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GEOMETRY OF 2D TOPOLOGICAL FIELD THEORIES

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Abstract. These lecture notes are devoted to the theory of "equations of associativity" describing geometry of moduli spaces of 2D topological field theories.

Contents.

Introduction.

In these lecture notes I consider one remarkable system of differential equations that appeared in the papers of physicists on two-dimensional topological field theory (TFT) in the beginning of '90 [156, 42]. Roughly speaking, the problem is to find a quasihomogeneous function $F = F(t)$ of the variables $t = (t^1, \ldots, t^n)$ such that the third derivatives of it

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

for any t are structure constants of an associative algebra A_t with a t-independent unity (the algebra will be automatically commutative) (see Lecture 1 for the precise formulation of the problem). For the function $F(t)$ one obtains a very complicated overdetermined system of PDEs. I call it WDVV equations. In the physical setting the solutions of WDVV describe moduli space of topological conformal field theories. One of the projects of these lectures is to try to reconstruct the building of 2D TFT on the base of WDVV equation.

From the point of view of a mathematician particular solutions of WDVV with certain "good" analytic properties are generating functions for the Gromov - Witten invariants of Kähler (and, more generally, of symplectic) manifolds [157]. They play the crucial role in the formulation (and, may be, in the future explanation) of the phenomenon of mirror symmetry of Calabi - Yau 3-folds [159]. Probably, they also play a central role in understanding of relations between matrix integrals, integrable hierarchies, and topology of moduli spaces of algebraic curves [157, 76, 96-108, 61].

We discuss briefly the "physical" and "topological" motivations of WDVV equations in Lecture 2. The other lectures are based mainly on the papers [48-50, 52-54] of the author. The material of Appendices was not published before besides Appendix D (a part of the preprint [54]).

In the abbreviated form our contribution to the theory of WDVV can be encoded by the following key words:

WDVV as Painlevé equations

Discrete groups and their invariants and particular solutions of WDVV

Symmetries of WDVV

To glue all these together we employ an amazingly rich (and nonstandard) differential geometry of WDVV. The geometric reformulation of the equations is given in Lecture 1. We observe then (on some simple but important examples) that certain analyticity conditions together with semisimplicity of the algebras A_t work as a very strong rule for selection of solutions (WDVV does almost not constrain the function $F(t)$ if the algebras A_t are nilpotent for any t). Probably, solutions with good analytic properties (in a sense to be formulated in a more precise way) are isolated points in the sea of all solutions of WDVV.

The main geometrical playing characters - the deformed affine connection and the deformed Euclidean metric - are introduced in Lecture 3. In this Lecture we find the general solution of WDVV for which the algebra A_t is semisimple for generic t. This is expressed via certain transcendental functions of the Painlevé-VI type and their higher order generalizations. The theory of linear differential operators with rational coefficients and of their monodromy preserving deformations plays an important role in these considerations.

In Appendix G we introduce another very important object: the monodromy group that can be constructed for any solution of WDVV. This is the monodromy group of a holonomic system of differential equations describing the deformation of Euclidean structure on the space of parameters t.

Our conjecture is that, for a solution of WDVV with good analytic properties the monodromy group is a discrete group. Some general properties of the monodromy group are obtained in Appendices G and H. We give many examples where the group is a finite Coxeter group, an extension of an affine Weyl group (Lecture 4) or of a complex crystallographic group (Appendix J). The solutions of WDVV for these monodromy groups are given by simple formulae in terms of the invariants of the groups. To apply this technique to the topological problems, like mirror symmetry, we need to find some natural groups related to, say, Calabi - Yau 3-folds (there are interesting results in this direction in the recent preprints [143]).

Bäcklund-type symmetries of the equations of associativity play an important technical role in our constructions. It turns out that the group of symmetries of WDVV is rich enough: for example, it contains elements that transform, in the physical notations, a solution with the given $d = \hat{c} = \frac{c}{3}$ to a solution with $d' = 2 - d$ (I recall that for topological sigma-models d is the complex dimension of the target space). Better understanding of the structure of the group of symmetries of WDVV could be useful in the mirror problem.

In the last Lecture we briefly discuss relation of the equations of associativity to integrable hierarchies and their semiclassical limits. Some of these relations were discussed also in [49, 88, 94-95, 104-105, 141] for the dispersionless limits of various integrable hierarchies of KdV type (the dispersionless limit corresponds to the tree-level approximation in TFT). Some of these observations were known also in the theory of Gauss - Manin equations [118]. Our approach is principally different: we construct an integrable hierarchy (in a semi-classical approximation) for *any* solution of WDVV. We obtain also for our hierarchies the semi-classical analogue of Lax representation. The problem of reconstruction of all the hierarchy (in all orders in the small dispersion expansion) is still open (see the recent papers [97, 61] where this problem was under investigation).

