ON ANALYTIC FAMILIES OF INVARIANT TORI FOR PDES

by

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Dedicated to J.-P.Ramis on the occasion of his 60th birthday

Abstract. — We propose to apply a version of the classical Stokes expansion method to the perturbative construction of invariant tori for PDEs corresponding to solutions quasiperiodic in space and time variables. We argue that, for integrable PDEs all but finite number of the small divisors arising in the perturbative analysis cancel. As an illustrative example we establish such cancellations for the case of KP equation. It is proved that, under mild assumptions about decay of the magnitude of the Fourier modes all analytic families of finite-dimensional invariant tori for KP are given by the Krichever construction in terms of theta-functions of Riemann surfaces. We also present an explicit construction of infinite dimensional real theta-functions and corresponding quasiperiodic solutions to KP as sums of infinite number of interacting plane waves.

1. Introduction

Quasiperiodic solutions of the equations of motion

$$\dot{u} = f(u)$$

in the form

$$u(t) = U(\phi_1, \dots, \phi_n), \quad \phi_j = \omega_j t + \phi_j^0, \; j = 1, \dots, n$$

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for a 2π-periodic in each $\phi_1, \ldots, \phi_n$ function U has been studied in the classical mechanics since 19th century. The associated geometric image of linear motion on an n-dimensional torus became widely accepted after creation of KAM theory and of the Arnold - Liouville theory of completely integrable Hamiltonian systems [**2**], although it was already familiar in the physics literature after the A.Einstein's treatment of the Bohr-Sommerfeld quantization rules for integrable systems with many degrees of freedom [**16**]. In particular, the Arnold - Liouville theory applied to a completely integrable Hamiltonian system on a $2n$-dimensional symplectic manifold $u \in M^{2n}$ establishes existence of families of n-dimensional invariant tori depending on n parameters $\mathbf{I} = (I_1, \ldots, I_n)$

$$u(t \,|\, \mathbf{I}) = U(\phi_1, \ldots, \phi_n \,|\, \mathbf{I}), \quad \phi_j = \omega_j(\mathbf{I})t + \phi_j^0, \; j = 1, \ldots, n. \tag{1.1}$$

Changing the values of the action variables $I_1, \ldots, I_n$ one represents a $2n$-dimensional domain in the symplectic manifold as a torus fibration. Under the nondegeneracy assumption [**2**] the frequencies $\omega_1(\mathbf{I}), \ldots, \omega_n(\mathbf{I})$ run through all possible directions. In particular, for generic values of the parameters $\mathbf{I}$ the solution (1.1) is a quasiperiodic function in time.

Systems of evolutionary PDEs

$$u_t^a = f^a(u, u_{\mathbf{x}}, u_{\mathbf{xx}}, \ldots), \quad \mathbf{x} = (x_1, x_2, \ldots, x_d), \quad a = 1, \ldots, r \tag{1.2}$$

can be considered as an infinite-dimensional analogue of dynamical systems define on a suitable space of functions of d spatial variables $x_1, \ldots, x_d$. Although in certain cases it is possible to develop an infinite-dimensional analogue of the Arnold-Liouville theory of completely integrable Hamiltonian systems and to construct families of infinite-dimensional invariant tori for certain nontrivial examples of nonlinear evolutionary PDEs, and, moreover, to develop an infinite-dimensional analogue of KAM theory (see [**25, 7, 21**]), in the most physical applications families of low dimensional invariant tori for PDEs play a prominent role.

For linear PDEs families of one-dimensional invariant tori can be readily found in the form of plane waves

$$u(\mathbf{x}, t) = A\cos(k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0). \tag{1.3}$$

The *wave numbers* $k_1, \ldots, k_d$ take arbitrary values within some domain of the d-dimensional space, the *frequency*

$$\omega = \omega(k_1, \ldots, k_d) \tag{1.4}$$

is determined from the so-called *dispersion relation* substituting the ansatz (1.3) into the equation (1.2). It will be assumed that all branches of the dispersion relation (1.3) are real-valued functions. For any such branch A is a r-component vector determined, in the generic situation, up to a scalar factor called the *amplitude*. The phase shift ϕ_0 can also take an arbitrary value. The solution (1.3) in general is quasiperiodic both in space and time variables. Multidimensional invariant tori for linear PDEs are obtained as linear superpositions of plane waves

$$u(\mathbf{x}, t) = \sum_{i=1}^{n} A_i \cos(k_1^i x_1 + \cdots + k_d^i x_d - \omega^i t + \phi_0^i)$$

with arbitrary amplitudes, phases and wave numbers, the frequencies determined as above

$$\omega^i = \omega(k_1^i, \ldots, k_d^i), \quad i = 1, \ldots, n.$$

Note that, in the discussion of invariant tori for PDEs, we need not specify the class of functions to be considered.

In many cases families of one-dimensional invariant tori can also be obtained for various nonlinear PDEs as travelling wave solutions

$$u(\mathbf{x}, t) = U(\phi \,|\, \mathbf{A}), \quad \phi = k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0. \tag{1.5}$$

Here $U(\phi \,|\, \mathbf{A})$ is a 2π-periodic function in ϕ depending on some number of parameters $\mathbf{A} = (A_1, A_2, \ldots)$ that determine the shape of the wave. The wave numbers and phases take arbitrary values. The shape of the wave does not depend on the phase shifts but it may depend on the wave numbers. It is convenient to subdivide the parameters $\mathbf{A}$ in two parts

$$\mathbf{A} = (k_1, \ldots, k_d; a) \tag{1.6}$$

where the parameter a is a nonlinear analogue of the amplitude. The frequency is to be determined from a nonlinear analogue of the dispersion relation. The latter involves also the amplitude parameters a,

$$\omega = \omega(k_1, \ldots, k_d; a). \tag{1.7}$$

For fixed t the solution (1.5) takes constant values along the hyperplanes

$$k_1 x_1 + \cdots + k_d x_d = \text{const}.$$

The points on the hyperplanes move in the orthogonal directions with the constant phase velocity

$$v = \frac{\omega}{|k|}, \quad |k| = \sqrt{k_1^2 + \cdots + k_d^2}.$$

Example 1.1. — The periodic travelling wave for the Kadomtsev - Petviashvili (KP) equation

$$u_{xt} + \frac{1}{4}(3u^2 + u_{xx})_{xx} + \frac{3}{4}u_{yy} = 0 \tag{1.8}$$

(here $d = 2$, $x = x_1$, $y = x_2$) can easily be obtained in terms of elliptic functions

$$\begin{aligned}
& u(x,y,t) = U(\phi), \quad \phi = kx + ly - \omega t + \phi_0 \\
& U(\phi) = \frac{2k^2}{\pi^2} K^2 \left(\kappa^2 \mathrm{cn}^2 \left[\frac{K}{\pi}\phi; \kappa \right] - \gamma \right) + \frac{c}{6} \\
& \omega = -\frac{c}{4}k + \frac{3}{4}\frac{l^2}{k} - k^3 \frac{K^2}{\pi^2} \left(3\frac{E}{K} + \kappa^2 - 2 \right) \\
& \gamma = \frac{E}{K} - 1 + \kappa^2.
\end{aligned} \tag{1.9}$$

Here $\mathrm{cn}\,[z;\kappa]$ is the Jacobi elliptic function with the modulus $0 \leq \kappa \leq 1$, $K = K(\kappa)$, $E = E(\kappa)$ are complete elliptic integrals of the first and second kind resp., c is an arbitrary constant.

The functions (1.9) are periodic travelling waves propagating with constant speed in the (x,t)-plane. For $l = 0$ the above formulae reduce to the so-called cnoidal waves for the Korteweg - de Vries (KdV) equation

$$u_t + \frac{1}{4}(3u^2 + u_{xx})_x = 0. \tag{1.10}$$

The KdV equation is known to arise in a fairly general setting of one-dimensional weakly nonlinear waves with small dispersion (see, e.g., [**29**]). In particular it describes one-dimensional shallow water waves of small amplitude. The y-dependence of solutions to the KP equation (1.8) describes[(1)] slow transversal perturbations of the KdV waves [**20**], [**29**].

The elliptic modulus κ plays the role of the amplitude parameter. At the limiting value $\kappa = 0$ one obtains trivial solution $u = 0$; the frequency takes the value $\omega = -\frac{1}{4}(c\,k + k^3)$. For small positive values of the parameter

$$\varepsilon^2 = k\left[\omega + \frac{1}{4}\left(c\,k + k^3 - 3\frac{l^2}{k}\right)\right] > 0$$

one obtains approximately the plane wave solution

$$u \simeq \frac{c}{6} + A\cos(kx + ly - \omega t + \phi_0), \quad \omega \simeq \frac{1}{4}\left(3\frac{l^2}{k} - c\,k - k^3\right)$$

with the small amplitude

$$A \simeq 2\sqrt{\frac{2}{3}}\varepsilon.$$

More accurate idea about the shape of the solution (1.9) for small amplitudes can be obtained by using *Stokes expansion* method [**32**]; see also Chapter 13 of the Whitham's book [**34**]. We will represent this classical method of the theory of water waves in a slightly modified version. Let us look for the solution to the KP equation in the form of Fourier series

$$u(x,y,t) = \frac{c}{6} + A_1\cos\phi + A_2\cos 2\phi + A_3\cos 3\phi + \dots, \quad \phi = kx + ly - \omega t + \phi_0 \tag{1.11}$$

depending on a small parameter ε assuming that

$$A_k = O(\varepsilon^k), \quad k = 1, 2, \dots. \tag{1.12}$$

Also the dispersion law must be expanded in a series with respect to the small parameter

$$\omega = \frac{1}{4}\left(3\frac{l^2}{k} - c\,k - k^3\right) + \omega_1 + \omega_2 + \dots, \quad \omega_k = O(\varepsilon^k). \tag{1.13}$$

The KP equation must hold for an arbitrary ϕ_0 as an identity for formal series in ε. Without loss of generality one can use the small amplitude

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(1)The equation (1.8) is often called KPII to distinguish it from the KPI case. The latter equation differs from (1.8) by the sign in front of the second derivative in y. It also has physical applications but not within the theory of water waves [**20**].

$A = A_1$ of the plane wave as the expansion parameter. Substituting the ansatz (1.11) - (1.13) into (1.8) yields, after simple calculation

$$u(x,y,t) = \frac{c}{6} + A\cos\phi + \frac{A^2}{2\,k^2}\left(1 - \frac{A^2}{8\,k^4} + O(A^4)\right)\cos 2\phi$$

$$+\left(\frac{3\,A^3}{16\,k^4} + O(A^5)\right)\cos 3\phi + \left(\frac{A^4}{16\,k^6} + O(A^6)\right)\cos 4\phi + \dots \tag{1.14}$$

$$\omega = \frac{1}{4}\left(3\frac{l^2}{k} - c\,k - k^3\right) + \frac{3\,A^2}{8\,k} + \frac{3\,A^4}{128\,k^5} + O(A^6). \tag{1.15}$$

For small amplitudes (1.14) - (1.15) gives a reasonably good uniform approximation to the cnoidal wave (1.9).

