Quantum gravity and critical phenomena

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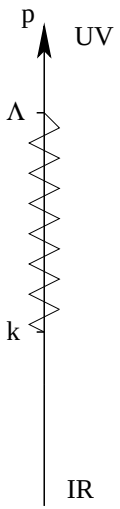
Outline

- The general idea:
  - A brief reminder of Wilson's RG and continuum limit ideas.
  - How could it possibly work for gravity
- A concrete realization:
  - Lattice approach: (causal) dynamical triangulations
- Outlook

Wilsonian renormalization group

Momentum-shell integration (or coarse graining)



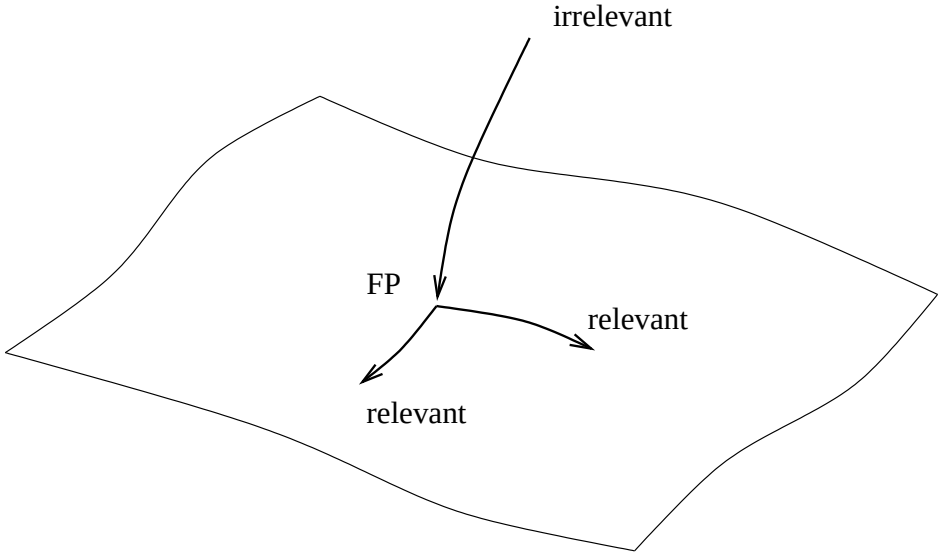
$$e^{-\Gamma_{k,\Lambda}} = \int_{k \leq p \leq \Lambda} [D\phi]\, e^{-\Gamma_\Lambda}$$

- It defines a flow of effective actions.
- Initial condition: $\Gamma_{k=\Lambda,\Lambda} = \Gamma_\Lambda$
- Crucial question in QFT:
  can we send $\Lambda \to \infty$ while holding fixed the physics at scale k?

Fixed point

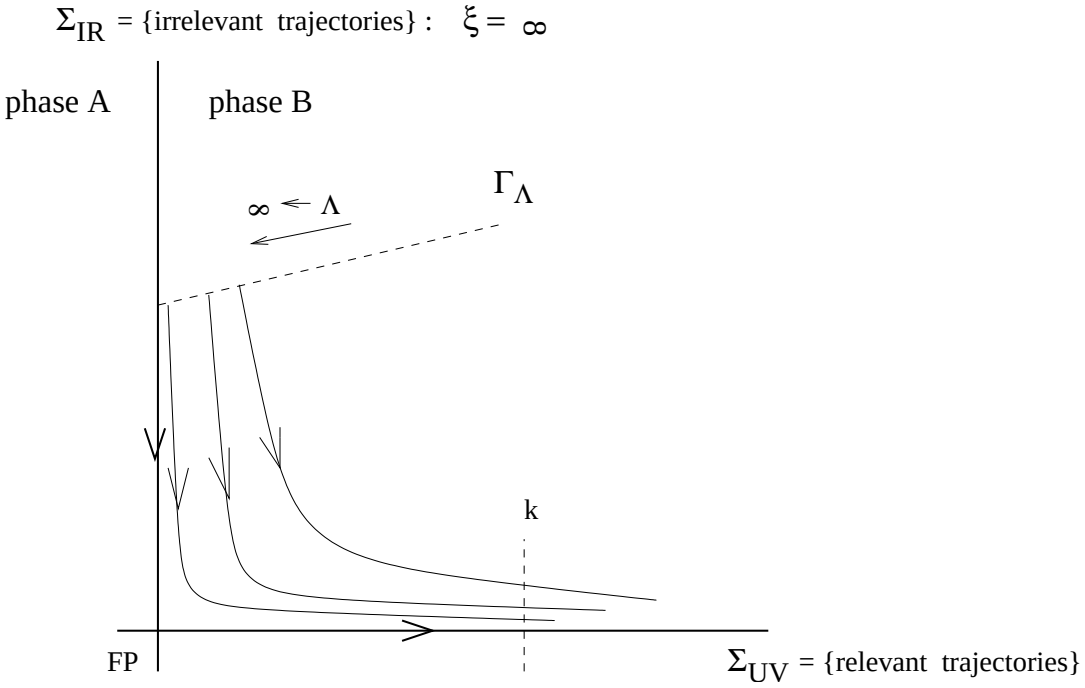
If the flow of effective actions has a Fixed Point
$\Rightarrow$ we can define a continuum limit, characterized by value of relevant couplings

[Wilson - $\sim$'70]