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Lecture 1.
WDVV equations and Frobenius manifolds.

I start with formulation of the main subject of these lectures: a system of differential equations arising originally in the physical papers on two-dimensional field theory (see below Lecture 2). We look for a function $F = F(t)$, $t = (t^1, \ldots, t^n)$ such that the third derivatives of it

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

obey the following equations

1) Normalization:

$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t) \tag{1.1}$$

is a constant nondegenerate matrix. Let

$$(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}.$$

We will use the matrices $(\eta^{\alpha\beta})$ and $(\eta_{\alpha\beta})$ for raising and lowering indices.

2) Associativity: the functions

$$c_{\alpha\beta}^\gamma(t) := \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t) \tag{1.2}$$

(summation over repeated indices will be assumed in these lecture notes) for any t must define in the n-dimensional space with a basis e_1, ..., e_n a structure of an assosciative algebra A_t

$$e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma(t) e_\gamma. \tag{1.3}$$

Note that the vector e_1 will be the unity for all the algebras A_t:

$$c_{1\alpha}^\beta(t) = \delta_\alpha^\beta. \tag{1.4}$$

3) $F(t)$ must be quasihomogeneous function of its variables:

$$F(c^{d_1} t^1, \ldots, c^{d_n} t^n) = c^{d_F} F(t^1, \ldots, t^n) \tag{1.5}$$

for any nonzero c and for some numbers d_1, ..., d_n, d_F.

It will be convenient to rewrite the quasihomogeneity condition (1.5) in the infinitesimal form introducing the *Euler vector field*

$$E = E^\alpha(t) \partial_\alpha$$

as

$$\mathcal{L}_E F(t) := E^\alpha(t) \partial_\alpha F(t) = d_F \cdot F(t). \tag{1.6}$$

For the quasihomogeneity (1.5) $E(t)$ is a linear vector field

$$E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha \tag{1.7}$$

generating the scaling transformations (1.5). Note that for the Lie derivative $\mathcal{L}_E$ of the unity vector field $e = \partial_1$ we must have

$$\mathcal{L}_E e = -d_1\, e. \tag{1.8}$$

Two generalizations of the quasihomogeneity condition will be important in our considerations:

1. We will consider the functions $F(t^1, \ldots, t^n)$ up to adding of a (nonhomogeneous) quadratic function in $t^1, \ldots, t^n$. Such an addition does not change the third derivatives. So the algebras A_t will remain unchanged. Thus the quasihomogeneity condition (1.6) could be modified as follows

$$\mathcal{L}_E F(t) = d_F F(t) + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C. \tag{1.9}$$

This still provides quasihomogeneity of the functions $c_{\alpha\beta\gamma}(t)$. Moreover, if

$$d_F \neq 0,\ d_F - d_\alpha \neq 0,\ d_F - d_\alpha - d_\beta \neq 0 \text{ for any } \alpha,\ \beta \tag{1.10}$$

then the extra terms in (1.9) can be killed by adding of a quadratic form to $F(t)$.

2. We will consider more general linear nonhomogeneous Euler vector fields

$$E(t) = \left(q^\alpha_\beta t^\beta + r^\alpha\right) \partial_\alpha. \tag{1.11}$$

If the roots of $E(t)$ (i.e., the eigenvalues of the matrix $Q = (q^\alpha_\beta)$) are simple and nonzero then $E(t)$ can be reduced to the form (1.7) by a linear change of the variables t. If some of the roots of $E(t)$ vanish then, in general the linear nonhomogeneous terms in (1.11) cannot be killed by linear transformations of t. In this case for a diagonalizable matrix Q the Euler vector field can be reduced to the form

$$E(t) = \sum_\alpha d_\alpha t^\alpha \partial_\alpha + \sum_{\alpha | d_\alpha = 0} r^\alpha \partial_\alpha \tag{1.12}$$

(here d_α are the eigenvalues of the matrix Q). The numbers r^α can be changed by linear transformations in the kernel Ker$Q$. However, in important examples the function $F(t)$ will be periodic (modulo quadratic terms) w.r.t. some lattice of periods in Ker$Q$ (note that periodicity of $F(t)$ can happen only along the directions with zero scaling dimensions). In this case the vector (r^α) is defined modulo the group of automorphisms of the lattice of periods. Particularly, in topological sigma models with non-vanishing first Chern class of the target space the vector (r^α) is always nonzero (see below Lecture 2).