Multidimensional invariant tori for PDEs is still a not completely understood phenomenon, although there are quite a few nontrivial examples of PDEs where families of finite-dimensional invariant tori have been constructed, mainly by applying the methods of algebraic geometry (see, e.g.,[**12, 23, 10**]). One can think of them as of the result of nonlinear interaction of travelling waves solutions, although this operation in general has to be defined. We suggest the following approach to the definition of the nonlinear interaction.

Let the PDE (or a system of PDEs) possess a family of travelling wave solutions of the form (1.5) depending on some vector parameter

$$\mathbf{A} = (k_1, \dots, k_d; \mathbf{a}).$$

It is assumed that the wave vector $k_1, \dots, k_d$ assumes arbitrary values in some domain of $\mathbb{R}^d$,

$$(k_1, \dots, k_d) \in \mathcal{K} \subset \mathbb{R}^d.$$

The amplitude parameter $\mathbf{a}$ belongs to a m-dimensional domain

$$\mathbf{a} \in \mathcal{D} \subset \mathbb{R}^m.$$

Denote

$$\mathcal{A} := \mathcal{K} \times \mathcal{D} \subset \mathbb{R}^{d+m}.$$

The solution (1.5) must satisfy the PDE identically in ϕ^0. Let us assume that, on a certain submanifold of codimension 1,

$$\mathbf{a} \in \mathcal{C} \subset \mathcal{D}, \quad \dim \mathcal{C} = m - 1$$

the solution (1.5) becomes constant. We will only consider the local situation where $\mathcal{D}$ is a small neighborhood of the manifold of constant solutions. Denote ε the distance from $\mathcal{C}$. So, the amplitude parameter is subdivided into

$$\mathbf{a} = (\varepsilon, \mathbf{c}), \quad \mathbf{c} \in \mathcal{C}.$$

For small ε the solution (1.5) must become close to the plane wave

$$u \simeq u_0(\mathbf{c}) + A(\varepsilon, \mathbf{c}) \cos\phi,$$

$$\phi = k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0, \quad \omega \simeq \omega_0(k_1, \ldots, k_d, \mathbf{c})) \tag{1.16}$$

where $\omega_0(k_1, \ldots, k_d, \mathbf{c})$ is the dispersion law of the linearized PDE near the manifold of constant solutions $\mathbf{c} \in \mathcal{C}$, $A(0, \mathbf{c}) = 0$.

Definition 1.2. — We say that the family of n-dimensional invariant tori of the form

$$u = U(\phi_1, \ldots, \phi_n | \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)}),$$

$$\phi_i = k_1^i x_1 + k_2^i x_2 + \cdots + k_d^i x_d - \omega^i t + \phi_i^0, \quad i = 1, \ldots, n \tag{1.17}$$

is obtained as the result of (nonlinear) interaction of n plane waves if the following conditions are fulfilled.
(i) The functions (1.17) are 2π-periodic in $\phi_1, \ldots, \phi_n$.
(ii) As functions of $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ they are analytic on a complement in $\mathcal{A} \times \cdots \times \mathcal{A}$ (n factors) to a collection of finite number of algebraic subvarieties $R_1, \ldots, R_N$

$$(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)}) \in \mathcal{A} \times \cdots \times \mathcal{A} \setminus \cup_{k=1}^N R_k \subset \mathbb{R}^{n\,(d+m)}. \tag{1.18}$$

(iii) Near the manifold of constant solutions the Fourier expansion of the functions (1.17) has the form

$$U(\phi_1, \ldots, \phi_n | \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)}) = u_0(\mathbf{c}_1, \ldots, \mathbf{c}_n; \varepsilon_1, \ldots, \varepsilon_n)$$
$$+ A(\varepsilon_1, \mathbf{c}_1) \cos\phi_1 + \cdots + A(\varepsilon_n, \mathbf{c}_n) \cos\phi_n$$
$$+ \sum_{\mathbf{m} \in \mathbb{Z}^n, |\mathbf{m}| > 1} A_{\mathbf{m}} e^{i\,(m_1\phi_1 + \cdots + m_n\phi_n)}$$
$$\phi_j = k_1^j x_1 + \cdots + k_d^j x_d - \omega^j + \phi_j^0, \quad j = 1, \ldots, n$$
$$\omega^j = \omega_0(k_1^j, \ldots, k_d^j, \mathbf{c}_j) + \sum_{k \geq 1} \omega_k^j$$

$$u_0(\mathbf{c}_1, \ldots, \mathbf{c}_n; \varepsilon_1, \ldots, \varepsilon_n) = u_0(\mathbf{c}_1) + \cdots + u_0(\mathbf{c}_n) + O(\varepsilon). \tag{1.19}$$

The Fourier coefficients

$$A_{\mathbf{m}} = A_{\mathbf{m}}(k_1^1, \dots, k_d^n, \varepsilon_1, \dots, \varepsilon_n, \mathbf{c}_1, \dots, \mathbf{c}_n)$$

must be analytic functions on (1.18). Their Taylor expansions in ε_1, $\dots \varepsilon_n$ near $\mathcal{C} \times \dots \times \mathcal{C}$ must begin with the terms of the order $|\mathbf{m}|$,

$$A_{\mathbf{m}} = O\left(\varepsilon^{|\mathbf{m}|}\right), \quad |\mathbf{m}| = |m_1| + \dots + |m_n|. \tag{1.20}$$

Also in the expansion of the dispersion law the k-th term

$$\omega_k^j = \omega_k^j(k_1^1, \dots, k_d^n, \varepsilon_1, \dots, \varepsilon_n, \mathbf{c}_1, \dots, \mathbf{c}_n) \tag{1.21}$$

must be of the order k in ε. The coefficients of the leading Fourier modes must coincide with the leading coefficients of the plane wave expansions (1.16).

We believe that existence, for any $n \geq 1$, of the analytic families of n-dimensional invariant tori satisfying the assumptions of the Definition 1.2 implies integrability of the PDE. It would be interesting to prove precise mathematical theorems in this direction.

In this paper we pursue a more modest goal. For the example of KP equation we want to prove that, indeed, the analytic families of invariant tori satisfying the conditions of the Definition 1.2 exist for any n. Actually, we will prove that the families of invariant tori obtained by the I.M.Krichever's construction [**22**]) satisfy the assumptions of the Definition. Moreover, we will prove that *all* such analytic families of invariant tori must be given by the Krichever's construction. Our main motivation was the mathematical understanding of the remarkable physical experiments of J.Hammack *et al.* [**18, 19**]. In these experiments the propagation of small amplitude shallow water waves was studied. In a water tank of the size approximately 13×27 m and depth 20 cm the waves were generated by a wavemaker programmed to create a superposition of two cnoidal waves with different directions of propagation and different amplitudes. The measurements of the resulting wave profile proved to be in a remarkable agreement with the two-dimensional invariant tori for KP given in terms of theta-functions (see below). Also some oceanic observation were analyzed in [**19**]; again the agreement with the theta-functional invariant tori looked encouraging.

To our opinion the experimental results suggest the following main question to be addressed: why the multidimensional invariant tori for KP created by Krichever [**22**] with sophisticated algebro-geometric technique are observable in the physical experiments? Putting this in a different way, the mathematical questions to be answered are

- does the Krichever's construction cover all finite-dimensional invariant tori for KP?

- are these tori stable?

One of the difficulties in proving exact statements in this direction is quasiperiodicity of the solutions with respect to the spatial directions. The extension of the finite-dimensional Arnold - Liouville and KAM theory to the infinite-dimensional situation developed in [**25, 7, 21**] mainly refers to the space of spatially periodic functions.

In Section 2 we prove a simple uniqueness statement (see Theorem 2.1 below): all finite-dimensional invariant tori for KP obtained as a result of nonlinear interaction of plane waves in the sense explained above are expressed in terms of theta-functions of Riemann surfaces via the Krichever construction.

In the last Section we extend the technique developed in the proof of Theorem 2.1 to the explicit construction of the moduli space of the KP theta-functions of infinite genus. They are obtained as infinite superpositions of plane waves satisfying certain requirements to ensure convergence of the infinite sums. The KP solutions given in terms of these theta-functions will be quasiperiodic in both space and time variables. For the case of hyperelliptic Riemann surfaces the theory of theta-functions of infinite genus and associated KdV periodic and quasiperiodic solutions was initiated by H.McKean and E.Trubowitz [**27**]. For the KP case, where arbitrary Riemann surfaces can appear in the finite genus case, the infinite genus theory for the *doubly periodic* in (x, y) KP solutions was created by I.Krichever [**24**] (see also [**6**]). The state-of-the-art of the theory of the associated infinite genus theta-functions can be found in the monograph of J.Feldman, H.Knörrer and E.Trubowitz [**15**]. Observe that our approach does not require any assumption about spatial periodicity.

2. Can one see the shape of Riemann surface looking at the water waves?

Let us begin with some preliminaries of the theory of KP equation. Although (1.8) is strictly speaking not an evolutionary PDE, our definition of nonlinear interaction of plane waves makes sense also for the KP case. Observe that the mean value

$$\int u(x,y,t)\,dx$$

is a first integral. We will always consider the solutions with zero mean value. This is not a serious constraint. Indeed, the KP equation is invariant with respect to the action of the group of scaling/Galilean transformations

$$\begin{aligned} x &= cx' + ac^2y' - \frac{1}{2}bc^3t' \\ y &= c^2y' - \frac{3}{2}ac^3t' \\ t &= c^3t' \\ u &= c^{-2}\left[u' + \frac{1}{2}a^2 - \frac{1}{3}b\right] \end{aligned} \tag{2.1}$$

depending on three arbitrary parameters $c \neq 0$, a, b. Using these transformations one can always kill the mean value.