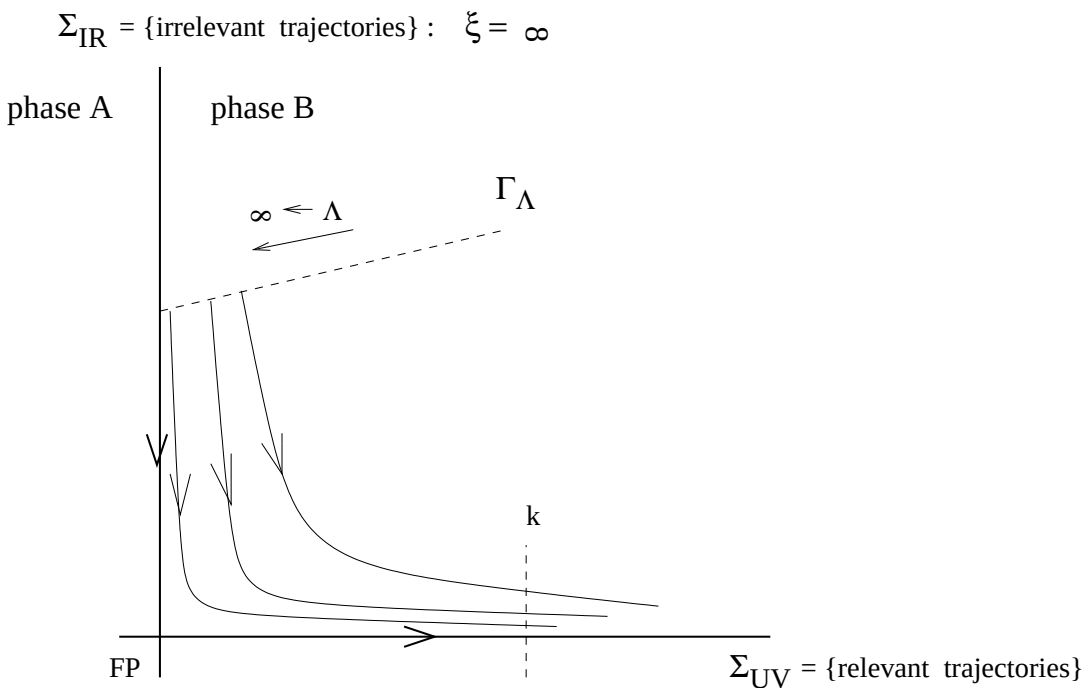
[Theory space: space of all possible initial conditions Γ_Λ.
Arrows point towards IR.]

Fixed points and continuum limit



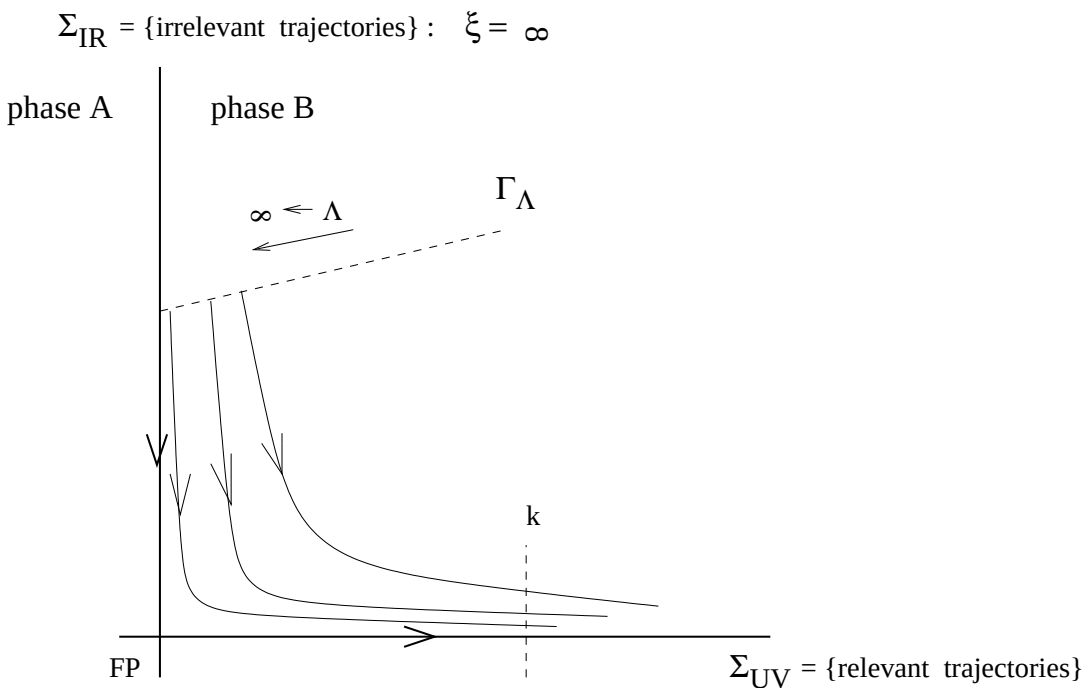
- Σ_{IR} is the locus of a 2nd order phase transition
- While sending $\Lambda \sim \frac{1}{a} \to \infty$, tune Γ_Λ to Σ_{IR}, where $\xi_L \to \infty$, such that $m = \frac{1}{a\,\xi_L} < \infty$.

Fixed points and continuum limit



- In the continuum limit the irrelevant couplings take their FP value
- Relevant couplings at scale k are fixed, as in renormalization conditions
- $\dim(\Sigma_{UV}) < \infty$ guarantees that only a finite number of couplings have to be fixed by hand (or by experiment)

Fixed points and continuum limit



- In discrete approach we study Γ_Λ and look for phase transitions; in continuum approach we study Γ_k and look for UV-FP.

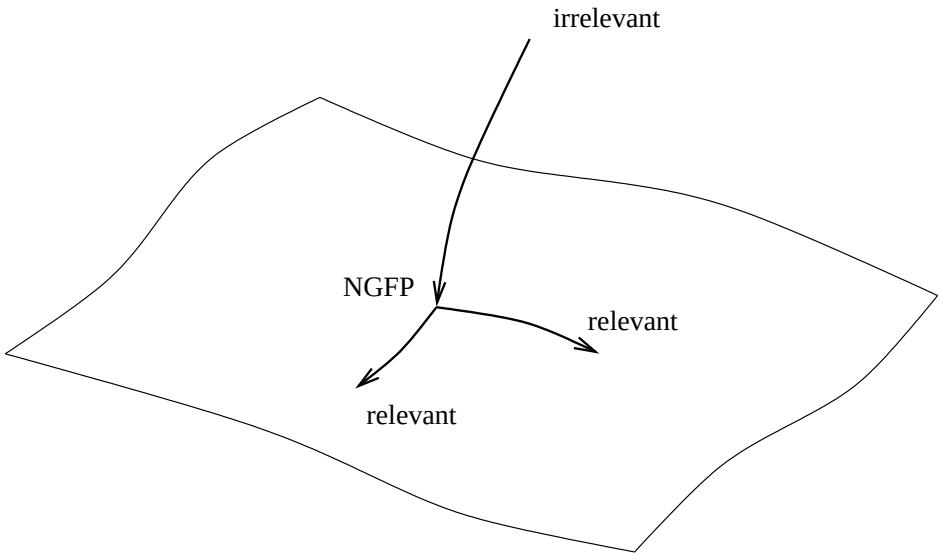
The Asymptotic Safety scenario

Gravity: G (Newton's constant) is an irrelevant coupling at the Gaussian FP (=perturbative non-renormalizability)