The degrees d_1, ..., d_n, d_F are well-defined up to a nonzero factor. We will consider only the case

$$d_1 \neq 0$$

(the variable t^1 is marked due to (1.1)). It is convenient in this case to normalize the degrees d_1, ..., d_n, d_F in such a way that

$$d_1 = 1. \tag{1.13a}$$

In the physical literature the normalized degrees usually are parametrized by some numbers $q_1 = 0$, q_2,..., q_n and d such that

$$d_\alpha = 1 - q_\alpha, \quad d_F = 3 - d. \tag{1.13b}$$

If the coordinates are normalized as in (1.18) then

$$q_n = d, \quad q_\alpha + q_{n-\alpha+1} = d. \tag{1.13c}$$

Mainly the case of real function $F(t)$ will be under consideration (although in the classification theorems we will work also with the complexified situation). All the numbers q_α, r_α, d in the real case are also to be real.

Associativity imposes the following system of nonlinear PDE for the function $F(t)$

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\delta \partial t^\mu} \tag{1.14}$$

for any α, β, γ, δ from 1 to n. The quasihomogeneity (1.9) determines the scaling reduction of the system. The normalization (1.1) completely specifies the dependence of the function $F(t)$ on the marked variable t^1. The resulting system of equations will be called *Witten - Dijkgraaf - E. Verlinde - H. Verlinde (WDVV)* system: it was first found in the papers [156, 42] in topological field theory (see Lecture 2 below). A solution of the WDVV equations will be called *(primary) free energy*. The system (1.14) (without the scaling (1.9)) will be called *associativity equations*.

Remark 1.1. More general reduction of (1.14) is given by *conformal* transformations of the metric $\eta_{\alpha\beta}$. The generator E of the corresponding one-parameter group of diffeomorphisms of the t-space for $n \geq 3$ due to Liouville theorem [59] must have the form

$$E = a \left\{ t_1 t^\gamma \partial_\gamma - \frac{1}{2} t_\sigma t^\sigma \partial_1 \right\} + \sum_{\epsilon=1}^{n} d_\epsilon t^\epsilon \partial_\epsilon \tag{1.15a}$$

for some constants $a, d_1, \ldots, d_n$ with

$$d_\alpha = 1 - q_\alpha, \quad q_1 = 0, \quad q_\alpha + q_{n-\alpha+1} = d$$

(in the normalization (1.17)). The function $F(t)$ must obey the equation

$$\sum_{\epsilon=1}^{n} (a t_1 + d_\epsilon) t^\epsilon \partial_\epsilon F = (3 - d + 2 a t_1) F + \frac{a}{8} \left(t_\sigma t^\sigma \right)^2 \tag{1.15b}$$

modulo quadratic terms. The equations of associativity (1.14) for F satisfying (1.15) also can be reduced to a system of ODE. I will not consider this system here. However, a *discrete* group of conformal symmetries of WDVV plays an important role in our considerations (see below Appendix B).

Observe that the system is invariant w.r.t. linear changes of the coordinates $t^1, \dots, t^n$. To write down WDVV system more explicitly we use the following

Lemma 1.1 *The scaling transformations generated by the Euler vector field E (1.9) act as linear conformal transformations of the metric* $\eta_{\alpha\beta}$

$$\mathcal{L}_E \eta_{\alpha\beta} = (d_F - d_1)\eta_{\alpha\beta} \tag{1.16}$$

where the numbers d_F *and* d_1 *are defined in (1.6) and (1.8).*

Proof. Differentiating the equation (1.9) w.r.t. t^1, t^α and t^β and using $\partial_1 E^\rho = d_1 \delta_1^\rho$ (this follows from (1.8)) we obtain

$$q_\alpha^\rho \eta_{\rho\beta} + q_\beta^\rho \eta_{\rho\alpha} = (d_F - d_1)\eta_{\alpha\beta}.$$

The l.h.s. of this equality coincides with the Lie derivative $\mathcal{L}_E\, \eta_{\alpha\beta}$ of the metric $<\ ,\ >$. Lemma is proved.

