since, as we have just said, the latter can be changed by adding local counterterms to the action.

We finally just mention about the existence of an elegant formalism, known as the Stora-Zumino descent equations, that allows, in any number of even dimensions, to get anomaly functionals $\mathcal{A}_a$ that automatically satisfy the conditions (7.5.6). The descent equations make also manifest the close relationship between chiral anomalies in $2n+2$ dimensions and gauge anomalies in $2n$ dimensions.

[4] Locality is crucial. By using non-local functionals, any anomaly can be cancelled. For instance, a gauge $U(1)^3$ anomaly would be cancelled by adding to $\Gamma(A)$ the non-local functional

$$-F(A)=\frac{g^2}{96\pi^2}\frac{1}{\Box}\partial^\mu A_\mu \epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,. \tag{7.5.10}$$

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7.6 ’t Hooft Anomaly Matching and the Wess-Zumino-Witten Term*

Asymptotically free gauge theories are strongly coupled in the IR and can give rise to confinement of its fermion constituents, like quarks in QCD. At low energies the propagating degrees of freedom are bound states of the elementary high energy (UV) states, such as mesons and hadrons in QCD. What is the fate at low energies of possible global anomalies coming from the elementary constituents at high energy ? t’Hooft has answered to this question by arguing that anomalies must arise in the low energy effective theory as well. More precisely, the so called ’t Hooft Anomaly Matching conditions state that if the UV theory has anomalous chiral symmetries – neither broken by gauge anomalies nor spontaneously broken (unlike chiral symmetries in QCD) – induced by the elementary UV fermion fields, then the IR theory must contain in its low energy spectrum massless fermion bound states that exactly reproduce the UV global anomaly. ’t Hooft’s argument is very simple. Let us assume of (weakly) gauging the unbroken global symmetries of the UV theory. In this way the theory will have gauge anomalies and would then be inconsistent. We might cancel the anomaly by adding suitable additional massless “spectator” fermions, neutral under the strong gauge group but charged under the weakly gauged symmetries. At low energies, when the strong gauge group confines, the spectrum of the theory will include the IR bound states plus the “spectator” fermions that, being neutral, are unaffected by the condensation of the strong gauge group. The UV theory with the spectators is, by construction, consistent and it has to remain so for all values of the gauge coupling constant. Hence, at low energies, there must be an anomaly contribution canceling that of the fermion spectators. Since, by assumption, no symmetry is spontaneously broken, no Goldstone bosons appear and only massless fermion bound states can possibly cancel the anomaly. The argument is valid for an arbitrarily weak gauging and thus apply also for global symmetries.

’t Hooft Anomaly Matching conditions do not apply if the global symmetries are spontaneously broken. Indeed, when (approximate) exact global symmetries are spontaneously broken, Goldstone’s theorem ensures that (light) massless scalar particles appear at low energies and these can replace in t’Hooft argument the massless fermion bound states that are no longer required. In this situation the anomaly of the UV fermions must be reproduced by the effective action of the (pseudo) Goldstone bosons. The latter situation is actually realized in Nature in QCD. We know that QCD with n_f massless quarks has an $SU_V(n_f)\times SU_A(n_f)$ global symmetry, spontaneously broken to $SU_V(n_f)$ by the quark condensate. Aside from being spontaneously broken, $SU_A(n_f)$ is also anomalous. Considering for simplicity $n_f=2$ and taking the neutral $U_A(1)\subset SU_A(2)$ subgroup generated by t^3, we have the following axial $U(1)_A U(1)^2_{EM}$ anomaly:

$$\begin{aligned}\partial_\mu J^\mu_A &= -\frac{N_c}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\left[\left(\frac{2e}{3}\right)^2\times 1+\left(\frac{-e}{3}\right)^2\times(-1)\right]\\ &= -\frac{N_c e^2}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,.\end{aligned}\tag{7.6.1}$$

Under the $U_A(1)$ chiral transformation above, the low-energy effective chiral Lagrangian $\mathcal{L}_\pi$ describing the dynamics of the π mesons interacting with photons should then not vanish, but rather reproduce the anomaly (7.6.1). Considering that $\delta_\epsilon\pi^0=\epsilon f_\pi$, $\mathcal{L}_\pi$ should include the coupling

$$\mathcal{L}_\pi \supset -\frac{N_c e^2}{48\pi^2 f_\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\pi^0\,.\tag{7.6.2}$$

The axial anomaly in QCD has allowed to resolve the puzzle of the $\pi^0\to 2\gamma$ decay. In absence of any anomaly, the term (7.6.2) would still appear in the chiral Lagrangian $\mathcal{L}_\pi$ but with a much more suppressed coupling. On the contrary, anomaly considerations uniquely fix its coefficient and it turns out that the experimental rate $\Gamma(\pi^0\to 2\gamma)$ is successfully reproduced with $N_c=3$. We have seen in chapter ??? that the chiral Lagrangian $\mathcal{L}_\pi$ should be described in terms of the matrix of fields $U=\exp(2i\pi^a t^a/f_\pi)$, rather than by the π's mesons themselves. The anomalous term (7.6.2) should then be rewritten in terms of the U's. This rewriting is not totally straightforward and will not be done here. The ending result goes under the name of the gauged version of the Wess-Zumino-Witten term. The latter includes many other couplings, including the term (7.6.2).