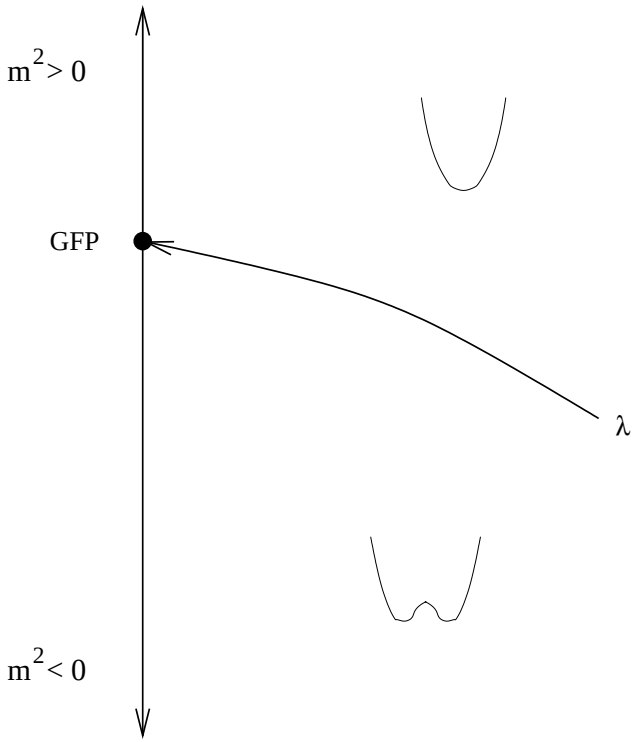
Definition of asymptotic safety

UV controlled by a non-Gaussian fixed point (NGFP)
with a finite number of relevant directions

[Weinberg - '79]

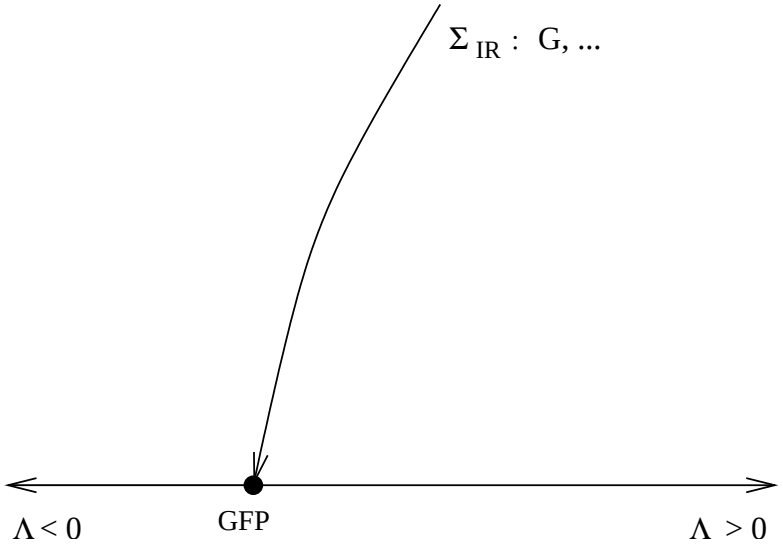


Tuning relevant parametrs



- Relevant parameters are tuned to obtain phase transitions
  e.g. $m^2 > 0 \to m^2 < 0$ for paramagnetic to ferromagnetic phase transition
- Is there an analogue picture in gravity?

Phases of gravity?



- We know of a fixed point for sure, the Gaussian one
- Λ: from compact to non-compact spacetime
  - Gaussian fixed point
  - only one relevant parameter: Λ
  - only useful for defining thermodynamic limit

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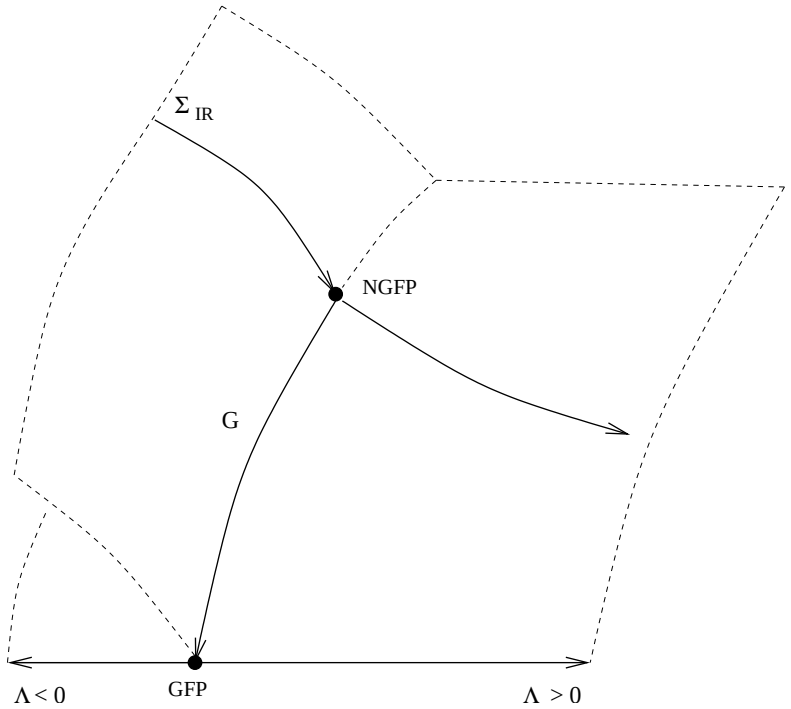