Technically it is more convenient to work with the so-called bilinear form of KP. The substitution

$$u = 2\partial_x^2 \log\tau(x,y,t) \tag{2.2}$$

reduces (1.8) to

$$3\tau_{xx}^2 - 4\tau_x\tau_{xxx} + \tau\,\tau_{xxxx} + 3(\tau_{yy}\tau - \tau_y^2) + 4(\tau_{xt}\tau - \tau_x\tau_t) + 2b\tau^2 = 0. \tag{2.3}$$

Here b is an integration constant. Actually what will be studied is the invariant tori for (2.3) of the form

$$\begin{aligned} \tau(x,y,t) &= A_0 + \sum_{\mathbf{m}\neq 0} A_{\mathbf{m}} e^{i\,(m_1\phi_1+\cdots+m_n\phi_n)}, \\ \phi_j &= k_jx + l_jy - \omega_jt + \phi_j^0, \quad j = 1,\ldots,n \end{aligned} \tag{2.4}$$

Without loss of generality one can assume

$$A_0 = 1.$$

Moreover, doing if necessary suitable shifts along $\phi_1^0, \ldots, \phi_n^0$ one can normalize the leading coefficients in such a way that

$$A_{(-1,0,\ldots,0)} = A_{(1,0,\ldots,0)}, \ \ldots, \ A_{(0,\ldots,0,-1)} = A_{(0,\ldots,0,1)}. \tag{2.5}$$

Let us first recall the construction of the algebro-geometric invariant tori for KP. They are parametrized by quadruples $(\Sigma_n, \infty, \zeta, \sigma)$ where Σ_n is a Riemann surface of genus n with a marked point $\infty \in \Sigma_n$, ζ is a 3-jet of a local parameter on Σ_n near ∞, $\zeta(\infty) = 0$. Finally, σ must be an anticomplex involution

$$\sigma : \Sigma_n \to \Sigma_n, \quad \sigma(\infty) = \infty, \ \sigma^* \zeta = \bar{\zeta} \tag{2.6}$$

such that the fixed-point set of the involution σ consists of $n+1$ components. Call $\mathbf{a}_1, \ldots, \mathbf{a}_n$ the (homology classes of) the suitably oriented components not containing the point ∞. These will be the basic a-cycles on the Riemann surface Σ_n. The conjugated b-cycles can be choosen arbitrarily provided that

$$\sigma_*(\mathbf{a}_j) = \mathbf{a}_j, \ \sigma_*(\mathbf{b}_j) = -\mathbf{b}_j, \quad j = 1, \ldots, n. \tag{2.7}$$

The Fourier coefficients of the algebro-geometric solutions have the form

$$A_{\mathbf{m}} = e^{-\pi \langle \mathbf{m}, \beta \, \mathbf{m} \rangle} \tag{2.8}$$

where $\beta = (\beta_{ij})$ is the real symmetric positive definite $n \times n$ matrix given by the periods of holomorphic differentials

$$\beta_{ij} = -i \oint_{\mathbf{b}_j} w_i, \quad \oint_{\mathbf{a}_j} w_i = \delta_{ij}. \tag{2.9}$$

The wave numbers and frequencies are given in terms of the coefficients of expansions of the basic holomorphic differentials near $\infty \in \Sigma_n$,

$$w_i(P) = \frac{1}{2\,\pi} \left(k_i + l_i \zeta + \omega_i \zeta^2 + O(\zeta^3) \right) d\zeta, \quad P \to \infty. \tag{2.10}$$

The phase shifts ϕ_j^0 can be arbitrary real numbers.

The formulae (2.8) - (2.7) is nothing but the Krichever's construction [**22**] of the algebro-geometric solutions to KP (see also [**10, 13**] regarding the reality conditions). We will refer to the class of quadruples $(\Sigma_n, \infty, \zeta, \sigma)$ described above as the KP Riemann surfaces, and their theta-functions as to the KP theta-functions. Recall that, besides the reality conditions no other constraints are to be imposed on the triple (Σ, ∞, ζ).

Let us call the wave numbers $k_1, \ldots, k_n, l_1, \ldots, l_n$ *resonant* if, for some $i \neq j$

$$k_i = \pm k_j \quad \text{and} \quad l_i k_j = l_j k_i. \tag{2.11}$$

If this is not the case and $k_1 \neq 0, \ldots, k_n \neq 0$ the wave numbers will be called *nonresonant*. From the definition of plane waves it follows that one can assume all wave numbers k_j to be positive.

Theorem 2.1. — *Let (2.4) be a family of solutions to (2.3), for arbitrary phase shifts $\phi_1^0, \ldots, \phi_n^0$, depending analytically on the small parameter ϵ and on the "amplitudes"*

$$a_1 = A_{(1,0,\ldots,0)} > 0,\ a_2 = A_{(0,1,0,\ldots,0)} > 0, \ldots, a_n = A_{(0,0,\ldots,1)} > 0 \tag{2.12}$$

and on arbitrary nonresonant wave numbers $k_1 \neq 0, \ldots, k_n \neq 0, l_1, \ldots, l_n$ such that

$$A_{\mathbf{m}} = O\left(\epsilon^{|m_1|+\cdots+|m_n|}\right). \tag{2.13}$$

Then this family is given by (2.8) - (2.7) for some quadruple $(\Sigma_n, \infty, \zeta, \sigma)$ of the above form.

Proof. — Let us begin with algebraic preliminaries. Denote

$$\mathcal{R} = \mathbb{C}\left[z_1^{\pm 1}, \ldots, z_n^{\pm 1}\right]$$

the ring of Laurent polynomials of n variables. The *degree* of a monomial in $\mathcal{R}$ is defined by

$$\deg z_1^{i_1} \ldots z_n^{i_n} = |i_1| + \cdots + |i_n|.$$

Denote $\mathcal{R}_m$ the subspace of Laurent polynomials of the degree m. The product of Laurent polynomials satisfies

$$\mathcal{R}_i \mathcal{R}_j \subset \oplus_{k=0}^{i+j} \mathcal{R}_k. \tag{2.14}$$

The ring of trigonometric polynomials in $\phi_1, \ldots, \phi_n$ is naturally identified with $\mathcal{R}$ by putting

$$z_j = e^{i\,\phi_j}, \quad j = 1, \ldots, n.$$

So, the above definition and properties of the degree holds true also for trigonometric polynomials.

We can now reformulate the assumptions of the Theorem in the following way. We are looking for a solution to the equation (2.3) in the form

$$\tau = 1 + \epsilon \sum_{j=1}^{n} a_j (z_j + z_j^{-1}) + \sum_{m \geq 2} \epsilon^m \tau^{[m]} \tag{2.15}$$

where

$$\tau^{[m]} = \sum_{k=2}^{m} \tau_k^{[m]}, \quad \tau_k^{[m]} \in \mathcal{R}_k. \tag{2.16}$$

In these formulae we use the superscript $[m]$ for labelling the terms of the order m with respect to ϵ. The coefficients of these trigonometric polynomials along with the coefficients of the expansions

$$\omega_j = \frac{1}{4}\left(3k_j\lambda_j^2 - k_j^3\right) + \sum_{m \geq 1} \epsilon^m \omega_j^{[m]}, \quad j = 1, \ldots, n \tag{2.17}$$

$$b = \sum_{m \geq 1} \epsilon^m b^{[m]} \tag{2.18}$$

are to be determined from the KP equation (2.3). Here we introduce the notation

$$\lambda_j := \frac{l_j}{k_j}, \quad j = 1, \ldots, n.$$

Let us now describe more precisely the result of substitution of the ansatz (2.15) to the KP equation (2.3). We need to introduce the following notations. Put

$$\partial_x = \sum_{j=1}^{n} k_j \frac{\partial}{\partial \phi_j}, \quad \partial_y = \sum_{j=1}^{n} k_j \lambda_j \frac{\partial}{\partial \phi_j}, \quad \partial_t = \sum_{j=1}^{n} (k_j^3 - 3k_j\lambda_j^2) \frac{\partial}{\partial \phi_j}.$$

We also introduce operators

$$\partial_t^{[m]} = \sum_{j=1}^{n} \omega_j^{[m]} \frac{\partial}{\partial \phi_j}, \quad m \geq 1.$$

Finally, the fourth order linear differential operator L will be defined by

$$L = \partial_x^4 + 3\partial_y^2 + 4\partial_x \partial_t. \tag{2.19}$$

Using these notations one can rewrite the result of substitution of the ansatz (2.15) to the KP equation at the order $m \geq 2$ approximation as the following equation:

$$\begin{aligned} &L\,\tau^{[m]} + b^{[m]} + {\sum_{k+l=m}}' \left[4\partial_t^{[k]}\partial_x \tau^{[l]} + 2b^{[k]}\tau^{[l]}\right] \\ &+ {\sum_{i+j=m}}' \left[3\partial_x^2\tau^{[i]}\partial_x^2\tau^{[j]} - 4\partial_x\tau^{[i]}\partial_x^3\tau^{[j]} + \tau^{[i]}\partial_x^4\tau^{[j]} + 4\tau^{[i]}\partial_x\partial_t\tau^{[j]} - 4\partial_x\tau^{[i]}\partial_t\tau^{[j]}\right] \\ &+ {\sum_{i+j+k=m}}' \left[4\partial_t^{[k]}\partial_x\tau^{[i]}\tau^{[j]} - 4\partial_x\tau^{[i]}\partial_t^{[k]}\tau^{[j]} + b^{[k]}\tau^{[i]}\tau^{[j]}\right] = 0. \end{aligned} \tag{2.20}$$

In this formula it is understood that, in the sums $\sum'$ all the summation indices are distinct from zero.

The left hand side of this equation is a trigonometric polynomial in $\phi_1^0, \ldots, \phi_n^0$. Because of the property (2.14), the degree of this differential polynomial is less or equal to m. Since $\phi_1^0, \ldots, \phi_n^0$ are arbitrary variables, we can determine the unknown coefficients just equating the coefficients of the trigonometric polynomials. More specifically, in order to determine the coefficient $a_{i_1\ldots i_n}^{[m]}$ of the trigonometric polynomial

$$\tau^{[m]} = \sum_{2 \leq |i_1| + \cdots + |i_n| \leq m} a_{i_1\ldots i_n}^{[m]} e^{i(i_1\phi_1^0 + \cdots + i_n\phi_n^0)}$$

one is to collect the coefficients of $z^{i_1}\ldots z^{i_n}$ in (2.20). Clearly the resulting expression will depend linearly on $a_{i_1\ldots i_n}^{[m]}$. It will also depend on the lower order coefficients $a_{j_1\ldots j_n}^{[m']}$, $b^{[m']}$ with $m' < m$, and on $\omega_j^{[m']}$ with $m' < m-1$. Here we use the assumption $|i_1| + \cdots + |i_n| \geq 2$. Similarly, in order to compute the coefficient $\omega_j^{[m-1]}$ of the expansion (2.17) one is to collect the terms containing the monomial z_j. Again, it is easy to see that all the coefficients of this monomial depend at most linearly on $\omega_j^{[m-1]}$ and also on $a^{[m']}_{j_1\cdots j_n}$, $b^{[m']}$ with $m' < m$, and on $\omega_j^{[m']}$ with $m' < m-1$. Finally, to determine $b^{[m]}$ it suffices to collect the constant term of the trigonometric polynomial (2.20).