Corollary 1.1. *If* $\eta_{11} = 0$ *and all the roots of* $E(t)$ *are simple then by a linear change (possibly, with complex coefficients for odd* n*) of coordinates* t^α *the matrix* $\eta_{\alpha\beta}$ *can be reduced to the antidiagonal form*

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}. \tag{1.17}$$

In these coordinates

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \dots, t^n) \tag{1.18}$$

for some function $f(t^2, \dots, t^n)$*, the sum*

$$d_\alpha + d_{n-\alpha+1} \tag{1.29}$$

does not depend on α *and*

$$d_F = 2d_1 + d_n. \tag{1.20}$$

If the degrees are normalized in such a way that $d_1 = 1$ *then they can be represented in the form*

$$d_\alpha = 1 - q_\alpha, \quad d_F = 3 - d \tag{1.21a}$$

where the numbers $q_1, \dots, d_n, d$ *satisfy*

$$q_1 = 0, \quad q_n = d, \quad q_\alpha + q_{n-\alpha+1} = d. \tag{1.21b}$$

Exercise 1.1. Show that if $\eta_{11} \neq 0$ (this can happen only for $d_F = 3d_1$) the function F can be reduced by a linear change of t^α to the form

$$F = \frac{c}{6}(t^1)^3 + \frac{1}{2} t^1 \sum_{\alpha=1}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \dots, t^n) \tag{1.22}$$

for a nonzero constant c where the degrees satisfy

$$d_\alpha + d_{n-\alpha+1} = 2d_1. \tag{1.23}$$

Proof of Corollary. If $< e_1, e_1 >= 0$ then one can chose the basic vector e_n such that $< e_1, e_n >= 1$ and e_n is still an eigenvector of Q. On the orthogonal complement of the span of e_1 and e_n we can reduce the bilinear form $< \, , \, >$ to the antidiagonal form using only eigenvectors of the scaling transformations. In these coordinates (1.18) follows from (1.1). Independence of the sum $d_\alpha + d_{n-\alpha+1}$ of α follows from (1.16). The formula for d_F follows from (1.16). Corollary is proved.

Remark 1.2. Over real numbers WDVV equations have an additional integer invariant: the signature of the quadratic form $\eta_{\alpha\beta} t^\alpha t^\beta$ restricted onto the subspace spanned by those basic vectors e_α with

$$q_\alpha = \frac{d}{2}.$$

I will mainly consider solutions of WDVV of the type (1.18). I do not know physical examples of the solutions of the second type (1.22). However, we are to take into account also the solutions with $\eta_{11} \neq 0$ for completeness of the mathematical theory of WDVV (see, e.g., Appendix B and Lecture 5).

Example 1.1. $n = 2$. Equations of associativity are empty. The other conditions specify the following general solution of WDVV

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^k, \quad k = \frac{3-d}{1-d}, \; d \neq -1, 1, 3 \tag{1.24a}$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^2 \log t_2, \quad d = -1 \tag{1.24b}$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + \log t_2, \quad d = 3 \tag{1.24c}$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + e^{\frac{2}{r} t_2}, \quad d = 1, \; r \neq 0 \tag{1.24d}$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2, \quad d = 1, \; r = 0 \tag{1.24e}$$

(in concrete formulae I will often label the coordinates t^α by subscripts for the sake of graphical simplicity). In the last two cases $d = 1$ the Euler vector field is $E = t_1 \partial_1 + r \partial_2$.

Example 1.2. $n = 3$. In the three-dimensional algebra A_t with the basis $e_1 = 1$, e_2, e_3 the law of multiplication is determined by the following table

$$\begin{aligned} e_2^2 &= f_{xxy} e_1 + f_{xxx} e_2 + e_3 \\ e_2 e_3 &= f_{xyy} e_1 + f_{xxy} e_2 \\ e_3^2 &= f_{yyy} e_1 + f_{xyy} e_2 \end{aligned} \tag{1.25}$$

where the function F has the form

$$F(t) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + f(t_2, t_3) \tag{1.26}$$

for a function $f = f(x, y)$ (the subscripts denote the corresponding partial derivatives). The associativity condition

$$(e_2^2)e_3 = e_2(e_2 e_3) \tag{1.27}$$

implies the following PDE for the function $f = f(x, y)$

$$f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy}. \tag{1.28}$$

It is easy to see that this is the only one equation of associativity for $n = 3$. The function f must satisfy also the following scaling condition

$$\left(1 - \frac{d}{2}\right) x f_x + (1-d) y f_y = (3-d) f, \quad d \neq 1,\ 2,\ 3 \tag{1.29a}$$

$$\frac{1}{2} x f_x + r f_y = 2f, \quad d = 1 \tag{1.29b}$$

$$r f_x - y f_y = f, \quad d = 2 \tag{1.29c}$$

$$\frac{1}{2} x f_x + 2y f_y = c, \quad d = 3 \tag{1.29d}$$

for some constants c, r. The corresponding scaling reductions of the equation (1.28)