Chapter 8

Final Project: The Abelian Higgs Model*

In this last chapter we study the abelian Higgs model, along the lines of [3]. This chapter should be seen as a sort of long exercise in which many of the notions and techniques introduced in these notes (effective potential, background field method, ghosts, gauge-fixing, CS equations, β-functions and anomalous dimensions) are considered together.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\Phi|^2 - \frac{\lambda}{6}(\Phi^\dagger\Phi)^2\,, \tag{8.0.1}$$

where $D_\mu\Phi = \partial_\mu\Phi - ieA_\mu\Phi$. Our aim will be to understand the vacuum of this theory, namely whether the $U(1)$ gauge symmetry is broken or not, and the RG flows of the two couplings e and λ. First of all, we have to gauge fix the theory. Since we want to study the effective potential as a function of the VEV of Φ, it is convenient to use a generalized ξ-gauge which is valid for any value of $\langle\Phi\rangle$:

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi}\Big[\partial_\mu A^\mu + ie\xi(\phi^\dagger\phi_0 - \phi\phi_0^\dagger)\Big]^2\,, \tag{8.0.2}$$

where $\Phi = \phi_0 + \phi$, with ϕ_0 the VEV of Φ and ϕ its quantum fluctuation. It is straightforward to verify that the quadratic mixing terms between ϕ and A_μ vanish when $\mathcal{L}_{g.f.}$ is added to the Lagrangian (8.0.1). Even if the local symmetry is abelian, the ghosts do not decouple in the ξ-gauge we have chosen. The ghost Lagrangian, as usual, is derived by taking the infinitesimal variation of $\mathcal{L}_{g.f.}$ with respect to a $U(1)$ transformation. One gets

$$\mathcal{L}_{ghosts} = \partial_\mu\omega^\star\partial^\mu\omega - e^2\xi\omega^\star\omega\Big[2|\phi_0|^2 + (\phi^\dagger\phi_0 + \phi_0^\dagger\phi)\Big]\,. \tag{8.0.3}$$

In the Landau gauge $\partial_\mu A^\mu = 0$, reached for $\xi \to 0$, the ghosts are free and decouple, whereas in the unitary gauge $\xi \to \infty$ they are infinitely massive and decouple again. We will not fix in the following a specific value of ξ, so that the ghosts should be taken into account. The total Lagrangian is

$$\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_{g.f.} + \mathcal{L}_{ghosts}. \tag{8.0.4}$$

8.1 One-loop Effective Potential*

The 1-loop effective potential is completely determined by the terms in $\mathcal{L}_{tot}$ quadratic in the field fluctuations. In momentum space, we have

$$\mathcal{L}_{tot,quad}(p) = -\frac{1}{2}A^{\mu}(-p)A^{\nu}(p)\mathcal{L}^{(A)}_{\mu\nu}(p) + \omega^{\star}(-p)\mathcal{L}^{(\omega)}(p)\omega(p) + \frac{1}{2}\phi_i(-p)\mathcal{L}^{(\phi)}_{ij}(p)\phi_j(p)\,, \tag{8.1.1}$$

where $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, and

$$\begin{aligned}
\mathcal{L}^{(A)}_{\mu\nu}(p) &= \eta_{\mu\nu}(p^2 - 2e^2|\phi_0|^2) - \Big(1 - \frac{1}{\xi}\Big)p_\mu p_\nu\,,\\
\mathcal{L}^{(\omega)}(p) &= p^2 - 2\xi e^2|\phi_0|^2\,,\\
\mathcal{L}^{(\phi)}_{ij}(p) &= \begin{pmatrix} p^2 - |\phi_0|^2(e\xi + \frac{2\lambda}{3}) + (\phi_0^2 + \phi_0^{\star 2})(\frac{e^2\xi}{2} - \frac{\lambda}{6}) & i(\phi_0^2 - \phi_0^{\star 2})(\frac{e^2\xi}{2} - \frac{\lambda}{6}) \\ i(\phi_0^2 - \phi_0^{\star 2})(\frac{e^2\xi}{2} - \frac{\lambda}{6}) & p^2 - |\phi_0|^2(e\xi + \frac{2\lambda}{3}) - (\phi_0^2 + \phi_0^{\star 2})(\frac{e^2\xi}{2} - \frac{\lambda}{6}) \end{pmatrix}.
\end{aligned} \tag{8.1.2}$$

Modulo irrelevant factors,

$$\begin{aligned}
\det \mathcal{L}^{(A)}_{\mu\nu}(p) &= (p^2 - 2e^2|\phi_0|^2)^3(p^2 - 2e^2\xi|\phi_0|^2)\,,\\
\det \mathcal{L}^{(\phi)}_{ij}(p) &= (p^2 - \lambda|\phi_0|^2)\Big[p^2 - |\phi_0|^2(2e^2\xi + \frac{\lambda}{3})\Big]\,.
\end{aligned} \tag{8.1.3}$$