We obtain a recursive procedure for computing the coefficients of the expansions (2.15) - (2.18). This procedure is an analogue of the classical Stokes expansion method explained in the introduction; it also resembles the Lindstedt series method of the classical mechanics (see Chapter XIII of the Poincaré book [**30**]). Let us prove that this procedure works to produce a unique solution for any m.

It is easy that the equations for $b^{[m]}$ and $\omega_j^{[m-1]}$ have unique solutions. Indeed, from the first line of (2.20) it follows that the coefficients of these unknowns are equal to 1. Let us prove that the coefficient of $a^{[m]}_{i_1\dots i_n}$ is not identically equal to zero.

Let us introduce the polynomial in $2n$ variables $k_1, \dots, k_n, \lambda_1, \dots, \lambda_n$ depending on n integer indices $i_1, \dots, i_n$,

(2.21)
$$D(i_1,\dots,i_n) := \left(\sum_{s=1}^n k_s i_s\right)^4 - 3\left(\sum_{s=1}^n k_s\lambda_s i_s\right)^2 - \sum_{s=1}^n k_s i_s \sum_{s=1}^n (k_s^3 - 3k_s\lambda_s^2) i_s.$$

Clearly, the following identity holds true

$$L\, e^{i\,(m_1\phi_1+\cdots+m_n\phi_n)} = D(m_1,\dots,m_n) e^{i\,(m_1\phi_1+\cdots+m_n\phi_n)} \tag{2.22}$$

if

$$\phi_j = k_j x + k_j\lambda_j y + \frac{1}{4}(k_j^3 - 3k_j\lambda_j^2)t + \phi_j^0, \quad j = 1,\dots,n.$$

For example,

$$D(\pm 1, 0, \dots, 0) = \cdots = D(0, \dots, 0, \pm 1) = 0$$
$$D(1,1,0,\dots,0) = 3k_1k_2\left[(k_1+k_2)^2 + (\lambda_1-\lambda_2)^2\right]$$
$$D(1,-1,0,\dots,0) = -3k_1k_2\left[(k_1-k_2)^2 + (\lambda_1-\lambda_2)^2\right]$$

etc. Let us prove that, for arbitrary integers $i_1, \dots, i_n$ satisfying

$$|i_1| + \cdots + |i_n| \geq 2 \tag{2.23}$$

the polynomial $D(i_1,\dots,i_n)$ is not an identical zero. Indeed, collecting the terms of the polynomial that contain the third and fourth powers of the variables $k_1, \ldots, k_n$ yields

$$D(i_1,\ldots,i_n)=\sum_{s=1}^n i_s^2(i_s^2-1)\,k_s^4+\sum_{s\neq t} i_s i_t(4i_s^2-1)k_s^3k_t+\ldots$$

where the periods stand for the terms of lower degree in k_j. If at least one of the indices $i_1, \ldots i_n$ is not equal to zero or to ± 1, then the sum of the fourth powers of k_j does not identically vanish. If this is not the case, at least two indices, say i_s and i_t, $s \neq t$ do not vanish, due to the assumption (2.23). In this case the coefficient of $k_s^3 k_t$ is not equal to zero.

From the above arguments it follows that, all the coefficients $a^{[m]}_{i_1\ldots i_n}$, $\omega_j^{[m-1]}$, $b^{[m]}$ for $m \geq 2$ are uniquely determined from the equation (2.20) in the form of polynomials in $a_1, \ldots, a_n$ with the coefficients being rational functions in $k_1, \ldots, k_n$.

We are now to prove existence of the analytic families of invariant tori of the described form. This will imply, last but not least, the proof of *cancellation of all the divisors* $D(i_1,\ldots,i_n)$ with $|i_1|+\cdots+|i_n|>2$.

To prove existence of the families of invariant tori with needed analytic properties we will use the Krichever construction [**22**] of algebro-geometric solutions of KP. According to this construction an arbitrary Riemann surface Σ_n of genus n with an arbitrary marked point $\infty \in \Sigma_n$ and a 3-jet of a local parameter ζ near ∞, $\zeta(\infty)=0$, gives rise to a family of solutions of KP of the form

$$u(x,y,t)=\partial_x^2\log\theta+\frac{c}{6}$$
$$\theta=\sum_{\mathbf{m}\in\mathbb{Z}^n} e^{-\pi\langle\mathbf{m},\beta\mathbf{m}\rangle}e^{i(m_1\phi_1+\cdots+m_n\phi_n)}$$
$$\phi_j=k_jx+l_jy-\omega_jt+\phi_j^0,\quad j=1,\ldots,n. \tag{2.24}$$

In this formulae $\beta=(\beta_{ij})$ is the period matrix (2.9) of holomorphic differentials on Σ_n with respect to a basis of cycles $\mathbf{a}_1,\ldots,\mathbf{a}_n,\mathbf{b}_1,\ldots,\mathbf{b}_n \in H_1(\Sigma_n,\mathbb{Z})$ normalized by the standard form of the intersection pairing matrix

$$\mathbf{a}_i\circ\mathbf{a}_j=\mathbf{b}_i\circ\mathbf{b}_j=0,\quad \mathbf{a}_i\circ\mathbf{b}_j=\delta_{ij}, \tag{2.25}$$

the wave numbers k_j, l_j and frequencies ω_j are given by the expansions (2.10) of the normalized holomorphic differentials w_j near ∞, $\phi_1^0, \ldots, \phi_n^0$

are arbitrary phase shifts, c is a certain constant. The constant c can be killed by the Galilean transformation

$$u \mapsto u - \frac{c}{6}, \quad x \mapsto x + \frac{c}{4}t$$

corresponding to a suitable change of the 3-jet of the local parameter ζ

$$\zeta \mapsto \zeta - \frac{c}{12}\zeta^3.$$

We will always assume $c = 0$.

The solution (2.24) in general is a complex valued meromorphic function of the variables x, y, t, $\phi_1^0, \ldots, \phi_n^0$. If the triple $(\Sigma_n, \infty, \zeta)$ admits an antiholomorphic involution σ satisfying (2.6) such that the fixed-point set of the involution consists of $n+1$ ovals, then the period matrix β_{ij} and the wave numbers and frequencies are all real provided the basis of cycles is chosen in the way described in the Theorem. Moreover [**14**], the theta-function in (2.24) takes positive values for all real phase shifts $\phi_1^0, \ldots, \phi_n^0$, and the solution $u(x, y, t)$ is real-valued and smooth. Therefore, in this case, the Krichever formulae (2.24) define a n-dimensional invariant torus for KP. It will also be invariant for all the flows of the KP hierarchy. Conversely, from reality and smoothness of the solution (2.24) on the torus generated by the flows of the KP hierarchy it follows that $(\Sigma_n, \infty, \zeta)$ must admit the antiholomorphic involution with the above properties [**13**].

We will now produce the needed analytic family of n-dimensional invariant tori for KP considering the families of solutions (2.24) with "small" a-cycles.

Let us consider the family of Riemann surfaces of the above form depending on n sufficiently small parameters $s_1, \ldots, s_n$ such that, in the limit $s_j \to 0$ the j-th cycle $\mathbf{a}_j$ is squeezed to zero such that

$$\Sigma_n(s)|_{s_j=0} \tag{2.26}$$

is a genus $n-1$ curve with an ordinary double point. Construction of such a deformation can be found in the Chapter III of the Fay's book [**14**]. The following statements proved in [**14**] will be essential for us.

First, denote $\hat{\Sigma}_n^j$ the normalization of (2.26) and $P_j^{\mp}$ the two points of the normalization to be identified on the nodal curve. The basic normalized holomorphic differential $w_j(s)$ on $\Sigma_n(s)$ in the limit $s_j \to 0$ goes to the normalized third kind differential on $\hat{\Sigma}_n^j$ with simple points with the residues $\pm\frac{1}{2\pi i}$ in the points $P_j^{\mp}$ resp. Other normalized holomorphic differentials w_k on $\Sigma_n(s)$ go to the normalized holomorphic differentials on $\hat{\Sigma}_n^j$. The same claim holds true for limits of normalized 2nd and 3d kind differentials on $\Sigma_n(s)$ with pole away from the pinched cycle. The diagonal entry β_{jj} has logarythmic behaviour as $s_j \to 0$,

$$\beta_{jj} = -\log s_j + O(1),$$

other matrix entries have regular expansions in s_j.

Iterating this procedure, in the limit $s_1 \to 0, \ldots, s_n \to 0$ the Riemann surface Σ_n goes to the rational nodal curve with n pairs of identified points $z_1^{\mp}, \ldots, z_n^{\mp}$. The basic holomorphic differentials take the limiting values

$$w_j = \frac{1}{2\pi i}\left(\frac{1}{z - z_j^-} - \frac{1}{z - z_j^+}\right) dz, \quad j = 1, \ldots, n. \tag{2.27}$$

We will assume that the marked point $\infty \in \Sigma_n(s)$ corresponds to the point $z = \infty$ of the limiting Riemann sphere and that the local parameter ζ on $\Sigma_n(s)$ goes to

$$\zeta = \frac{1}{z}$$

on the Riemann sphere near infinity. Comparing the expansions

$$w_j = -\frac{1}{2\pi i}\left[(z_j^- - z_j^+) + ((z_j^-)^2 - (z_j^+)^2)\zeta + ((z_j^-)^3 - (z_j^+)^3)\zeta^2 + O(\zeta^3)\right] d\zeta$$

with the formulae (2.10) expressing the wave numbers and frequencies in terms of expansion near ∞ of the basic normalized holomorphic differentials we conclude that the identified points must have the form

$$z_j^{\pm} = \frac{1}{2}(\lambda_j \pm i k_j), \quad j = 1, \ldots, n \tag{2.28}$$

Observe that the nonresonancy condition (2.11) means that all $2n$ points (2.28) are pairwise distinct.

The crucial point in proving cancellation of all small divisors but those corresponding to the resonances (2.11) is in proving that arbitrary configuration of the pairwise distinct double points (2.28) on the Riemann sphere can be obtained by the above n-parametric degeneration procedure within the family of KP Riemann surfaces.