$$f(x, y) = \frac{x^4}{y} \phi\left(\log(y x^q)\right), \ q = -\frac{1-d}{1 - \frac{1}{2}d}, \quad d \neq 1,\ 2,\ 3 \tag{1.30a}$$

$$f(x, y) = x^4 \phi(y - 2r \log x), \quad d = 1 \tag{1.30b}$$

$$f(x, y) = y^{-1} \phi(x + r \log y), \quad d = 2 \tag{1.30c}$$

$$f(x, y) = 2c \log x + \phi(y x^{-4}), \quad d = 3 \tag{1.30d}$$

are the following third order ODEs for the function $\phi = \phi(z)$,

$$\begin{gathered}
-6\,\phi + 48\,\phi^2 + 11\,\phi' + 88\,q\,\phi\,\phi' - (144 + 144\,q - 3\,q^2)\,\phi'^2 - 6\,\phi'' + 48(2 + 2\,q + q^2)\,\phi\,\phi'' \\
-4\,q(16 + 16\,q + q^2)\,\phi'\,\phi'' - (13\,q^2 + 13\,q^3 + q^4)\,\phi''^2 \\
+\phi''' + 8q(3 + 3q + q^2)\,\phi\,\phi''' + 2\,q^2(1 + q + q^2)\,\phi'\,\phi''' - q^3(1+q)\,\phi''\,\phi''' = 0
\end{gathered} \tag{1.31a}$$

for $d \neq 1,\ 2,\ 3$,

$$-144\,\phi'^2 + 96\,\phi\,\phi'' + 128\,r\,\phi'\,\phi'' - 52\,r^2\,\phi''^2 + \phi''' - 48\,r\,\phi\,\phi''' + 8\,r^2\,\phi'\,\phi''' + 8\,r^3\,\phi''\,\phi''' = 0 \tag{1.31b}$$

for $d = 1$,

$$\phi'''[r^3 + 2\phi' - r\phi''] - (\phi'')^2 - 6r^2\phi'' + 11r\phi' - 6\phi = 0 \tag{1.31c}$$

for $d=2$,

$$\phi''' = 400\,\phi'^2 + 32\,c\,\phi'' + 1120\,z\,\phi'\,\phi'' + 784\,z^2\,\phi''^2 + 16\,c\,z\,\phi''' + 160\,z^2\,\phi'\,\phi''' + 192\,z^3\,\phi''\,\phi''' \tag{1.31d}$$

for $d=3$.

Any solution of WDVV for $n=3$ can be obtained from a solution of these ODEs. Later these will be shown to be reduceable to a particular case of the Painlevé-VI equation.

Remark 1.3. Let us compare (1.28) with the WDVV equations for the prepotential F of the second type (1.22). Here we look for a solutions of (1.14) in the form

$$F = \frac{1}{6}t_1^3 + t_1 t_2 t_3 + f(t_2, t_3). \tag{1.32}$$

The three-dimensional algebra with a basis $e_1 = e$, e_2, e_3 has the form

$$\begin{aligned} e_2^2 &= f_{xxy}e_2 + f_{xxx}e_3 \\ e_2e_3 &= e_1 + f_{xyy}e_2 + f_{xxy}e_3 \\ e_3^2 &= f_{yyy}e_2 + f_{xyy}e_3. \end{aligned} \tag{1.33}$$

The equation of associativity has the form

$$f_{xxx}f_{yyy} - f_{xxy}f_{xyy} = 1. \tag{1.34}$$

It is interesting that this is the condition of unimodularity of the Jacobi matrix

$$\det\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = 1$$

for

$$P = f_{xx}(x,y), \quad Q = f_{yy}(x,y).$$

The function f must satisfy the quasihomogeneity condition

$$f(c^{1-a}x, c^{1+a}y) = c^3 f(x,y). \tag{1.35}$$

Example 1.3. $n=4$. Here we have a system of 6 equations for the function $f = f(x,y,z)$ where

$$F(t_1,t_2,t_3,t_4) = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 + f(t_2,t_3,t_4) \tag{1.36}$$

$$\begin{aligned} -2\,f_{xyz} - f_{xyy}\,f_{xxy} + f_{yyy}\,f_{xxx} &= 0 \\ -f_{xzz} - f_{xyy}\,f_{xxz} + f_{yyz}\,f_{xxx} &= 0 \\ -2\,f_{xyz}\,f_{xxz} + f_{xzz}\,f_{xxy} + f_{yzz}\,f_{xxx} &= 0 \\ -f_{yyy}\,f_{xxz} + f_{yzz} + f_{yyz}\,f_{xxy} &= 0 \\ f_{zzz} - {f_{xyz}}^2 + f_{xzz}\,f_{xyy} - f_{yyz}\,f_{xxz} + f_{yzz}\,f_{xxy} &= 0 \\ f_{yyy}\,f_{xzz} - 2\,f_{yyz}\,f_{xyz} + f_{yzz}\,f_{xyy} &= 0. \end{aligned} \tag{1.37}$$

It is a nontrivial exercise even to verify compatibility of this overdetermined system of equations.