Summing over all contributions (gauge, ghosts and scalar fields), we get

$$\begin{aligned}
V_{1-loop}(\rho) &= \frac{1}{2}\int \frac{d^4p_E}{(2\pi)^4}\Big[3\log(p_E^2 + e^2\rho^2) - \log(p_E^2 + \xi e^2\rho^2)\\
&\quad + \log(p_E^2 + \frac{\lambda}{2}\rho^2) + \log\Big(p_E^2 + \big(\xi e^2 + \frac{\lambda}{6}\big)\rho^2\Big)\Big]\,,
\end{aligned} \tag{8.1.4}$$

where $\rho^2 = 2|\phi_0|^2$. We renormalize V_{1-loop} by demanding that

$$\frac{d^2V_{1-loop}}{d\rho^2}\bigg|_{\rho=0} = 0\,, \quad \frac{d^4V_{1-loop}}{d\rho^4}\bigg|_{\rho=\mu} = 0\,. \tag{8.1.5}$$

After some simple algebra, we obtain

$$V_{eff}(\rho) = V_{tree}(\rho) + V_{1-loop}(\rho) = \frac{\lambda}{4!}\rho^4 + \frac{\rho^4}{64\pi^2}\Big(3e^4 + \frac{5}{18}\lambda^2 + \frac{1}{3}\xi\lambda e^2\Big)\Big(\log\frac{\rho^2}{\mu^2} - \frac{25}{6}\Big). \tag{8.1.6}$$

Let us study the minima of V_{eff}, assuming that $\lambda \sim e^4$, so that at leading order we can

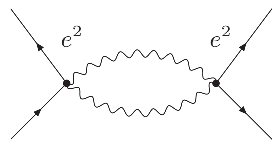


Figure 8.1: One-loop graph leading to a $\lambda\rho^4$ coupling.

neglect the λ^2 and $\xi\lambda e^2$ terms in eq.(8.1.6). It is important to emphasize here that at tree-level (or, alternatively, at any given energy scale) we can assume any relation we like of the form $\lambda \sim e^n$ for any n, but such relations at the quantum level cannot generally hold at any energy scale for any n. Indeed, radiative corrections will nevertheless generate the $\lambda\rho^4$ coupling in the theory. The leading one-loop correction arises from virtual photons as illustrated in fig.8.1. Being this correction of $\mathcal{O}(e^4)$, we see that $\lambda \sim e^n$, with $n \leq 4$, are the only radiatively stable assumptions we can make. The extrema of V_{eff} are

$$\frac{dV_{eff}}{d\rho} = \rho^3\left(\frac{\lambda}{6} + \frac{e^4}{16\pi^2}\left(3\log\frac{\rho^2}{\mu^2} - 11\right)\right) = 0\,. \tag{8.1.7}$$

Taking $\mu = \langle\rho\rangle$ in eq.(8.1.7), we get $\rho = 0$ and

$$\lambda(\langle\rho\rangle) = \frac{33}{8\pi^2}e^4(\langle\rho\rangle)\,. \tag{8.1.8}$$

Equation (8.1.8) is an instance of dimensional trasmutation: we have traded the VEV of ρ for the coupling λ. Plugging back in V_{eff} gives

$$V_{eff} \simeq \frac{3e^4}{64\pi^2}\rho^4\left(\log\frac{\rho^2}{\langle\rho\rangle^2} - \frac{1}{2}\right). \tag{8.1.9}$$

The extremum (8.1.8) is a minimum. The photon and scalar masses are

$$m_\gamma^2 = e^2\langle\rho\rangle^2\,, \qquad m_\rho^2 = \frac{3e^4}{8\pi^2}\langle\rho\rangle^4\,. \tag{8.1.10}$$

We conclude that in this theory a dynamical spontaneous symmetry breaking of the $U(1)$ gauge symmetry can occur. In order to firmly establish that, we have to compute the RG evolutions of λ and of the charge e to check the existence of an energy scale $\langle\rho\rangle$ where eq.(8.1.8) is valid. This computation will be the subjects of the following sections.

8.2 The Quantum Effective Action*

An instructive way of computing the β-functions of λ and e, as well as the anomalous dimensions of A_μ and ϕ, makes use of a functional form of the CS equations (4.4.1), which reads:

$$\mu\frac{d\Gamma}{d\mu} = \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - \gamma\int d^4x\phi(x)\frac{\delta}{\delta\phi(x)}\right)\Gamma(\phi) = 0\,, \tag{8.2.1}$$

where $\Gamma(\phi)$ is the quantum effective action. For a single real scalar field, keeping up to two derivative terms, the latter reads

$$\Gamma(\phi) = \int d^4x\left(\frac{1}{2}Z(\phi)(\partial_\mu\phi)^2 - V_{eff}(\phi)\right)\,, \tag{8.2.2}$$

where V_{eff} is the Coleman-Weinberg potential and Z is the radiative correction to the kinetic term. Once Γ is known, eq.(8.2.1) gives us β and γ. It is straightforward to see that eq.(8.2.1) encodes all eqs.(4.4.1) for any n, by recalling that

$$\Gamma(\phi) = \sum_{n=0}^\infty\frac{1}{n!}\int d^4x_1\ldots d^4x_n\Gamma^{(n)}(x_1,\ldots,x_n)\phi(x_1)\ldots\phi(x_n)\,. \tag{8.2.3}$$