- NGFP?
- Tune also G_Λ (and more?) to obtain cont.lim. with finite G_k
- Old hypothesis for role of such transition:
  from extended ($\langle g_{\mu\nu} \rangle \neq 0$) to crumpled ($\langle g_{\mu\nu} \rangle = 0$) geometry

The lattice approach: Dynamical Triangulations

A statistical approach to lattice quantum gravity

- The main idea is to discretize the path integral

$$\int_{\text{geom}} \mathcal{D}[g] e^{-S[g]} \,,$$

directly sampling the space of (diffeo equivalent) Riemannian geometries, by means of piecewise flat geometries:

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- Define a statistical ensemble

  "equilateral triangulations" T of fixed topology

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  - Define a statistical ensemble
  - Define the Boltzmann weight

$$A(T) = e^{-S[T]} \,, \quad S[T] = \kappa_D N_D - \kappa_{D-2} N_{D-2}$$

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  - Construct partition function and observables

$$Z = \sum_{T} e^{-S(T)}$$

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  - Construct partition function and observables
  - Study the phase diagram of the model

Thermodynamic limit

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  $\Rightarrow$ Thermodynamic limit, or infinite (lattice) volume limit: tune $\kappa_D \to \kappa_D^c$

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$$a \to 0\,, \quad \text{s.t.} \quad a^D \langle N_D \rangle = V \sim \frac{1}{\Lambda_{IR}} < \infty$$

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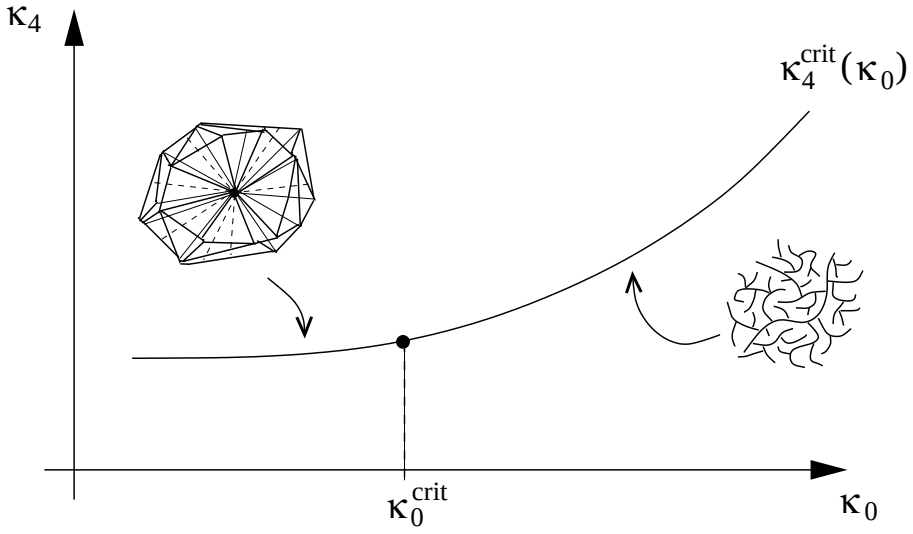
- Usually not enough: in order to have propagating degrees of freedom we need to tune some other coupling to a 2nd order phase transition

Phases of dynamical triangulations

- Hausdorff dimension as an order parameter:

$$\langle n(r) \rangle \sim r^{d_H - 1}$$



$$d_H \sim \infty \qquad\qquad d_H \sim 2$$

[Agishtein, Migdal; Ambjørn, Jurkiewicz; Bialas et al.; Catteral, Kogut, et al.; de Bakker, Smit; etc.]

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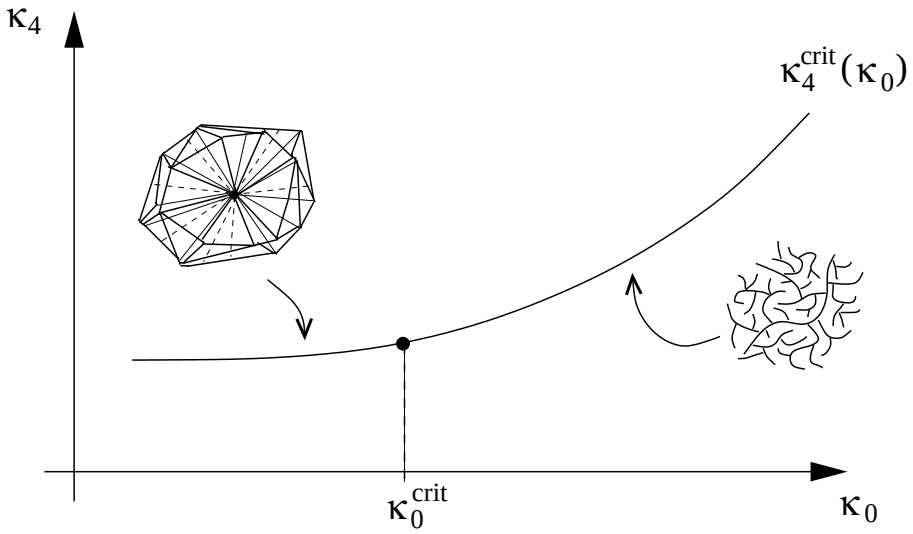

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- No classical geometry
- Phase transition is 1st order: no diverging correlation length :-(

Changing the phase diagram by non-local constraints

The "causality" condition: [Ambjørn, Loll]

- introduce a foliation in triangulation

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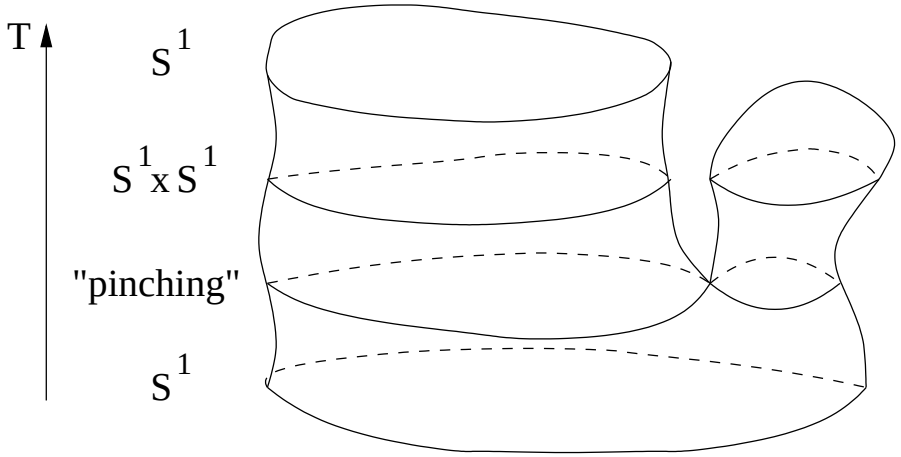