Particularly, any function f depending only on t_2 gives a solution of these equations. The multiplication in the corresponding algebras A_t have then the form

$$e_1 \cdot e_\alpha = e_\alpha,\ e_2 \cdot e_3 = e_4,\ e_2^2 = f'''(t_2) e_3, \tag{1.38}$$

other products vanish. Thus A_t are nilpotent algebras for any t. These solutions of WDVV are compatible with the scaling (1.9) if the degree of t_2 equals 0.

I am going now to give a coordinate-free formulation of WDVV. Let me give first more details about the algebras A_t.

Definition 1.1. An algebra A over $\mathbf{C}$ is called (commutative) *Frobenius algebra* if:
1) It is a commutative associative $\mathbf{C}$-algebra with a unity e.
2) It is supplied with a $\mathbf{C}$-bilinear symmetric nondegenerate inner product

$$A \times A \to \mathbf{C}, \quad a,\, b \mapsto < a, b >$$

being invariant in the following sense:

$$< ab, c >=< a, bc > . \tag{1.39}$$

Remark 1.4. Let $\omega \in A^*$ be the linear functional

$$\omega(a) :=< e, a > .$$

Then

$$< a, b >= \omega(ab). \tag{1.40}$$

This formula determines a bilinear symmetric invariant inner product for arbitrary linear functional ω. It will be nondegenerate (for finite-dimensional Frobenius algebras) for generic $\omega \in A^*$. Note that we consider a Frobenius algebra with a *marked* invariant inner product.

Example 1.4. A is the direct sum of n copies of one-dimensional algebras. This means that a basis $e_1, \ldots, e_n$ can be chosen in the algebra with the multiplication law

$$e_i\, e_j = \delta_{ij}\, e_i, \quad i,\, j = 1, \ldots, n. \tag{1.41}$$

Then

$$< e_i, e_j >= 0 \text{ for } i \neq j, \tag{1.42}$$

the nonzero numbers $< e_i, e_i >$, $i = 1, \ldots, n$ are the parameters of the Frobenius algebras of this type. This algebra is semisimple (it has no nilpotents).

Exercise 1.2. Prove that any Frobenius algebra over $\mathbf{C}$ without nilpotents is of the above form. Prove this statement not assuming *a priori* existence of a unity in the algebra.

I recal that a nonzero element $a \in A$ is called nilpotent if $a^m = 0$ for some m. The algebra A is semisimple if it contains no nilpotents. A basis of idempotents $\pi_1, \ldots, \pi_n$ exists in a semisimple n-dimensional Frobenius algebra

$$\pi_i \, \pi_j = \delta_{ij} \pi_i, \quad i, j = 1, \ldots, n.$$

The invariaant inner product is diagonal in the basis $\pi_1, \ldots, \pi_n$

$$< \pi_1, \pi_j > = < e, \pi_i \pi_j > = \delta_{ij} \eta_{ii}$$

where

$$\eta_{ii} := < e, \pi_i > .$$

Introducing the orthonormalized basis

$$f_i = \frac{\pi_i}{\sqrt{\eta_{ii}}}$$

we define the transition matrix

$$e_\alpha = \sum_{i=1}^n \psi_{i\alpha} f_i.$$

It is clear that the matrix $(\psi_{i\alpha})$ can be computed in a pure algebraic way in terms of the structure constants $c_{\alpha\beta}^\gamma$ and of the invariant inner product $\eta_{\alpha\beta}$. We have

$$\eta_{ii} = \psi_{i1}^2$$

(due to the normalization $e_1 = e$)

$$\eta_{\alpha\beta} = \sum_{i=1}^n \psi_{i\alpha} \psi_{i\beta}$$

and

$$c_{\alpha\beta\gamma} = < e_\alpha e_\beta, e_\gamma > = \sum_{i=1}^n \frac{\psi_{i\alpha} \psi_{i\beta} \psi_{i\gamma}}{\psi_{i1}}.$$

For a solution of WDVV satisfying the semisimplicity condition for a generic t the matrix $\psi_{i\alpha}$ depends in a nice way on the parameters t. This will be used in Lecture 3 for local classification of these solutions.