In the case at hand, with two fields and two couplings, eq.(8.2.1) generalizes to

$$\begin{aligned}&\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_e\frac{\partial}{\partial e} - \gamma_A\int d^4xA_{\mu,0}(x)\frac{\delta}{\delta A_{\mu,0}(x)}\right.\\ &\left.-\gamma_\phi\int d^4x\Big(\phi_0(x)\frac{\delta}{\delta\phi_0(x)} + \phi_0^\dagger(x)\frac{\delta}{\delta\phi_0^\dagger(x)}\Big)\right)\Gamma(\phi_0, A_{\mu,0}) = 0\,,\end{aligned} \tag{8.2.4}$$

where ϕ_0, $\phi_0^\dagger$ and $A_{\mu,0}$ are the background field configurations. Invariance under the background $U(1)$ gauge invariance implies that the lowest dimensional operators appearing in Γ are of the form

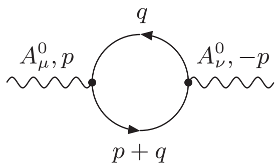
$$\Gamma(\phi_0, A_{\mu,0}) = \int d^4x\left[-\frac{1}{4}H(\rho)F_{\mu\nu,0}^2 + Z(\rho)|D_\mu\phi_0|^2 - V_{eff}(\rho)\right]\,. \tag{8.2.5}$$

The effective potential V_{eff} has been already computed and is given by eq.(8.1.6). We have then to determine H and Z only. This can be done by decomposing the gauge field as well in terms of background and fluctuation fields: $A_\mu \to A_\mu^0 + A_\mu$.

Let us start by computing H, in which case we can take $\phi_0 = constant$. The relevant interaction terms are

$$\mathcal{L} \supset ieA_0^\mu(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi)\,, \tag{8.2.6}$$

and H is determined by the contraction of the two scalar currents. The contractions of the form $\langle\phi\phi\rangle$ or $\langle\phi^\dagger\phi^\dagger\rangle$, although non-vanishing for $\phi_0 \neq 0$ due to mass terms of the form ϕ^2 and $(\phi^\dagger)^2$, give only rise to irrelevant divergent contact terms. In dimensional regularization, for instance, they all trivially vanish. The only relevant contraction are the usual of the form $\langle\phi^\dagger\phi\rangle$. The computation essentially boils down to the one-loop photon vacuum polarization in scalar QED. Two diagrams contribute. The first reads



$$= i\Pi^1_{\mu\nu}(p) = (ie)^2 \int \frac{d^dq}{(2\pi)^d} \frac{i^2(p+2q)_\mu(p+2q)_\nu}{(q^2-m^2)[(p+q)^2-m^2]}\,, \tag{8.2.7}$$

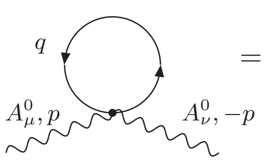
where, according to eq.(8.1.2),

$$m^2 = m^2(\rho) = \rho^2\Big(\frac{2\lambda}{3} + \frac{e^2\xi}{2}\Big)\,. \tag{8.2.8}$$

By performing the usual manipulations (introduce the Feynman parameter x, shift $q \to q - xp$ and Wick rotate to euclidean momentum), we get

$$\begin{aligned} i\Pi^1_{\mu\nu}(p) = (ie)^2 \int \frac{d^dq}{(2\pi)^d} \frac{i^2(p+2q)_\mu(p+2q)_\nu}{(q^2-m^2)[(p+q)^2-m^2]} = \frac{ie^2}{(4\pi)^{d/2}} \int_0^1 dx\,[m^2 - p^2x(1-x)]^{\frac{d-4}{2}} \\ \times \Gamma\Big(\frac{4-d}{2}\Big)\Big(p_\mu p_\nu(1-2x)^2 - \frac{4}{2-d}\eta_{\mu\nu}[m^2 - p^2x(1-x)]\Big) \end{aligned} \tag{8.2.9}$$

The second diagram is a tadpole, that does not depend on the external momentum p. It can be cast in a form close to eq.(8.2.9) by multiplying and dividing it by $(p+q)^2 - m^2$. Using the same manipulations as before, we get



$$= i\Pi^2_{\mu\nu}(p) = 2ie^2\eta_{\mu\nu} \int \frac{d^dq}{(2\pi)^d} \frac{i}{q^2-m^2} \frac{(p+q)^2-m^2}{(p+q)^2-m^2} = -\frac{2ie^2}{(4\pi)^{d/2}}\eta_{\mu\nu} \tag{8.2.10}$$

$$\times \int_0^1 dx\,[m^2 - p^2x(1-x)]^{\frac{d-4}{2}}\Gamma\Big(\frac{4-d}{2}\Big)\Big((1-x)^2p^2 - m^2 - \frac{d}{2-d}[m^2 - p^2x(1-x)]\Big)\,.$$