Let $\beta_{ij}(s)$ be the period matrix (2.9) of the family of Riemann surfaces with respect to the basis of cycles that will be assumed to be continuosly depending on the parameter s. Denote

$$a_j(s) = e^{-\pi\beta_{jj}(s)}, \quad j = 1, \ldots, n. \tag{2.29}$$

At $s = 0$ one has

$$a_1(0) = \cdots = a_n(0) = 0.$$

The off-diagonal entries of the matrix $\beta_{ij}(s)$ admit finite limits at $s \to 0$ and

$$e^{-2\pi\beta_{ij}(0)} = \frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2}, \quad i \neq j. \tag{2.30}$$

The wave numbers $k_j(s), l_j(s)$ and the frequencies $\omega_j(s)$ defined from the expansions (2.10) also admit the limits as $s \to 0$ of the form

$$k_j(0) = k_j, \quad l_j(0) = k_j\lambda_j, \quad \omega_j(0) = \frac{1}{4}(3k_j\lambda_j^2 - k_j^3), \quad j = 1, \ldots, n. \tag{2.31}$$

We are now to prove that, for arbitrary nonresonant real numbers k_1, $\ldots$, k_n and arbitrary real numbers l_1, $\ldots$, l_n and for arbitrary sufficiently small positive numbers a_1, $\ldots$, a_n there exists a family of triples $(\Sigma_n, \infty, \zeta)$ of the above form depending analytically on the parameters $a_1, \ldots, a_n, k_1, \ldots, k_n, l_1, \ldots, l_n$. To this end we are to introduce theta-functions of the second order.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be a vector with all components $\nu_j = 0$ or 1. Such a vector will be called *characteristic*. Define second order theta-function $\tilde\theta[\nu](\phi|\beta)$ with the characteristic ν by

$$\tilde\theta[\nu](\phi|\beta) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \prod_{i=1}^{n} a_i^{2(m_i^2 + m_i\nu_1)} \prod_{i<j} Z_{ij}^{2m_im_j + m_i\nu_j + m_j\nu_i} e^{i((2m_1+\nu_1)\phi_1 + \cdots + (2m_n+\nu_n)\phi_n)}. \tag{2.32}$$

Here

$$a_j = e^{-\pi\beta_{jj}}, \quad Z_{ij} = e^{-2\pi\beta_{ij}}, \quad i \neq j. \tag{2.33}$$

Our definition of the second order theta-functions differs from the standard one (see, e.g. [**14**]) by the factor

$$\frac{1}{2}\prod_{i=1}^{n} a_i^{-\frac{\nu_i^2}{2}} \prod_{i<j} Z_{ij}^{-\frac{\nu_i\nu_j}{2}}.$$

The advantage of our normalization is that, the functions (2.32) are real analytic in the variables $Z_{ij}>0$, $a_j\geq 0$, $\phi_k\in\mathbb{R}$ provided that the lowest eigenvalue ρ of the symmetric off-diagonal matrix

$$\log Z_{ij}$$

satisfies

$$\rho < 2\pi \log a_j^{-2}, \quad j=1,\dots,n. \tag{2.34}$$

Actually, (2.32) are even functions in a_1, ..., a_n. In particular,

$$\tilde\theta[0] = 1 + 2\sum_{i=1}^{n} a_i^2 \cos 2\phi_i + O(a^4)$$

(2.35)

$$\tilde\theta[\mathbf{n}_i] = \cos\phi_i + \sum_{j\neq i} a_j^2 \left[Z_{ij}\cos(2\phi_j+\phi_i) + Z_{ij}^{-1}\cos(2\phi_j-\phi_i)\right] + O(a^4)$$

(2.36)

$$\begin{aligned}\tilde\theta[\mathbf{n}_{ij}] &= \cos(\phi_i+\phi_j) + Z_{ij}^{-1}\cos(\phi_i-\phi_j)\\ &+\sum_{k\neq i,j} a_k^2 \left[Z_{ik}Z_{jk}\cos(2\phi_k+\phi_i+\phi_j) + Z_{ik}^{-1}Z_{jk}^{-1}\cos(2\phi_k-\phi_i-\phi_j)\right.\\ &\left.+Z_{ij}^{-1}\left(Z_{ik}Z_{jk}^{-1}\cos(\phi_i-\phi_j+2\phi_k) + Z_{ik}^{-1}Z_{jk}\cos(\phi_i-\phi_j-2\phi_k)\right)\right]\\ &+O(a^4)\end{aligned}$$

(2.37)

In these formulae $\mathbf{n}_i$ stands for the characteristic with the i-th component 1 and all others 0,

$$\mathbf{n}_{ij} = \mathbf{n}_i + \mathbf{n}_j, \quad i\neq j.$$

The following statement was proven in [**8**] (cf. also [**28**], [**10**]).

Lemma 2.2. — *The function*

$$\tau(x,y,t) = \theta(\phi|\beta) = \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{-\pi\langle\mathbf{m},\beta\mathbf{m}\rangle} e^{i(m_1\phi_1+\dots+m_n\phi_n)},$$

$$\phi_j = k_j x + l_j y - \omega_j t + \phi_j^0, \quad j = 1, \dots, n$$

satisfies (2.3) for arbitrary phase shifts $\phi_1^0, \dots, \phi_n^0$ iff *the vectors* $k = (k_1, \dots, k_n)$, $l = (l_1, \dots, l_n)$, $\omega = (\omega_1, \dots, \omega_n)$ *and the matrix* $\beta = (\beta_{ij})$ *satisfy the following system of equations*

$$f[\nu](k, l, \omega, \beta) := \left(\partial_k^4 + 3\partial_l^2 - 4\partial_k\partial_\omega + b\right) \tilde{\theta}[\nu](\phi|\beta)|_{\phi=0} = 0 \tag{2.38}$$

for some constant $b = b(k, l, \omega, \beta)$ *and for arbitrary characteristic* $\nu \in (\mathbb{Z}/2\mathbb{Z})^n$. *Here*

$$\partial_k := \sum k_j \frac{\partial}{\partial \phi_j}, \quad \partial_l := \sum l_j \frac{\partial}{\partial \phi_j}, \quad \partial_\omega := \sum \omega_j \frac{\partial}{\partial \phi_j}.$$

In particular, the equations (2.38) remain valid for the values

$$k = k(s), \quad l = l(s), \quad \omega = \omega(s), \quad \beta = \beta(s)$$

of our family of Riemann surfaces for a suitable constant $b = b(s)$. Indeed, it can be readily checked that, at the limit $s = 0$ the equations (2.38) hold true by substituting $a_1^2 = \cdots = a_n^2 = 0$ and the values Z_{ij}, k_j, l_j, ω_j from (2.30), (2.31) and $b = 0$.

We will now prove that the system (2.38) has unique solution of the form

$$\begin{aligned}
&Z_{ij} = Z_{ij}(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n), \\
&\omega_j = \omega_j(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n), \\
&b = b(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n) \\
&Z_{ij}(0, \dots, 0, k_1, \dots, k_n, l_1, \dots, l_n) = \frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2} \\
&\omega_j(0, \dots, 0, k_1, \dots, k_n, l_1, \dots, l_n) = \frac{1}{4}\left(3k_j\lambda_j^2 - k_j^3\right) \\
&b(0, \dots, 0, k_1, \dots, k_n, l_1, \dots, l_n) = 0
\end{aligned}$$

(2.39)

analytic for sufficiently small $a_1^2, \dots, a_n^2$ and for arbitrary nonresonant vectors k and l. Let us first construct such analytic solution for the subsystem

$$f[0] = 0, \quad f[\mathbf{n}_i] = 0, \; i = 1, \dots, n, \quad f[\mathbf{n}_{ij}] = 0, \; 1 \le i < j \le n. \tag{2.40}$$

To this end let us fix the nonresonant vectors k^0 and λ^0 and choose a real positive number A such that the symmetric matrix β^0_{ij} with

$$\beta^0_{jj} = -2\log a_j, \quad \beta^0_{ij} = -\log\frac{(k_i^0-k_j^0)^2+(\lambda_i^0-\lambda_j^0)^2}{(k_i^0+k_j^0)^2+(\lambda_i^0-\lambda_j^0)^2}, \ i\neq j$$

is positive definite for

$$0 < a_j < A, \quad j = 1, \dots, n.$$

Then the functions $f[0]$, $f[\mathbf{n}_i]$, $f[\mathbf{n}_{ij}]$ will be real analytic in a, Z, k, λ, ω, b for

$$0 \leq a_j < A', \quad j = 1, \dots, n$$

for some $A' < A$ and for Z, k, l, ω, b sufficiently close to

$$Z^0_{ij} = \frac{(k_i^0-k_j^0)^2+(\lambda_i^0-\lambda_j^0)^2}{(k_i^0+k_j^0)^2+(\lambda_i^0-\lambda_j^0)^2}, \ k^0, \ \lambda^0, \ \omega^0_j = \frac{k^0_j}{4}\left(3(\lambda^0_j)^2-(k^0_j)^2\right), b^0 = 0$$

respectively. For $a_1 = \cdots = a_n = 0$ the system (2.40) has unique solution given by (2.39). We derive existence of such solution to (2.40) for positive small a by applying the implicit function theorem (cf [**9**]). Indeed, from the formulae (2.35) - (2.37) it readily follows that, at $a_1^2 = 0, \dots, a_n^2 = 0$

$$\begin{aligned}
&\frac{\partial f[0]}{\partial b} = 1,\\
&\frac{\partial f[0]}{\partial \omega_j} = 0, \quad \frac{\partial f[\mathbf{n}_i]}{\partial \omega_j} = k_i\delta_{ij},\\
&\frac{\partial f[0]}{\partial Z_{pq}} = 0, \quad \frac{\partial f[\mathbf{n}_i]}{\partial Z_{pq}} = 0,\\
&\frac{\partial f[\mathbf{n}_{ij}]}{\partial Z_{pq}} = 3Z_{ij}^{-2}k_ik_j\left[(k_i-k_j)^2+(\lambda_i-\lambda_j)^2\right]\delta_{ip}\delta_{jq}, \quad i<j, \ p<q.
\end{aligned}$$

(2.41)

We obtain a triangular Jacobi matrix with the nonvanishing diagonal. This proves existence of the needed analytic solution.