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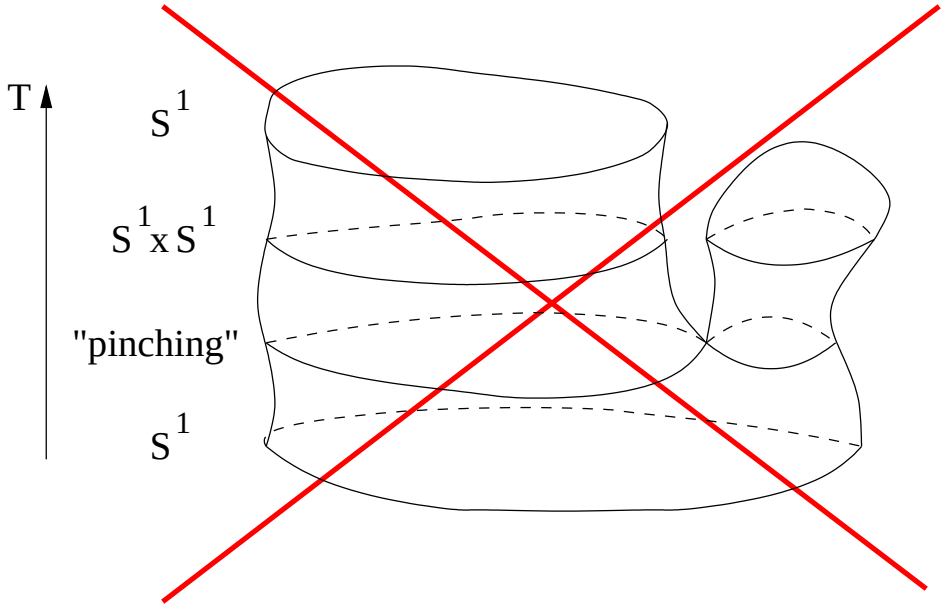

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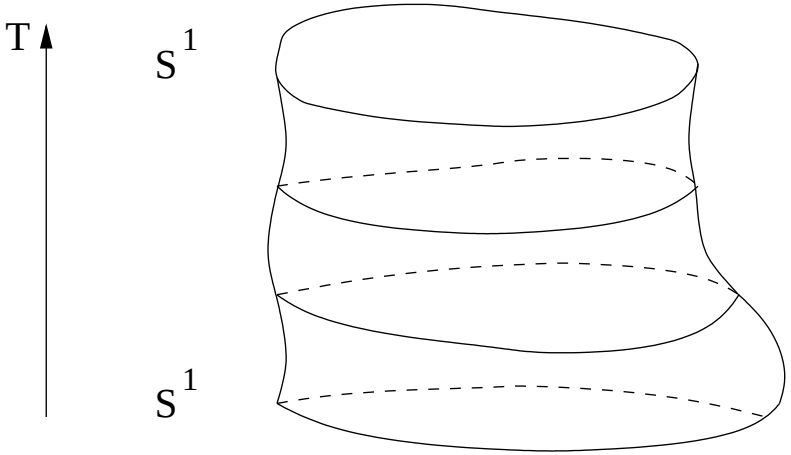
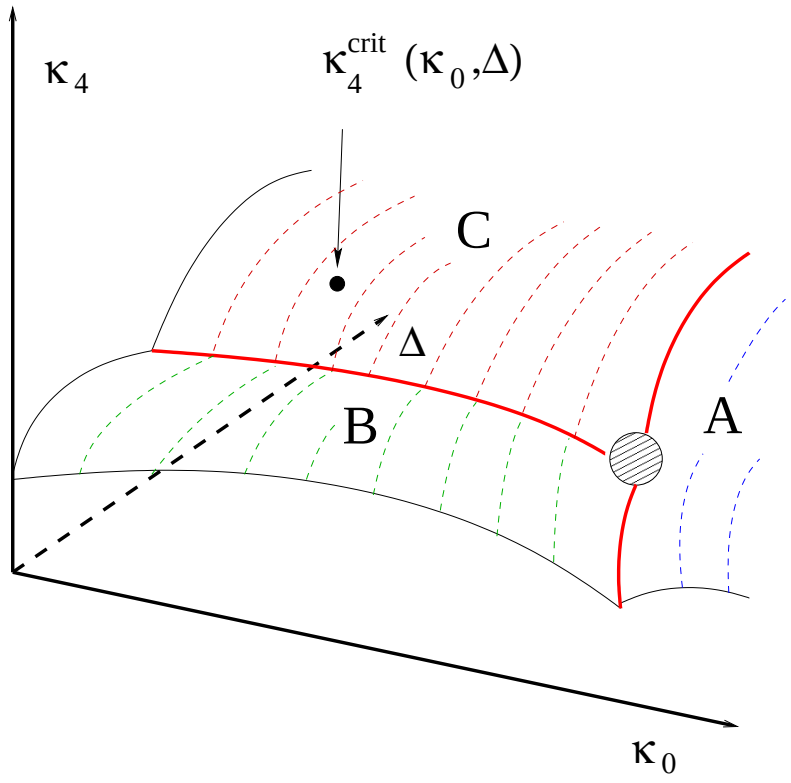


Phases of causal dynamical triangulations (CDT)