Summing the two contributions, we have

$$\begin{aligned} i\Pi_{\mu\nu} = i\Pi^1_{\mu\nu} + i\Pi^2_{\mu\nu} = \frac{ie^2}{24\pi^2}(p_\mu p_\nu - \eta_{\mu\nu}p^2)\Big(\frac{1}{\epsilon} + \text{const.}\Big) - \frac{ie^2}{16\pi^2}\int_0^1 dx \log\Big(m^2 - p^2x(1-x)\Big) \\ \times \Big(p_\mu p_\nu(1-2x)^2 - \eta_{\mu\nu}p^2(4x^2 - 6x + 2)\Big)\,. \end{aligned} \tag{8.2.11}$$

We are interested in computing H, which is the coefficient of the F^2 term, quadratic in the external momentum p. The last term in eq.(8.2.11) is already quadratic in p, so we can safely neglect the p^2 term inside the log and keep only the mass term m^2. In this way we get

$$i\Pi_{\mu\nu} = \frac{ie^2}{24\pi^2}(p_\mu p_\nu - \eta_{\mu\nu}p^2)\Big(\frac{1}{\epsilon} + \text{const.} - \log\rho\Big)\,, \tag{8.2.12}$$

where we have reabsorbed in the arbitrary constant the ρ-independent factors coming from eq.(8.2.8). As expected, $\Pi_{\mu\nu}$ is transverse. We now perform a non-minimal subtraction, i.e. we add the counterterm[1] $i(Z-1)(p_\mu p_\nu - \eta_{\mu\nu}p^2)$ and require that

$$i\Pi_{\mu\nu} + i(Z-1)(p_\mu p_\nu - \eta_{\mu\nu}p^2) = 0,\ \text{at}\ \rho = \mu\,. \tag{8.2.13}$$

The renormalized one-loop photon vacuum polarization is then

$$i\Pi^R_{\mu\nu} = \frac{ie^2}{24\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\log\frac{\rho}{\mu} \equiv iAp^2\mathcal{P}_{\mu\nu}\,, \tag{8.2.14}$$

where $A = e^2/(24\pi^2)\log\phi/\mu$ and we have defined the projector

$$\mathcal{P}_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\,. \tag{8.2.15}$$

The tree-level photon propagator is

$$G_{\mu\nu}(p) = -\frac{i}{p^2}\Big(\mathcal{P}_{\mu\nu} + \xi\frac{p_\mu p_\nu}{p^2}\Big)\,. \tag{8.2.16}$$

Iterating the one-loop correction (8.2.14) in the tree-level expression (8.2.16) we get

$$G_{\mu\nu}(p) \to \frac{-i\mathcal{P}_{\mu\nu}}{p^2(1-A)} - \frac{i}{p^2}\frac{\xi p_\mu p_\nu}{p^2} \tag{8.2.17}$$

from which we see that

$$-\frac{1}{4}F^2_{\mu\nu,0} \to -\frac{1}{4}(1-A)F^2_{\mu\nu,0} \tag{8.2.18}$$

finally giving the desired $H(\rho)$:

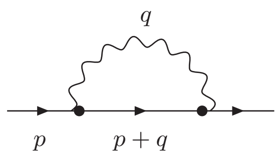
$$H(\rho) = 1 - A(\rho) = 1 - \frac{e^2}{24\pi^2}\log\frac{\rho}{\mu}\,. \tag{8.2.19}$$

Let us now determine Z, the wave-function renormalization of ϕ_0. In this case we can set $A_{\mu,0} = 0$ but of course ϕ_0 can no longer be taken constant. The tadpole graph given by the quartic scalar interaction induces only a mass renormalization, so we need only to consider the contribution induced by the gauge interactions. These are of the form $A_\mu\phi\phi_0$, given by

$$\mathcal{L} \supset 2ieA^\mu(\phi^\dagger\partial_\mu\phi_0 - \partial_\mu\phi_0^\dagger\phi)\,. \tag{8.2.20}$$

[1] Of course, the counterterm Z should not be confused with the finite $Z(\rho)$ appearing in the effective action (8.2.5).