Explicitly, the expansion of the needed solution reads

$$Z_{ij} = \frac{\rho_{ij}^-}{\rho_{ij}^+}\left\{1 + 32\frac{k_i k_j}{\left[\rho_{ij}^+\rho_{ij}^-\right]^2}\left[a_i^2 k_i^2 p_{ij} + a_j^2 k_j^2 p_{ji}\right]\right.$$

$$\left. +256\frac{k_i k_j}{\rho_{ij}^+\rho_{ij}^-}\sum_{k\neq i,j}\frac{a_k^2 k_k^4 q_{ijk}}{\rho_{ik}^+\rho_{ik}^-\rho_{jk}^+\rho_{jk}^-}\right\} + O(a^4) \tag{2.42}$$

where

$$\rho_{ij}^{\pm} = (k_i \pm k_j)^2 + (\lambda_i - \lambda_j)^2, \quad i \neq j \tag{2.43}$$

$$p_{ij} = \left(k_i^2 - k_j^2\right)^2 + 2\left(3k_i^2 - k_j^2\right)(\lambda_i - \lambda_j)^2 - 3(\lambda_i - \lambda_j)^4 \tag{2.44}$$

$$\begin{aligned} q_{ijk} &= \\ &= \left[(\lambda_k - \lambda_j)(k_i^2 - 3\lambda_i^2) + (\lambda_j - \lambda_i)(k_k^2 - 3\lambda_k^2) + (\lambda_i - \lambda_k)(k_j^2 - 3\lambda_j^2)\right] \\ &\times \left[(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)\right. \\ &\left.+\lambda_i(k_j^2 - k_k^2) + \lambda_j(k_k^2 - k_i^2) + \lambda_k(k_i^2 - k_j^2)\right] \end{aligned} \tag{2.45}$$

$$\omega_i = \frac{1}{4}\left(3k_i\lambda_i^2 - k_i^3\right) + 6k_i\left[a_i^2 k_i^2 + 8\sum_{j\neq i}\frac{a_j^2 k_j^4(\lambda_i - \lambda_j)^2}{\rho_{ij}^+\rho_{ij}^-}\right] + O(a^4) \tag{2.46}$$

$$b = -6\sum a_i^2 k_i^4 + O(a^4). \tag{2.47}$$

Let us now prove that the solution (2.42) - (2.47) to the subsystem (2.40) also satisfies the whole system (2.38).

Lemma 2.3. — *Let Z_{ij}^0 be the value of the functions (2.42) at a point a^0, k^0, l^0 (nonresonancy of k_j^0, l_j^0 is assumed). Then the system of equations*

$$Z_{ij}(a_1^2, \ldots, a_n^2, k_1, \ldots, k_n, l_1, \ldots, l_n) = Z_{ij}^0, \quad 1 \leq i < j \leq n \tag{2.48}$$

for sufficiently small

$$\sum\left(a_j^2 - (a_j^0)^2\right)^2 + \sum(k_j - k_j^0)^2 + \sum(\lambda_j - \lambda_j^0)^2$$

has three-dimensional variety of solutions.

Proof. — Let us first establish validity of the claim of the Lemma for $a_1^0 = \cdots = a_n^0$. Let us rewrite the formula (2.30) in the form of the cross-ratio

$$\frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2} = (z_i^+, z_i^-, z_j^+, z_j^-) \equiv \frac{z_i^+ - z_j^+}{z_j^+ - z_i^-} \cdot \frac{z_j^- - z_i^-}{z_i^+ - z_j^-} \tag{2.49}$$

where the complex numbers $z_i^\pm$ are defined in (2.28). Because of invariance of the cross-ratio with respect to the Möbius group

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

the space of complex solutions to the system

$$\frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2} = \frac{(k_i^0 - k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}{(k_i^0 + k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}, \quad 1 \leq i < j \leq n$$

is at least three-dimensional. The subgroup $PSL_2(\mathbb{R})$ of the Möbius group preserves reality of the numbers k_j, λ_j. So the dimension of the space of real solutions is also greater or equal to three. It is easy to see that this dimension cannot be greater than 3. This proves the Lemma in the limiting case $a^0 = 0$.

Let us now extend the $PSL_2(\mathbb{R})$-symmetry onto the whole space of solutions to the equations (2.40). We first rewrite the symmetry in the infinitesimal form with the generator

$$X_0 = \sum \left[\frac{1}{4}p(\lambda_j^2 - k_j^2) + q\lambda_j + r\right]\frac{\partial}{\partial \lambda_j} + \sum \left[\frac{1}{2}p\lambda_j k_j + qk_j\right]\frac{\partial}{\partial k_j}. \tag{2.50}$$

Here p, q, r are arbitrary real parameters. The one-parameter subgroups corresponding to q and r have a clear meaning: these are the groups of scaling transformations of k and λ and diagonal shifts of λ respectively,

$$\begin{aligned} &k_j \mapsto c\,k_j, \ \lambda_j \mapsto c\,\lambda_j, \ j = 1, \dots, n, \quad c \neq 0 \\ &\lambda_j \mapsto \lambda_j + a, \ j = 1, \dots, n. \end{aligned} \tag{2.51}$$

They are clearly also symmetries of the full system inducing the transformation

$$l_j \mapsto l_j + ak_j, \ \omega_j \mapsto \omega_j + \frac{3}{2}al_j + \frac{3}{4}a^2 k_j, \ j = 1, \dots, n.$$

The generator of the one-parameter subgroup corresponding to p can be recast into the form

$$X_0^{(p)} = \sum \frac{1}{2} l_j \frac{\partial}{\partial k_j} + \omega_j^0 \frac{\partial}{\partial l_j}, \quad \omega_j^0 = \frac{1}{4}(3k_j\lambda_j^2 - k_j^3). \tag{2.52}$$

Remarkably, in this form the transformations (2.52) yield symmetries of the full system (2.38) when ω_j^0 is replaced by the exact solution ω_j of the system. This deep result is one of the important steps in the proof of the Shiota theorem [**31**]. It follows from the following claim [**31**]: compatibility of the system (2.38) implies compatibility of the system

$$\left(2\partial_k^3\partial_l + 4\partial_l\partial_\omega - 4\partial_k\partial_{\dot\omega} + \dot b\right)\tilde\theta[\nu](\phi|\beta)|_{\phi=0} = 0 \tag{2.53}$$

for some vector $\dot\omega$ and some constant $\dot b$. From uniqueness of such a vector it follows that $\dot\omega$ coincides with the derivative of ω along the vector field

$$X^{(p)} = \sum \frac{1}{2} l_j \frac{\partial}{\partial k_j} + \omega_j \frac{\partial}{\partial l_j}. \tag{2.54}$$

The lemma is proved. □

We are now ready to complete the proof of the Theorem. According to Lemma 2.3 combined with Torelli theorem [**17**], the dimension of the space of solutions to the system (2.38) is equal to the dimension of the moduli space of (real) Riemann surfaces of genus n plus 3, i.e., it is equal to $3n$ for $n \geq 2$. We have described the $3n$-dimensional manifold of solutions (2.42), (2.46) to the subsystem (2.40) that by construction contains the solutions of the form (2.9) - (2.7) for Riemann surfaces with sufficiently small real ovals $\mathbf{a}_1, \ldots, \mathbf{a}_n$. The dimension counting proves coincidence of these two families. In particular this implies that all the remaining equations of the system (2.38) hold true on the space of solutions (2.42) - (2.47). Therefore the unique solution to KP defined in (2.15) - (2.17) starting from a given nonresonant wave numbers $k_1, \ldots, k_n, l_1, \ldots, l_n$ and arbitrary sufficiently small amplitudes $a_1, \ldots, a_n$ must have the form (2.8) - (2.7). Uniform convergence of the series (2.24) for theta-functions together with cancellation of all the divisors but $D(\mathbf{n}_i \pm \mathbf{n}_j)$, $i \neq j$ implies analyticity of the family of invariant tori. The Theorem is proved. □

Remark 2.4. — Explicitly, the extension of the symmetry (2.52) onto the full space of solutions to (2.38) reads

$$X^{(p)} = \sum \frac{1}{2}\lambda_j k_j \frac{\partial}{\partial k_j} + \frac{1}{4}(\lambda_j^2 - k_j^2)\frac{\partial}{\partial \lambda_j}$$

$$+6\sum \left[a_i^2 k_i^2 + 8 \sum_{j\neq i} \frac{a_j^2 k_j^4 (\lambda_i - \lambda_j)^2}{\rho_{ij}^+ \rho_{ij}^-} + O(a^4)\right] \frac{\partial}{\partial \lambda_j} \tag{2.55}$$

Together with the fields

$$X^{(q)} = \sum k_j \frac{\partial}{\partial k_j} + \lambda_j \frac{\partial}{\partial \lambda_j}$$

and

$$X^{(r)} = \sum \frac{\partial}{\partial \lambda_j}$$

it generates the action of $PSL_2(\mathbb{R})$ on the space of solutions of the system (2.38):

$$\left[X^{(q)}, X^{(p)}\right] = X^{(p)}, \quad \left[X^{(r)}, X^{(p)}\right] = \frac{1}{2} X^{(q)}, \quad \left[X^{(r)}, X^{(q)}\right] = X^{(r)}.$$

The vector field $X^{(p)}$ generates infinitesimal changes of the marked point $\infty \in \Sigma_n$. In other words, integrating the vector field (2.55) one obtains, for $n > 1$, the Riemann surface with the parameters $a_1^2, \ldots, a_n^2$, $k_1, \ldots, k_n$, $\lambda_1, \ldots, \lambda_n$. This construction gives an answer to the question put in the title of the Section. It would be important however to elaborate more practical tools in the analysis of the experimental water wave data in order to measure the moduli of the Riemann surface "hidden" behind the water wave picture. For the case of two interacting plane waves such tools has been developed in [**18, 19**].

3. Infinite genus theta-functions of Riemann surfaces without Riemann surfaces

The invariant tori for KP identified in the previous Section as the result of nonlinear interaction of n plane waves with small amplitudes $2a_1, \ldots, 2a_n$ can be represented as infinite sums of homogeneous polynomials in $a_1, \ldots, a_n$ of various degrees with coefficients depending on the phases

$\phi_1, \ldots, \phi_n$ and on the wave numbers $k_1, \ldots, k_n$, $l_1 = k_1\lambda_1, \ldots, l_n = k_n\lambda_n$. Let us recast this sum in the following way. For any subset

$$I = \{i_1, \ldots, i_g\} \subset \{1, 2, \ldots, n\}, \quad g > 0$$

denote

$$\theta_I = a_{i_1} \ldots a_{i_g} \Delta\theta_I \tag{3.1}$$

the sum of all monomials that contain only a_i for $i \in I$. We put

$$\theta_\emptyset = 1.$$

Denote also

$$\phi_I = \{\phi_{i_1}, \ldots, \phi_{i_g}\}, \quad a_I = \{a_{i_1}, \ldots, a_{i_g}\},$$
$$k_I = \{k_{i_1}, \ldots, k_{i_g}\}, \quad \lambda_I = \{\lambda_{i_1}, \ldots, \lambda_{i_g}\}.$$

Lemma 3.1. — *The genus n KP theta-function described in the Theorem 2.1 can be represented in the form*

$$\theta(\phi_1, \ldots, \phi_n|\beta) = \sum_I \theta_{|I|}(\phi_I|a_I, k_I, \lambda_I) \tag{3.2}$$

where the summation takes place over all subsets $I \subset \{1, 2, \ldots, n\}$. The functions $\theta_{|I|}(\phi_I|a_I, k_I, \lambda_I)$ are real analytic for all real nonresonant vectors k_I, λ_I and for sufficiently small nonnegative amplitudes a_I. The terms of this expansion can be uniquely determined from the system of the form (2.38) with $n \mapsto |I|$ by requiring that the sum

$$\sum_{J \subset I} \theta_{|J|}(\phi_J|a_J, k_J, \lambda_J) \tag{3.3}$$

with

$$\phi_i = k_i(x + \lambda_i y) - \omega_i^I t + \phi_i^0, \quad i \in I$$

with some vector ω^I satisfies KP.