An operation of *rescaling* is defined for an algebra with a unity e: we modify the multiplication law and the unity as folllows

$$a \cdot b \mapsto k \, a \cdot b, \quad e \mapsto k^{-1} \, e \tag{1.43}$$

for a given nonzero constant k. The rescalings preserve Frobenius property of the algebra.

Back to the main problem: we have a family of Frobenius algebras depending on the parameters $t = (t^1, \ldots, t^n)$. Let us denote by M the space of the parameters. We have thus a fiber bundle

$$\begin{array}{cc} & \downarrow \quad A_t \\ t \in & M \end{array} \tag{1.44}$$

The basic idea is to identify this fiber bundle with the tangent bundle TM of the manifold M.

We come thus to our main definition. Let M be a n-dimensional manifold.

Definition 1.2. M is *Frobenius manifold* if a structure of Frobenius algebra is specified on any tangent plane T_tM at any point $t \in M$ smoothly depending on the point such that

1. The invariant inner product $< \, , \, >$ is a flat metric on M.
2. The unity vector field e is covariantly constant w.r.t. the Levi-Civitá connection ∇ for the metric $< \, , \, >$
$$\nabla e = 0. \tag{1.45}$$
3. Let
$$c(u, v, w) := < u \cdot v, w > \tag{1.46}$$
(a symmetric 3-tensor). We require the 4-tensor
$$(\nabla_z c)(u, v, w) \tag{1.47}$$
to be symmetric in the four vector fields u, v, w, z.
4. A vector field E must be determined on M such that
$$\nabla(\nabla E) = 0 \tag{1.48}$$
and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric $< \, , \, >$ and by rescalings on the Frobenius algebras T_tM.

In these lectures the word 'metric' stands for a $\mathbf{C}$-bilinear quadratic form on M.

Note that the requirement 4 makes sense since we can locally identify the spaces of the algebras T_tM using the Euclidean parallel transport on M. We will call E *Euler vector field* (see formula (1.9) above) of the Frobenius manifold. The covariantly constant operator

$$Q = \nabla E(t) \tag{1.49}$$

on the tangent spaces T_tM will be called *the grading operator* of the Frobenius manifold. The eigenvalues of the operator Q are constant functions on M. The eigenvalues q_α of $(\mathrm{id} - Q)$ will be called *scaling dimensions* of M. Particularly, as it follows from (1.45), the unity vector field e is an eigenvector of Q with the eigenvalue 1.

The infinitesimal form of the requirement 4 reads

$$\nabla_\gamma \left(\nabla_\beta E^\alpha\right) = 0 \tag{1.50a}$$
$$\mathcal{L}_E c_{\alpha\beta}^\gamma = c_{\alpha\beta}^\gamma \tag{1.50b}$$
$$\mathcal{L}_E e = -e \tag{1.50c}$$
$$\mathcal{L}_E \eta_{\alpha\beta} = D\eta_{\alpha\beta} \tag{1.50d}$$

for some constant $D = 2 - d$. Here $\mathcal{L}_E$ is the Lie derivative along the Euler vector field. In a coordinate-free way (1.50b) and (1.50d) read

$$\mathcal{L}_E(u \cdot v) - \mathcal{L}_E u \cdot v - u \cdot \mathcal{L}_E v = u \cdot v \tag{1.50b'}$$

$$\mathcal{L}_E < u, v > - < \mathcal{L}_E u, v > - < u, \mathcal{L}_E v > = D < u, v > \tag{1.50d'}$$

for arbitrary vector fields u and v.

Exercise 1.3. Show that the operator

$$\hat{V} := \nabla E - \frac{1}{2}(2 - d)\mathrm{id} \tag{1.51}$$

is skew-symmetric w.r.t. $< \ , \ >$

$$< \hat{V} x, y > = - < x, \hat{V} y > . \tag{1.52}$$

Lemma 1.2. *Any solution of WDVV equations with $d_1 \neq 0$ defined in a domain $t \in M$ determines in this domain the structure of a Frobenius manifold by the formulae*

$$\partial_\alpha \cdot \partial_\beta := c_{\alpha\beta}^\gamma(t)\partial_\gamma \tag{1.53a}$$

$$< \partial_\alpha, \partial_\beta > := \eta_{\alpha\beta} \tag{1.53b}$$

where

$$\partial_\alpha := \frac{\partial}{\partial t^\alpha} \tag{1.53c}$$

etc.,

$$e := \partial_1 \tag{1.53d}$$

and the Euler vector field has the form (1.9).