We have



$$= i\Sigma(p) = (2ie)^2 p^\mu p^\nu \int \frac{d^d q}{(2\pi)^d} G_{\mu\nu}(q) \frac{i}{(p+q)^2 - m^2} \,. \quad (8.2.21)$$

For $\xi \neq 1$, up to three denominators appear in eq.(8.2.21). Since two of them are equal, it is enough to introduce a single Feynman parameter, using the relation

$$\frac{1}{a^2 b} = 2 \int_0^1 dx \frac{1-x}{[a(1-x)+bx]^3} \,. \quad (8.2.22)$$

After the usual manipulations and omitting trivial steps, we get

$$\begin{aligned} i\Sigma(p) &= -4ie^2p^2 \int \frac{d^d q_E}{(2\pi)^d} \int_0^1 dx \left(\frac{1}{[q_E^2 + m^2 x - p^2 x(1-x)]^2} + \frac{2(1-x)(1-\xi)(x^2p^2 - q^2/d)}{[q_E^2 + m^2 x - p^2 x(1-x)]^3} \right) \\ &= -\frac{4ie^2p^2}{(4\pi)^{d/2}} \int_0^1 dx \, [m^2 x - p^2 x(1-x)]^{\frac{d-4}{2}} \left[\Gamma\Big(\frac{4-d}{2}\Big) \Big(1 - \frac{(1-x)(1-\xi)}{2}\Big) \right. \\ &\qquad \left. + \Gamma\Big(\frac{6-d}{2}\Big) \frac{p^2 x^2}{m^2 x - p^2 x(1-x)} \right] . \end{aligned} \quad (8.2.23)$$

Focusing on the terms up to quadratic order in p, eq.(8.2.23) gives

$$i\Sigma(p) = -\frac{4ie^2p^2}{(4\pi)^{d/2}} \Gamma\Big(\frac{4-d}{2}\Big) \int_0^1 dx \, (m^2 x)^{\frac{d-4}{2}} \Big(1 - \frac{(1-x)(1-\xi)}{2}\Big) \,. \quad (8.2.24)$$

Similarly to the previous computation of H, we add the counter term $i(\tilde{Z}-1)p^2$ and perform a non-minimal subtraction requiring

$$i\Sigma(p) + i(\tilde{Z}-1)p^2 = 0, \ \text{at} \ \rho = \mu \,. \quad (8.2.25)$$

The renormalized scalar two-point function becomes

$$i\Sigma^R(p) = \frac{ie^2}{8\pi^2}(3+\xi)p^2 \log\frac{\rho}{\mu} \equiv ip^2 B \,. \quad (8.2.26)$$

Iterating the one-loop correction in the tree-level scalar propagator, as in the photon case, gives

$$\frac{i}{p^2} \to \frac{i}{p^2(1+B)} \,, \quad (8.2.27)$$

from which we extract $Z(\rho)$:

$$Z(\rho) = 1 + B(\rho) = 1 + \frac{e^2}{8\pi^2}(3+\xi)\log\frac{\rho}{\mu} \,. \quad (8.2.28)$$

The form of the effective action (8.2.5) is finally determined and we can proceed to use it to compute the RG evolution of the couplings e and λ. It is worth to emphasize that the effective action (8.2.5) should be gauge invariant, but on the contrary it seems gauge-dependent, since ξ enters in both $Z(\rho)$ and in $V_{eff}(\rho)$. This apparent paradox is explained by noticing that ϕ_0 is not a canonically normalized field and the rescaling

$$\phi_0 \to \frac{1}{\sqrt{Z(\rho)}}\phi_0 \tag{8.2.29}$$

is needed. The potential (8.1.6) is then rescaled by a factor $1/Z(\rho)^2$ and we have

$$\begin{aligned}\frac{1}{Z(\rho)^2}V_{eff}(\rho) &= \frac{\lambda}{4!}\rho^4\Big(1-\frac{e^2}{4\pi^2}(3+\xi)\log\frac{\rho}{\mu}\Big)+\frac{\rho^4}{32\pi^2}\Big(3e^4+\frac{5}{18}\lambda^2+\frac{1}{3}\xi\lambda e^2\Big)\log\frac{\rho}{\mu} \\ &= \frac{\lambda}{4!}\rho^4+\frac{\rho^4}{32\pi^2}\Big(3e^4+\frac{5}{18}\lambda^2-e^2\lambda\Big)\log\frac{\rho}{\mu}\,,\end{aligned} \tag{8.2.30}$$

and the ξ-dependence cancels. In eq.(8.2.30) we have focused on the log terms only, since the constant term is scheme-dependent and we have not been careful in systematically using a given scheme.

8.3 RG Equations and Their Solutions*

For a single real scalar field, the functional RG equation (8.2.1) splits into two ordinary differential equations for Z and V:

$$\begin{aligned}\Big(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}-\gamma\phi\frac{\partial}{\partial\phi}-2\gamma\Big)Z&=0\,,\\ \Big(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}-\gamma\phi\frac{\partial}{\partial\phi}\Big)V_{eff}&=0\,.\end{aligned} \tag{8.3.1}$$

In dealing with quartic potentials, like in our case, it is actually convenient to write an RG equation for $V^{(4)}\equiv\partial^4 V_{eff}/\partial\phi^4$ rather than V_{eff} itself. Given that $\partial^4_\phi(\gamma\phi\partial_\phi)=4\gamma\partial^4_\phi+\gamma\phi\partial_\phi\partial^4_\phi$, we have

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}-4\gamma-\gamma\phi\frac{\partial}{\partial\phi}\Big)V^{(4)}=0\,. \tag{8.3.2}$$