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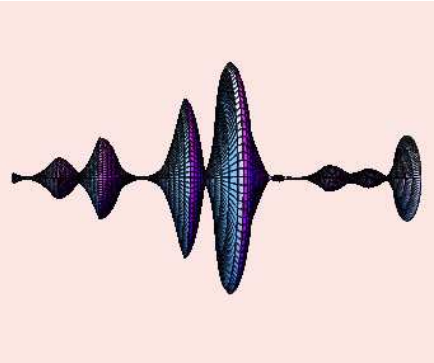
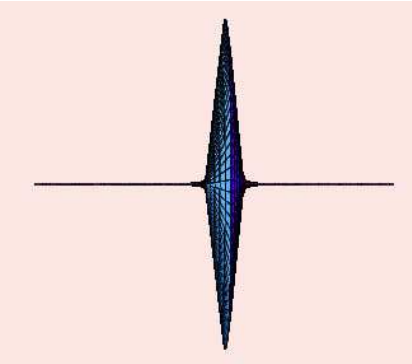
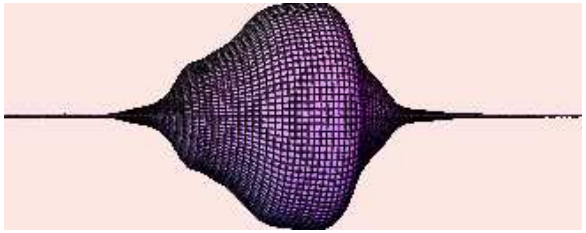


- phase A: $d_H \sim 2$.



- phase B: $d_H \sim \infty$.



- phase C: $d_H \sim 4$; profile averages to 4-sphere.

[Ambjørn, Jurkiewicz, Loll, Görlich, Jordan]

Comparison to Hořava-Lifshitz gravity

- Hořava has proposed an anisotropic theory of gravity.
  In $m+1$ dimensions he proposed an action of the type:

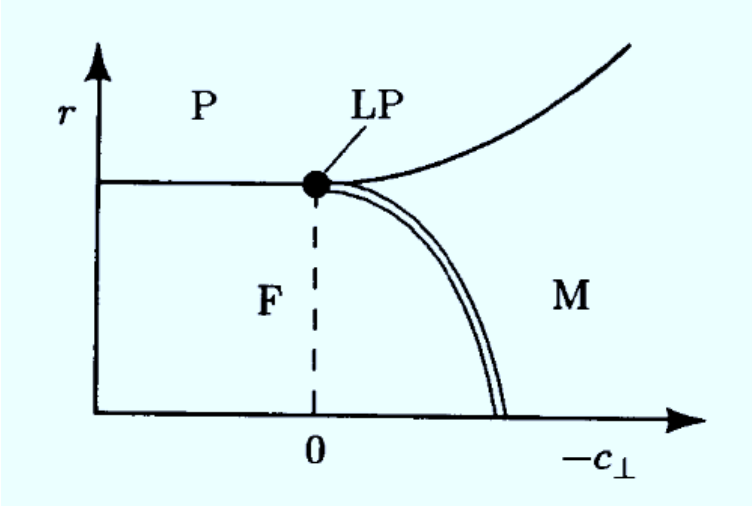
$$S = \int dt\, d^m x\, N \sqrt{g} \left\{ \frac{2}{\kappa^2} \left[\lambda K^2 - K_{ij} K^{ij} \right] + \mathcal{V}(g_{ij}) \right\}$$

  - Pro:
    For a potential $\mathcal{V}$ containing up to $2z = 2m$ spatial derivatives of the fluctuations of the spatial metric g_{ij},
    the theory is power-counting renormalizable,
    and with no higher-derivatives in time.
  - Con:
    Spacetime diffeomorphisms are reduced to foliation-preserving diffeomorphisms.
    $\Rightarrow$ new scalar *dof*

Comparison to Hořava-Lifshitz gravity

- The Landau free energy for a generic (d, m)-Lifshitz point with scalar order parameter is

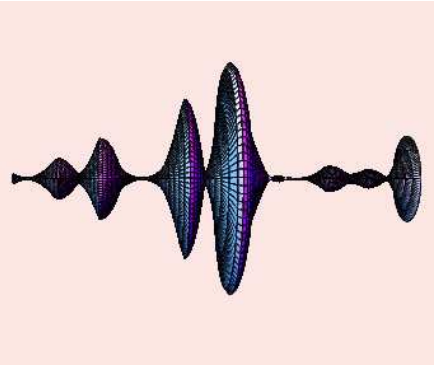
$$F = \frac{1}{2} \int d^d x \left[c_\parallel (\nabla_\parallel \phi)^2 + c_\perp (\nabla_\perp \phi)^2 + D(\nabla_\perp^2 \phi)^2 + r\phi^2 + u\phi^4 \right]$$



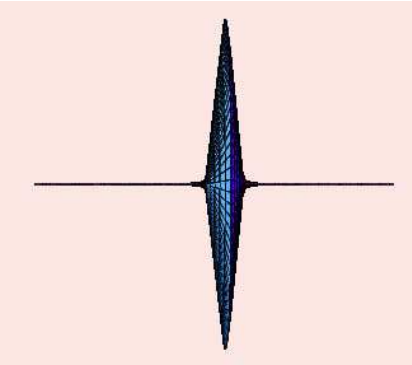
The point $r = 0$, $c_\perp = 0$ is a Lifshitz point.

- It is tempting to compare to the CDT phase diagram.

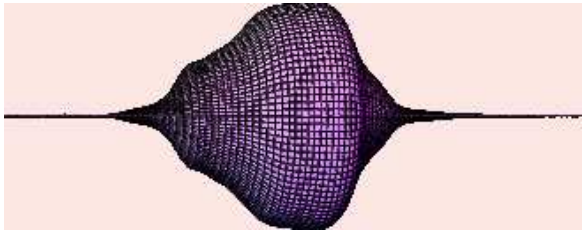
Phases of gravity?



- phase A $\leftrightarrow$ modulated phase:
  Lorentzian geometries?



- phase B $\leftrightarrow$ paramagnetic phase:
  "$\langle g_{\mu\nu} \rangle = 0$" ?



- phase C: $\leftrightarrow$ ferromagnetic phase:
  "$\langle g_{\mu\nu} \rangle \neq 0$" ?