Here $|I|$ is the cardinality of the set I. It should be emphasized that the radii of convergence

$$a_{i_1} < r_{i_1}, \quad a_{i_g} < r_{i_g}$$

of the series depend on k_I, λ_I.

Proof. — This statement is almost obvious since, supressing all the amplitudes $a_j = 0$ for $j \in \{1, 2, \ldots, n\} \setminus I$ one reduces a theta-function of the genus n to another one of the genus $|I|$. $\square$

We will also redenote the functions $\Delta\theta_I$ by $\Delta\theta_g$ with $g = |I|$. Explicitly, from (2.42) - (2.45) it follows that

$$\Delta\theta_1(\phi|a) = 2\sum_{n>0} a^{n^2-1}\cos n\phi \tag{3.4}$$

$$\begin{aligned}
&\Delta\theta_2(\phi_1,\phi_2|a_1,a_2,k_1,k_2,l_1,l_2) = 2\left[\frac{\rho_{12}^-}{\rho_{12}^+}\cos(\phi_1+\phi_2) + \frac{\rho_{12}^+}{\rho_{12}^-}\cos(\phi_1-\phi_2)\right]\\
&+64\frac{k_1k_2}{[\rho_{12}^+\rho_{12}^-]^2}\left(a_1^2k_1^2p_{12} + a_2^2k_2^2p_{21}\right)\left[\frac{\rho_{12}^-}{\rho_{12}^+}\cos(\phi_1+\phi_2) - \frac{\rho_{12}^+}{\rho_{12}^-}\cos(\phi_1-\phi_2)\right]\\
&+O(a^6)
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\Delta\theta_3(\phi_1,\phi_2,\phi_3|a_1,a_2,a_3,k_1,k_2,k_3,l_1,l_2,l_3)\\
&= 2\left[\frac{\rho_{12}^-\rho_{23}^-\rho_{31}^-}{\rho_{12}^+\rho_{23}^+\rho_{31}^+}\cos(\phi_1+\phi_2+\phi_3) + \frac{\rho_{12}^-\rho_{23}^+\rho_{31}^+}{\rho_{12}^+\rho_{23}^-\rho_{31}^-}\cos(\phi_1+\phi_2-\phi_3)\right.\\
&\left.+\frac{\rho_{12}^+\rho_{23}^+\rho_{31}^-}{\rho_{12}^-\rho_{23}^-\rho_{31}^+}\cos(\phi_1-\phi_2+\phi_3) + \frac{\rho_{12}^+\rho_{23}^-\rho_{31}^+}{\rho_{12}^-\rho_{23}^+\rho_{31}^-}\cos(-\phi_1+\phi_2+\phi_3)\right]\\
&+512\frac{k_1k_2k_3q_{123}}{\rho_{12}^+\rho_{12}^-\rho_{23}^+\rho_{23}^-\rho_{31}^+\rho_{31}^-}\left\{a_1k_1^3\left[\frac{\rho_{23}^-}{\rho_{23}^+}\cos(\phi_2+\phi_3) - \frac{\rho_{23}^+}{\rho_{23}^-}\cos(\phi_2-\phi_3)\right]\right.\\
&+a_2k_2^3\left[\frac{\rho_{31}^-}{\rho_{31}^+}\cos(\phi_3+\phi_1) - \frac{\rho_{31}^+}{\rho_{31}^-}\cos(\phi_3-\phi_1)\right]\\
&\left.+a_3k_3^3\left[\frac{\rho_{12}^-}{\rho_{12}^+}\cos(\phi_1+\phi_2) - \frac{\rho_{12}^+}{\rho_{12}^-}\cos(\phi_1-\phi_2)\right]\right\} + O(a^5)
\end{aligned} \tag{3.6}$$

In these formulae we use the same notations as in the previous Section, i.e., the polynomials $\rho_{ij}^{\pm}$, p_{ij}, q_{ijk} in the variables $k_1, \ldots, k_n, \lambda_1, \ldots, \lambda_n$ whith

$$\lambda_j = l_j/k_j$$

are defined in (2.43) - (2.45). Recall that, in order to obtain a solution $\tau(x, y, t)$ to the KP equation (2.3) one is to substitute in (3.2)

$$\phi_j = k_j x + l_j y - \omega_j t + \phi_j^0$$

with arbitrary phase shifts ϕ_j^0 and the frequencies represented by a decomposition similar to (3.2)

$$\omega_j = \omega_j^0(k_j, \lambda_j) + \Delta\omega_j^1(a_j, k_j) + \sum_{i \neq j} \Delta\omega_j^2(a_j, a_i, k_i, k_j, \lambda_i, \lambda_j) + \dots \tag{3.7}$$

In this expansion

$$\omega_j^0(k_j, \lambda_j) = \frac{1}{4}(k_j\lambda_j^2 - k_j^3)$$

is the dispersion law of the linearized KP,

$$\omega_j^n(a_1, \dots, a_n, k_1, \dots, k_n, \lambda_1, \dots, \lambda_n)$$

is the "pure genus n" contribution into the nonlinear dispersion law (3.7) to be found from the system (2.40) of the genus n and then subtracting the lower genera contributions. Explicitly,

$$\omega_i^1 = 6k_i^3 \left(a_i^2 + 3a_i^4 + O(a_i^6)\right) \tag{3.8}$$

$$\omega_i^2 = 48k_i \sum_{j \neq i} \frac{a_j^2 k_j^4 (\lambda_i - \lambda_j)^2}{\rho_{ij}^+ \rho_{ij}^-} + O(a^4) \tag{3.9}$$

etc. The genus g term

$$\begin{aligned}
&a_{i_1} \dots a_{i_g} \Delta\theta_g(\phi_{i_1}, \dots, \phi_{i_g} | a_{i_1}, \dots, a_{i_g}, k_{i_1}, \dots, k_{i_g}, \lambda_{i_1}, \dots, \lambda_{i_g}) \\
&\phi_j = k_j x + k_j \lambda_j y - \omega_j t + \phi_j^0 \\
&\omega_j = \omega_j^0(k_j, \lambda_j) \\
&+\Delta\omega_j^1(a_j, k_j) + \dots + \Delta\omega_i^g(a_{i_1}, \dots, a_{i_g}, k_{i_1}, \dots, k_{i_g}, \lambda_{i_1}, \dots, \lambda_{i_g})
\end{aligned} \tag{3.10}$$

is created as the result of interaction of g plane waves

$$\begin{aligned}
&2a_{i_1} \cos[k_{i_1}(x + \lambda_{i_1} y) - \omega_{i_1}^0(k_{i_1}, \lambda_{i_1})t + \phi_{i_1}^0] + \dots \\
&+2a_{i_g} \cos[k_{i_g}(x + \lambda_{i_g} y) - \omega_{i_g}^0(k_{i_g}, \lambda_{i_g})t + \phi_{i_g}^0]
\end{aligned}$$

and their harmonics. If the amplitudes of the plane waves are of order ϵ then their g-tuple interaction is of the order ϵ^g. In other words, to compute the solution $\tau(x, y, t)$ of (2.3) of genus $N >> 1$ with the accuracy

ϵ^n for $n < N$ it suffices to sum the expansions of the form (3.10) with $g \leq n$ truncating them at the order n. The result of the truncation will give uniform in the whole plane $(x, y) \in \mathbb{R}^2$ approximation of the genus N solution for the times $|t| < O(\epsilon^{-n})$.

We want to generalize the expansion (3.2) to the case of interaction of infinite number of plane waves. Given infinite sequences of real numbers (3.11)

$$a = (a_1, a_2, \dots),\ a_j > 0, \quad k = (k_1, k_2, \dots),\ k_j > 0, \quad \lambda = (\lambda_1, \lambda_2, \dots)$$

we can construct a formal Fourier series of infinite number of variables $\phi = (\phi_1, \phi_2, \dots)$ representing it as the following power series in a

$$\theta(\phi|a, k, \lambda) = \sum_{g=0}^{\infty} \sum_{|I|=g} a_I \Delta\theta_g(\phi_I | a_I, k_I, \lambda_I). \tag{3.12}$$

The summation takes place over all finite subsets $I \subset \mathbb{N}$. This formal expression makes sense for finite sequences of amplitudes a, i.e., assuming that $a_j = 0$ for $j \geq N$ for some big N. In that case it reduces, for sufficiently small $a_1, \dots, a_N$, to the KP theta function of genus N.

If all the amplitudes $a_1, a_2, \dots$ do not vanish, then, at each order in a one is to summate infinite series. E.g., at the order one (3.12) gives

$$2 \sum_{i=1}^{\infty} a_i \cos \phi_i,$$

at the order two

$$2 \sum_{i<j} a_i a_j \left[\frac{\rho_{ij}^-}{\rho_{ij}^+} \cos(\phi_i + \phi_j) + \frac{\rho_{ij}^+}{\rho_{ij}^-} \cos(\phi_i - \phi_j) \right]$$

etc. We will now give simple sufficient conditions for convergence of the series (3.12) for infinite sequencies of the data (3.11). To this end we are to recall some important points of the theory of infinite dimensional theta-functions, following the book [**15**].

Let $\beta = \beta_{ij}$ be an infinite symmetric matric with real values, $i, j = 1, 2, \dots$. We say that the matrix β satisfies FKT condition if there exists a sequence of positive numbers $\sigma = (\sigma_1, \sigma_2, \dots)$ and a number κ satisfying

$$0 < \kappa < \pi$$

such that

(i) the following series converges

$$\sum_{j=1}^{\infty} e^{-\kappa\sigma_j} < \infty; \tag{3.13}$$

(ii) for all finite sequencies of integers $\mathbf{m} = (m_1, m_2, \dots)$, $|\mathbf{m}| = |m_1| + |m_2| + \cdots < \infty$ the following inequality holds true

$$\langle \mathbf{m}, \beta\mathbf{m}\rangle \equiv \sum_{ij} \beta_{ij} m_i m_j \geq \sum_j \sigma_j m_j^2. \tag{3.14}$$

For a given sequence σ introduce the Banach space B_σ given by

$$B_\sigma = \left\{ \mathbf{z} = (z_1, z_2, \dots,) \in \mathbb{C}^\infty \mid \lim_{j\to\infty} \frac{|z_j|}{\sigma_j} = 0. \right\} \tag{3.15}$$

with the norm

$$\|\mathbf{z}\| = \sup_j \frac{|z_j|}{\sigma_j}. \tag{3.16}$$

According to the Theorem 4.6 of [**15**] for a symmetric matrix β satisfying FKT condition for some σ the theta-series

$$\theta(\phi|\beta) = \sum_{\mathbf{m}\in\mathbb{Z}^\infty,\ \ |\mathbf{m}|<\infty} e^{-\pi\langle \mathbf{m}, \beta\mathbf{m}\rangle} e^{i\langle \mathbf{m}, \phi\rangle} \tag{3.17}$$

converges absolutely and uniformely on a sufficiently small ball around any point $\phi \in B_\sigma$ to a holomorphic function.