The RG equation (8.3.2) can further be simplified by noting that $V^{(4)}$ depends on ϕ only through the dimensionless combination ϕ/μ. Then $\phi\partial_\phi V^{(4)}=-\mu\partial_\mu V^{(4)}=\partial_t V^{(4)}$, where $t=\log\phi/\mu$ and thus we get

$$\Big(-\frac{\partial}{\partial t}+\bar\beta\frac{\partial}{\partial\lambda}-4\bar\gamma\Big)V^{(4)}=0\,. \tag{8.3.3}$$

where

$$\bar{\beta} = \frac{\beta}{1+\gamma}\,, \qquad \bar{\gamma} = \frac{\gamma}{1+\gamma}\,. \tag{8.3.4}$$

Using the same manipulations, the RG equation for Z becomes

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}\frac{\partial}{\partial \lambda} - 2\bar{\gamma}\right) Z = 0\,. \tag{8.3.5}$$

In the situation at hand, the functional RG equation is given by eq.(8.2.4). It is clear that there can be no cancellations between the three terms (8.2.5) appearing in Γ, so eq.(8.2.4) splits into three independent equations. Let us first focus on the scalar kinetic term $|D\phi_0|^2$:

$$\begin{aligned}&|D_\mu\phi_0|^2\Big(-\frac{\partial}{\partial t} + \beta_e\frac{\partial}{\partial e} - \gamma_\phi\rho\partial_\rho\Big)Z + Z\Big(\beta_e\frac{\partial}{\partial e}|D_\mu\phi_0|^2 - \gamma_A\int d^4x A_{\nu,0}(x)\frac{\delta|D_\mu\phi_0|^2}{\delta A_{\nu,0}(x)}\\ &-\gamma_\phi\int d^4x\Big(\phi_0(x)\frac{\delta}{\delta\phi_0(x)} + \phi_0^\dagger(x)\frac{\delta}{\delta\phi_0^\dagger(x)}\Big)\Big)|D_\mu\phi_0|^2 = 0\,,\end{aligned} \tag{8.3.6}$$

where we have used the fact that Z depends on constant ρ only, with $\phi\partial_\phi + \phi^\dagger\partial_{\phi^\dagger} = \rho\partial_\rho$. The term in the second line of eq.(8.3.6) equals $-2\gamma_\phi|D_\mu\phi_0|^2$, while the last two terms in the first line of eq.(8.3.6) gives rise to a different operator. As such, two independent equations arise from eq.(8.3.6). Requiring the vanishing of the coefficient proportional to $|D_\mu\phi_0|^2$ gives

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}_e\frac{\partial}{\partial e} - 2\bar{\gamma}_\phi\right) Z = 0, \tag{8.3.7}$$

with

$$\bar{\beta}_e = \frac{\beta_e}{1+\gamma_\phi}\,, \qquad \bar{\gamma}_\phi = \frac{\gamma_\phi}{1+\gamma_\phi}\,. \tag{8.3.8}$$

It is straightforward to see that

$$e\int d^4x A_{\nu,0}(x)\frac{\delta|D_\mu\phi_0|^2}{\delta A_{\nu,0}(x)} = \frac{\partial}{\partial e}|D_\mu\phi_0|^2 \tag{8.3.9}$$

and hence the vanishing of the coefficient multiplying $\partial|D_\mu\phi_0|^2/\partial e$ gives

$$\beta_e = e\gamma_A\,. \tag{8.3.10}$$

Consider now the gauge kinetic term $F^2_{\mu\nu,0}$. We get

$$-\frac{1}{4}F^2_{\mu\nu,0}\Big(-\frac{\partial}{\partial t} + \beta_e\frac{\partial}{\partial e} - \gamma_\phi\rho\partial_\rho\Big)H - 2\gamma_A H\Big(-\frac{1}{4}F^2_{\mu\nu,0}\Big) = 0\,, \tag{8.3.11}$$

that gives rise to

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}_e\frac{\partial}{\partial e} - 2\bar{\gamma}_A\right) H = 0, \quad \bar{\gamma}_A = \frac{\gamma_A}{1+\gamma_\phi}\,. \tag{8.3.12}$$

Finally we have the potential term. As explained before, we write an RG equation for $V^{(4)}$ rather than V_{eff}, which is the obvious generalization of eq.(8.3.3):

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}_\lambda\frac{\partial}{\partial\lambda}+\bar{\beta}_e\frac{\partial}{\partial e}-4\bar{\gamma}_\phi\right)V^{(4)}=0\,. \tag{8.3.13}$$

Equations (8.3.7), (8.3.10), (8.3.12) and (8.3.13) are enough to determine β_e, β_λ, γ_A and γ_ϕ. Let us recall below the explicit form of Z, H and $V^{(4)}$, the latter computed from eq.(8.1.6):

$$\begin{aligned} Z(t)&=1+\frac{e^2}{8\pi^2}(\xi+3)t\\ H(t)&=1-\frac{e^2}{24\pi^2}t\\ V^{(4)}(t)&=\lambda+\frac{1}{4\pi^2}\Big(9e^4+\frac{5}{6}\lambda^2+\xi\lambda e^2\Big)t \end{aligned} \tag{8.3.14}$$