Spectral dimension

- We can say a little bit more for the comparison
  - Spectral dimension measured in $d = 4$ CDT:
    from 4 (IR) to 2 (UV)

    [Ambjorn et al.]
  - Simple argument in AS: from d (IR) to $d/2$ (UV)

    [Lauscher, Reuter]
  - Simple argument in HL: from d (IR) to 2 (UV)

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  [DB, Henson]

$$D_s(\infty, N_3 = 200{,}000) = 3.05 \pm 0.04$$
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- Caveat: we are only in the thermodynamic limit

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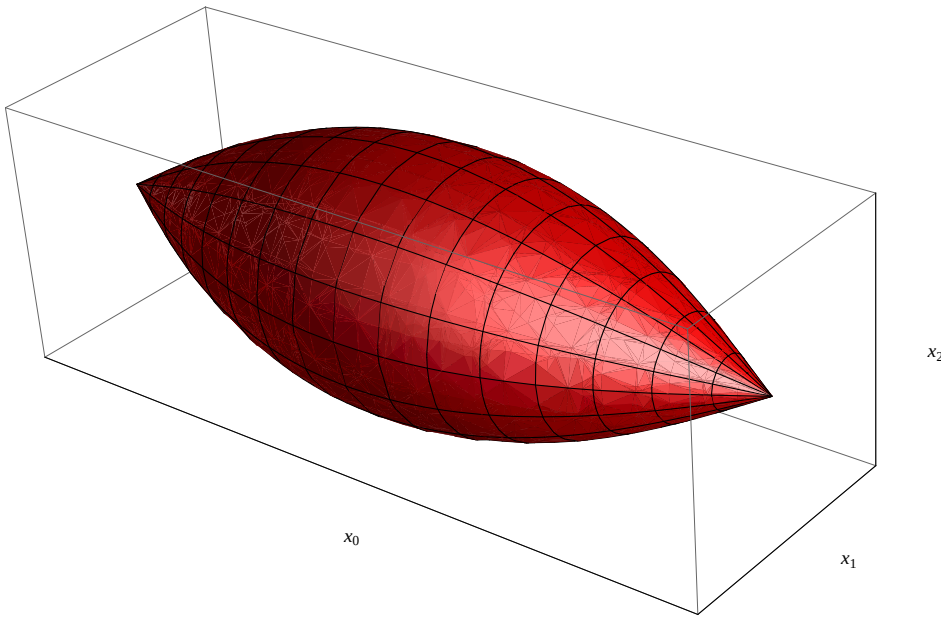


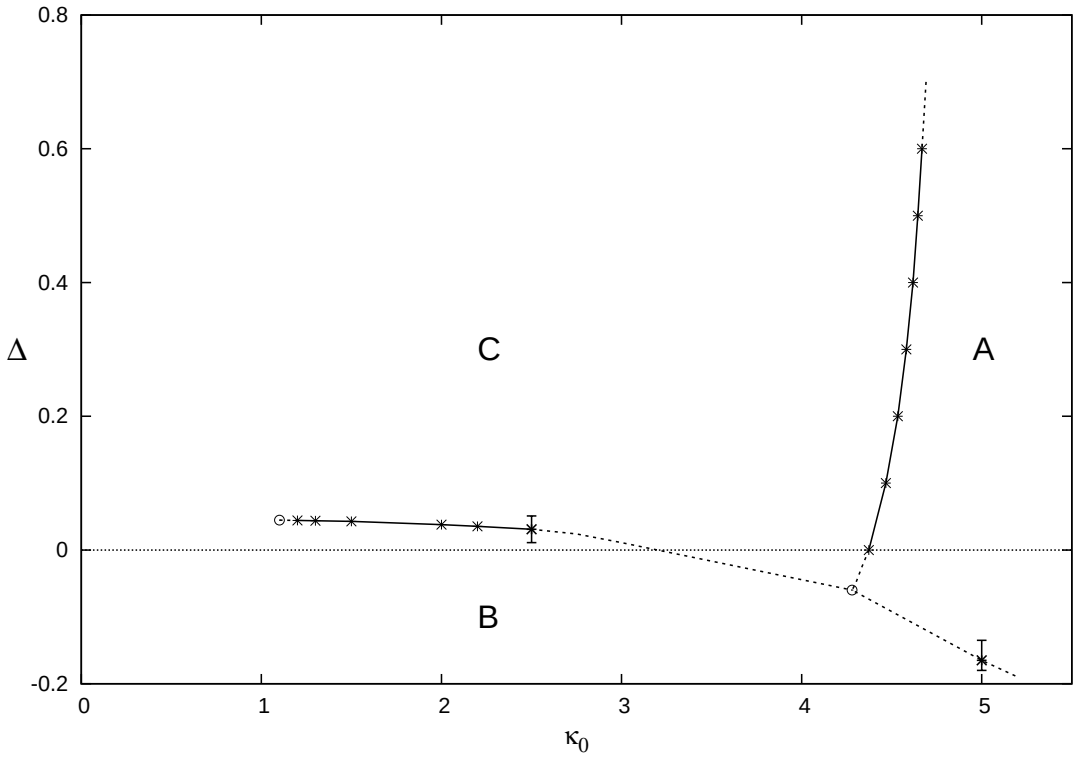
- At large diffusione time, spectral curve matches that of a stretched sphere ($s \neq 1$):
$$ds^2_{S^3_s} = r^2(s^2 d\psi^2 + \sin^2\psi\ (d\theta^2 + \sin^2\theta\ d\phi^2))$$
- This is actually a solution of HL-gravity in $2+1$ dimensions

HL or o HL?

- Recapitulating, evidence for a CDT-HL relation comes from
  - Presence of a foliation
  - Analogy in phase diagram
  - Short-scale spectral dimension
  - Large-scale shape (stretched sphere)

HL or o HL?

- Neverthelss CDT results are not at continuum limit...



- Triple-point and end-point are different fixed points?
- Different universality classes? (Isotropic and anisotropic?)

Alternative modifications of the ensemble and/or of the amplitudes?