It is clear that, for a given symmetric matrix β_{ij} satisfying FKT condition, another symmetric matrix β'_{ij} with the same off-diagonal terms $\beta'_{ij} = \beta_{ij}$ for $i \neq j$ and with arbitrary diagonal terms satisfying $\beta'_{jj} \geq \beta_{jj}$ for all $j = 1, 2, \dots$ will also satisfy FKT condition with the same σ.

Let us first give a simple sufficient condition for an *off-diagonal* symmetric matrix β_{ij} to ensure a possibility to choose positive numbers β_{11}, $\beta_{22}, \dots$ in such a way that the whole symmetric matrix β_{ij} satisfies FKT condition for some sequence σ.

Lemma 3.2. — *Let the real symmetric off-diagonal matrix β_{ij} satisfies the condition*

$$\mu_i^2 := \sum_{j>i} \beta_{ij}^2 < \infty, \quad i = 1,\, 2,\, \dots. \tag{3.18}$$

Let σ be any sequence of positive numbers satisfying the convergence condition (3.13) with some positive $\kappa < \pi$. Let β^0_{jj} be another sequence of positive numbers defined by

$$\beta^0_{jj} = \sigma_j + 2\sum_{k=1}^{j} \mu_k, \quad j \geq 1$$

(it is assumed that all numbers μ_j are nonnegative). Then, for any choice of the diagonal entries satisfying

$$\beta_{jj} > \beta^0_{jj}, \quad j \geq 1 \tag{3.19}$$

the matrix β satisfies FKT condition.

Proof. — Because of the obvious inequality

$$\sum_{i,j} \beta_{ij} m_i m_j \geq \sum_j \beta_{jj} m_j^2 - 2\sum_i |m_i \sum_{j>i} \beta_{ij} m_j|$$

it suffices to obtain upper estimate for the second term. Let us consider the Hilbert space of square summable sequencies

$$L_2^{(i)} = \{(x_i, x_{i+1}, \dots) \mid \sum_{j\geq i} x_j^2 < \infty\}.$$

Applying the standard inequality

$$|(x, A\,x)| \leq \|A\|_{L_2}(x, x), \quad x \in L_2^{(i)}$$

valid for an arbitrary Hilbert - Schmidt operator A to the rank one operator

$$(x_i, x_{i+1}, \dots) \mapsto (\sum_{j>i} \beta_{ij} x_j, 0, \dots)$$

we obtain

$$|x_i \sum_{j>i} \beta_{ij} x_j| \leq \mu_i \sum_{j\geq i} x_j^2.$$

Finite sequencies of integers give vectors in $L_2^{(i)}$. Applying to these vectors the last inequality yields

$$\sum_{i,j} \beta_{ij} m_i m_j \geq \sum_j \beta_{jj} m_j^2 - 2\sum_{j=1}^{\infty} \mu_j \sum_{k\geq j} m_k^2.$$

This proves the Lemma. $\square$

Using the Lemma, we will give a simple sufficient condition for an infinite sequence of plane waves to generate, via the formula (3.12), an infinite genus KP theta-function for arbitrary sufficiently small amplitudes a_j and given wave numbers k_j, l_j.

Lemma 3.3. — *Let*

$$z_j = \frac{1}{2}(\lambda_j + ik_j), \quad k_j > 0 \quad j \geq 1$$

be a sequence of complex numbers satisfying the following conditions.
(i) *There exists a small positive number $r > 0$ such that*

$$|z_i - z_j| > r, \quad i \neq j, \quad |z_i - \bar{z}_j| > r, \quad i, j = 1, 2, \ldots. \tag{3.20}$$

(ii) *The series*

$$\sum_{j=1}^{\infty} |z_j|^{-2} < \infty \tag{3.21}$$

converges.

Then there exists a sequence of positive numbers β^0_{jj} such that the matrix β with the off-diagonal entries

$$\beta^0_{ij} = -\frac{1}{2\pi} \log \frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2}, \quad i \neq j \tag{3.22}$$

satisfies FKT condition for arbitrary diagonal entries such that

$$\beta_{jj} > \beta^0_{jj}, \quad j = 1, 2, \ldots.$$

Proof. — The formula for β^0_{ij} can be rewritten in the form

$$\beta^0_{ij} = -\frac{1}{2\pi} \log \left| \frac{z_j - z_i}{z_j - \bar{z}_i} \right|^2.$$

Using the elementary inequality

$$\left| \log \left| \frac{z - w}{z - \bar{w}} \right|^2 \right| < \frac{4}{|z|} |\mathrm{Im}\, w| \quad \text{for} \quad \left| \frac{w}{z} \right| < \frac{1}{2}$$

we derive that

$$|\beta^0_{ij}| < \frac{2}{\pi} \frac{k_i}{\sqrt{k_j^2 + \lambda_j^2}}$$

for a fixed i and any sufficiently large $j >> i$. Applying Lemma 3.2 we complete the proof of the Lemma. $\square$

We are now ready to prove convergence of the series (3.12) for a suitable class of parameters a, k, λ. Let the vectors k, λ satisfy the conditions of the Lemma 3.3. Choose positive numbers σ_j in such a way that the series (3.13) converges for some positive $\kappa < \pi$. Choose numbers β^0_{jj} in such a way that

$$\beta^0_{jj} > \sigma_j + 2\sum_{k=1}^{j} \mu^0_k, \quad j = 1, 2, \dots \tag{3.23}$$

where

$$\mu^0_i := \left(\sum_{j>i} \left(\beta^0_{ij}\right)^2\right)^{\frac{1}{2}} \tag{3.24}$$

and the off-diagonal matrix $\beta^0_{ij} = \beta^0_{ij}(k, \lambda)$ is defined in (3.22).

Theorem 3.4. — *For arbitrary positive numbers $a = (a_1, a_2, \dots)$ satisfying*

$$a_j < e^{-\pi\,\beta^0_{jj}}, \quad j = 1, 2, \dots \tag{3.25}$$

the series (3.12) converges absolutely and uniformely on a sufficiently small ball around any point $\phi \in B_\sigma$ to a holomorphic function. The series expansion (3.7) also converges to a sequence of frequencies $(\omega_1, \omega_2, \dots)$. The theta-function (3.12), after the substitution

$$\phi_j = k_j(x + \lambda_j y) - \omega_j t + \phi^0_j, \quad j = 1, 2, \dots$$

for arbitrary real phase shifts, yields a quasiperiodic solution to the KP equation (2.3) for some constant $b = b(a, k, \lambda)$.

Proof. — Let us consider the space of off-diagonal matrices β_{ij} satisfying the following inequalities

$$\sum_{j=1}^{k} \left(\sum_{j>i} \beta_{ij} r\right)^{\frac{1}{2}} < \frac{1}{2}\left(\beta^0_{kk} - \sigma_k\right), \quad k = 1, 2, \dots . \tag{3.26}$$

For any a satisfying (3.25) and any off-diagonal β_{ij} satisfying (3.26) the theta-series (3.17) converges to an analytic function on B_σ. It will also depend analytically on the period matrix, moreover, it satisfies the heat equations

$$a_k \frac{\partial \theta}{\partial a_k} = -\frac{\partial^2 \theta}{\partial \phi_k^2}, \quad Z_{ij} \frac{\partial \theta}{\partial Z_{ij}} = -\frac{\partial^2 \theta}{\partial \phi_i \partial \phi_j}.$$

One can also prove analyticity of the theta-functions of the second order (2.32). Like in the proof of the Theorem 2.1, we consider the system of equations (2.40). The functions $f[\nu]$ vanish at $a = 0$ for

$$\beta_{ij} = \beta_{ij}^0, \; i < j, \quad \omega_j = \frac{1}{4}(3k_j\lambda_j^2 - k_j^3), \quad b = 0.$$

The inverse to the Jacobi matrix (2.41) is a bounded operator due to our assumptions about the wave numbers. Applying the implicit function theorem we obtain convergence of the series (3.12), (3.7). The Theorem is proved. □

Example 3.5. — Let $\lambda_j = 0$ for all $j \geq 1$ and k_j be arbitrary positive numbers satisfying

$$|k_i - k_j| > r, \quad i \neq j$$

for some positive r. Then the assumptions of the Theorem 3.4 are fulfilled. In this way one obtains the theta-functions of the hyperelliptic Riemann surfaces of infinite genus (cf. [**27**], [**15**]). In particular, if k_j grows linearly with j, then the series (3.12) will converge for all a with exponential decay

$$a_j < e^{-cj}$$

for some positive constant c. The formulae (3.12), (3.7) define quasiperiodic solutions to the KdV equations.

More generally, our approach describes some neighborhood of the manifold of hyperelliptic Riemann surfaces of infinite genus. In particular, assuming that the points z_j satisfying (3.20) belong to a strip of a finite width along the imaginary axis, one obtain slow transversal perturbations of the KdV quasiperiodic solutions. The condition (3.21) in this case holds automatically true. It would be interesting to prove that the intersection with this neighborhood of the so-called heat curves of [**24**], [**6**], [**15**] associated with doubly periodic in x, y solutions $u(x, u, t)$ of KP form a dense subset. For the case of finite genus density this was proved in [**5**].

Some of our assumptions about behaviour of the sequence of wave numbers can in fact be relaxed. We will consider more general situation in a subsequent publication. The assumption (3.20) that prevent the interacting waves to be close to resonant seems however to be essential. For example, as it was shown by S.Venakides [**33**], the limits of hyperelliptic theta-functions with the parameters k_j accumulating in the interval $[0,1]$ are weird functions described by a minimization principle of the Lax - Levermore type [**26**]. It would be also interesting to prove that our infinite genus theta-functions (3.12) come from a parabolic Riemann surfaces in the sense of Ahlfors and Sario [**1**].

We also plan to study in subsequent publications the relationship of our approach to the approach of V.Zakharov and E.Schulman to the problem of classification of integrable PDEs.

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