One immediately gets from eqs.(8.3.7), (8.3.10) and (8.3.12)

$$\begin{aligned} \gamma_\phi&=\bar{\gamma}_\phi+\mathcal{O}(e^4)=-\frac{e^2}{16\pi^2}(\xi+3)+\mathcal{O}(e^4)\,,\\ \gamma_A&=\bar{\gamma}_A+\mathcal{O}(e^4)=\frac{e^2}{48\pi^2}+\mathcal{O}(e^4)\,,\\ \beta_e&=\bar{\beta}_e+\mathcal{O}(e^5)=\frac{e^3}{48\pi^2}+\mathcal{O}(e^5)\,, \end{aligned} \tag{8.3.15}$$

Plugging the values (8.3.15) in eq.(8.3.13) allows us to determine β_λ:

$$\beta_\lambda=\bar{\beta}_\lambda+\mathcal{O}(e^6,e^4\lambda,e^2\lambda^2)=\frac{1}{4\pi^2}\Big(9e^4+\frac{5}{6}\lambda^2-3\lambda e^2\Big)+\mathcal{O}(e^6,e^4\lambda,e^2\lambda^2)\,. \tag{8.3.16}$$

Notice how all ξ-dependent factors have cancelled in β_λ as it should be, being the latter gauge invariant, like β_e (and γ_A). The scalar field anomalous dimension γ_ϕ, instead, does depend on ξ. This is expected since ϕ changes by a phase under a gauge transformation and at the quantum level there is no gauge invariant notion of γ_ϕ.

The RG flow of e is easily computed from β_e. We get

$$e^2(t)=\frac{e_0^2}{1-\frac{e_0^2}{24\pi^2}t}\,. \tag{8.3.17}$$

The RG flow of λ requires some more work. It is convenient to define $R(t)=\lambda(t)/e^2(t)$ and write an RG equation for R. One gets

$$e^2(t)\dot{R}(t)=\frac{e^4(t)}{4\pi^2}\Big(\frac{5}{6}R^2(t)-\frac{19}{6}R(t)+9\Big)\,, \tag{8.3.18}$$

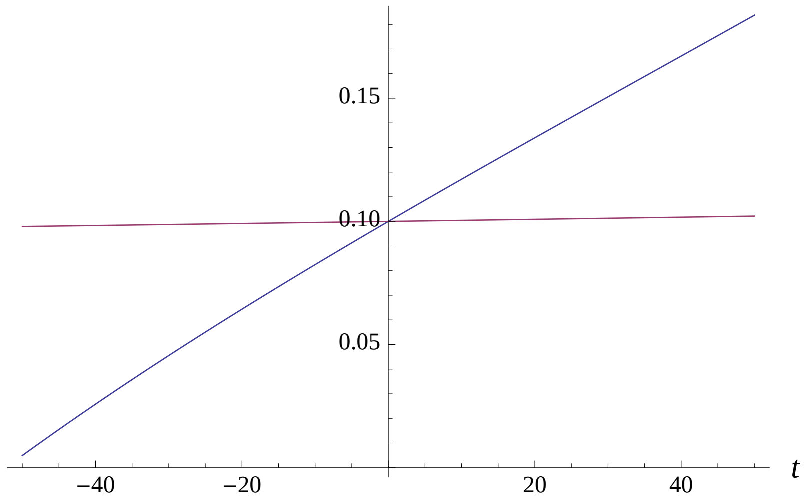


Figure 8.2: Comparison between the RG behaviour of $e^2(t)$ (blue line) and $\lambda(t)$ (red line) over 100 orders of magnitude. We have taken $e^2(0) = \lambda(0) = 1/10$.

which is further simplified by considering $R = R(e^2)$, so that

$$\dot{R} = \frac{dR}{de^2} 2e\dot{e} = \frac{e^4}{24\pi^2} \frac{dR}{de^2} \,. \tag{8.3.19}$$

Our desired final equation reads

$$e^2 \frac{dR(e^2)}{de^2} = 5R^2(e^2) - 19R(e^2) + 54 \,, \tag{8.3.20}$$

whose solution is

$$R(e^2) = \frac{1}{10}\left(19 + \sqrt{719}\tan\left(\frac{1}{2}\sqrt{719}\log e^2 + \theta\right)\right) \tag{8.3.21}$$

giving

$$\lambda(t) = \frac{e^2(t)}{10}\left(19 + \sqrt{719}\tan\left(\frac{1}{2}\sqrt{719}\log e^2(t) + \theta\right)\right) , \tag{8.3.22}$$

where θ is an integration constant. Both e and λ grow in the UV but, as explicitly shown in fig.(8.2), the quartic coupling λ varies significantly over a range in which the electric charge remains essentially constant. We can then conclude that for a wide range of initial conditions for e and λ there exists an energy scale where $\lambda \sim e^4$, and in particular eq.(8.1.8) is valid.

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