- For example, (non-causal) DT models with modified amplitude:

$$A(T) = e^{-\kappa_D N_D + \kappa_{D-2} N_{D-2}} \prod_i q_i^{-\beta} \,,$$

$q_i = \#$ of D-simplices sharing $(D-2)$-simplex i

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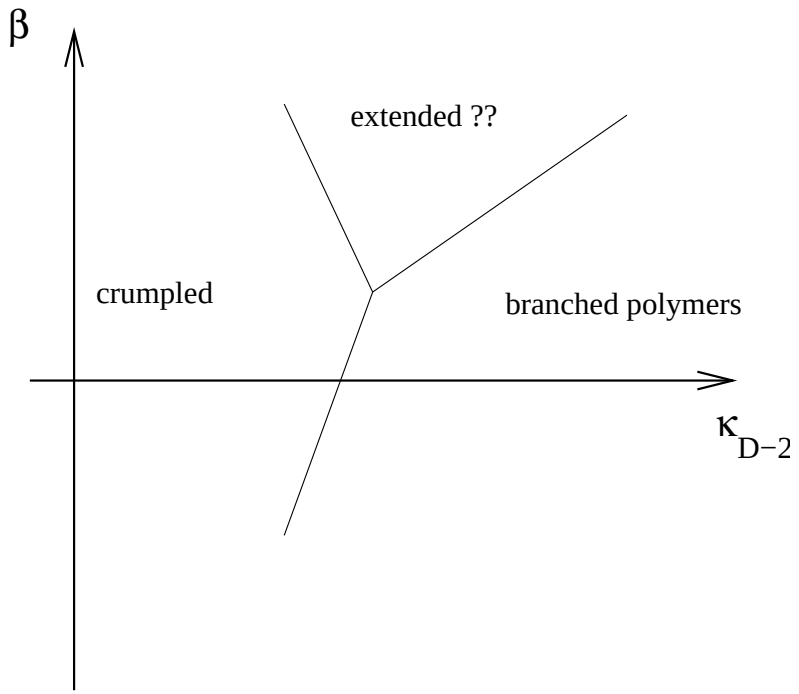
  [Bilke et. al. '98; Laiho, Coumbe 2011]



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- Exact solution (from tensor model) at large $\kappa_{D-2} \sim \ln N$: [DB, Gurau]

$$\begin{aligned}
e^{N^D E} &= \int d\bar{\psi}\, d\psi\; e^{-S(\psi,\bar{\psi})} \,, \\
S(\psi,\bar{\psi}) &= \sum_{i=0}^{D} \sum_{\vec{p}_i, \bar{\vec{n}}_i} \psi^i_{\vec{p}_i} \Big(\prod_j (C^{-1})_{p_{ij} \bar{n}_{ij}} \Big) \bar{\psi}^i_{\bar{\vec{n}}_i} \\
&\quad + \frac{\lambda}{N^{D(D-1)/4}} \sum_n \prod_{i=0}^{D} \psi^i_{\vec{n}_i} + \frac{\bar{\lambda}}{N^{D(D-1)/4}} \sum_{\bar{n}} \prod_{i=0}^{D} \bar{\psi}^i_{\bar{\vec{n}}_i} \,, \\
\mathrm{Tr}[(C^T C)^q] &= N q^{-\beta}
\end{aligned}$$

Alternative modifications of the ensemble and/or of the amplitudes?

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Susceptibility exponent ($E_{\text{non-analytic}} \sim (g - g_c(\beta))^{2-\gamma}$):

$$\gamma = \begin{cases} \frac{1}{2} & \text{for} \quad \beta < \beta_c \\ \frac{1}{2} - \beta & \text{for} \quad \beta > \beta_c \end{cases}$$

$$\frac{d^3 g_c}{d\beta^3} \sim (\beta_c - \beta)^{\frac{2-2\beta_c}{\beta_c - \frac{1}{2}}}$$

$\beta_c \approx 1.216 \Rightarrow$ 3rd order phase transition !

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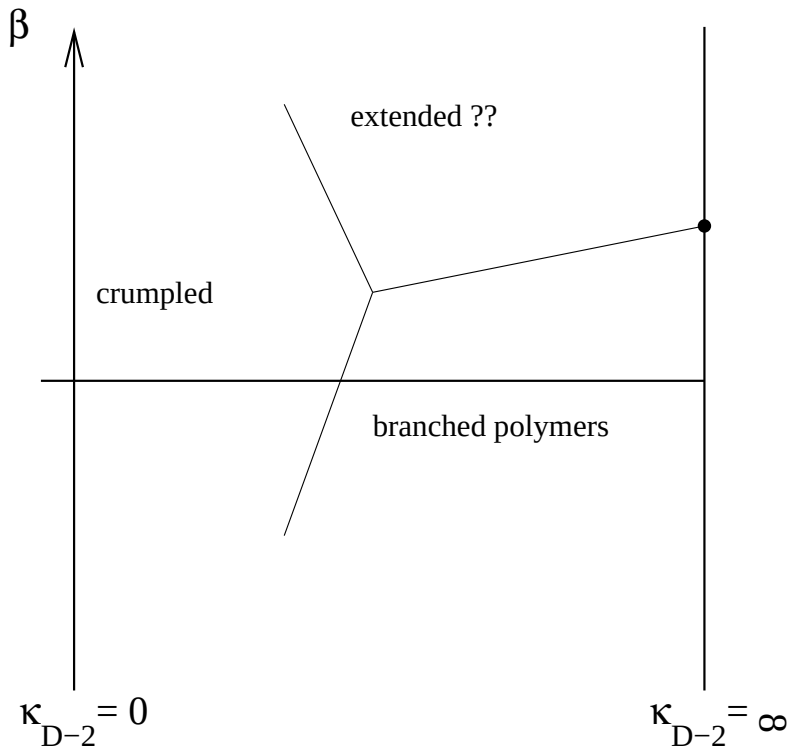

Outlook

The continuum limit?

- Analytical work in the continuum suggests existence of a NGFP – ongoing work aimed at giving more robustness to such results

  [Reuter, ...; Percacci, ...; Litim, ...; Saueressig, ...; DB, ...]

- Numerical work in the discrete provides evidence for emergence of classical geometries and for two candidate critical points for c.l.

  [Ambjørn, Jurkiewicz, Loll, Görlich, Jordan; DB, Henson]

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     $\Rightarrow$ asymptotic safety
  2. Anisotropy survives in continuum limit?
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- The (non-causal) models might also come back from their tomb: We proved that in DT the measure term is *not* inessential

  [DB, Gurau; see also results of Laiho, Coumbe]