Conversely, locally any Frobenius manifold has the structure (1.53), (1.11) for some solution of WDVV equations.

Proof. The metric (1.53b) is manifestly flat being constant in the coordinates t^α. In these coordinates the covariant derivative of a tensor coincides with the corresponding partial derivatives. So the vector field (1.53d) is covariantly constant. For the covariant derivatives of the tensor (1.46)

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{1.54}$$

we have a completely symmetric expression

$$\partial_\delta c_{\alpha\beta\gamma}(t) = \frac{\partial^4 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma \partial t^\delta}. \tag{1.55}$$

This proves the property 3 of our definition. The property 4 is obvious since the one-parameter group of diffeomorphisms for the vector field (1.9) acts by rescalings (1.43).

Conversely, on a Frobenius manifold locally one can chose flat coordinates t^1, ..., t^n such that the invariant metric $< \, , \, >$ is constant in these coordinates. The symmetry condition (1.47) for the vector fields $u = \partial_\alpha$, $v = \partial_\beta$, $w = \partial_\gamma$ and $z = \partial_\delta$ reads

$$\partial_\delta c_{\alpha\beta\gamma}(t) \text{ is symmetric in } \alpha,\ \beta,\ \gamma,\ \delta$$

for

$$c_{\alpha\beta\gamma}(t) = < \partial_\alpha \cdot \partial_\beta, \partial_\gamma > .$$

Together with the symmetry of the tensor $c_{\alpha\beta\gamma}(t)$ this implies local existence of a function $F(t)$ such that

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma.}$$

Due to covariant constancy of the unity vector field e we can do a linear change of coordinates in such a way that $e = \partial_1$. This gives (1.53d).

We are to prove now that the function $F(t)$ satisfies (1.9). Due to (1.48) in the flat coordinates $E(t)$ is a linear vector field of a form (1.11). From the definition of rescalings we have in the coordinates t^α

$$[\partial_1, E] = \partial_1. \tag{1.56}$$

Hence ∂_1 is an eigenvector of the operator $Q = \nabla E$ with the eigenvalue 1. I.e. $d_1 = 1$. The constant matrix $\left(Q^\alpha_\beta\right)$ must obey the equation

$$Q_{\alpha\beta} = D\, \eta_{\alpha\beta} \tag{1.57}$$

(this follows from (1.50d)) for some constant D. The last step is to use the condition (1.50b) (the definition of rescalings). From (1.50b) and (1.50d) we obtain

$$\mathcal{L}_E c_{\alpha\beta\gamma} = (1 + D) c_{\alpha\beta\gamma}.$$

Due to (1.54) this can be rewritten as

$$\partial_\alpha \partial_\beta \partial_\gamma \left[E^\epsilon \partial_\epsilon F - (1 + D) F\right] = 0.$$

This gives (1.9). Lemma is proved.

The function $F(t)$ determined by (1.54) will be called *prepotential* or *free energy* of the Frobenius manifold. In general it is well-defined only locally.

Exercise 1.4. In the case $d_1 = 0$ show that a two-dimensional commutative group of diffeomorphisms acts locally on the space of parameters t preserving the multiplication (1.53a) and the metric (1.53b).

This symmetry provides integrability in quadratures of the equation (1.31a) for $d_1 = 0$ (observation of [32]). Indeed, (1.31a) for $d_1 = 0$ ($q = -2$) reads

$$12 z^2 (\phi'')^2 + 8 z^3 \phi'' \phi''' - \phi''' = 0.$$

The integral of this equation is obtained in elliptic quadratures from

$$\phi'' = \frac{1}{8z^3} \pm \frac{\sqrt{cz^3 + \frac{1}{64}}}{z^3}.$$

Definition 1.3. Two Frobenius manifolds M and $\tilde{M}$ are *equivalent* if there exists a diffeomorphism

$$\phi : M \to \tilde{M} \tag{1.58a}$$

being a linear conformal transformation of the corresponding invariant metrics ds^2 and $d\tilde{s}^2$

$$\phi^* d\tilde{s}^2 = c^2\, ds^2 \tag{1.58b}$$

(c is a nonzero constant) with the differential ϕ_* acting as an isomorphism on the tangent algebras

$$\phi_* : T_t M \to T_{\phi(t)} \tilde{M}. \tag{1.58c}$$

If ϕ is a local diffeomorphism with the above properties then it will be called *local equivalence.*

Note that for an equivalence ϕ not necessarily $F = \phi^* \tilde{F}$. For example, if the coordinates $(t^1, \dots, t^n)$ and $(\tilde{t}^1, \dots, \tilde{t}^n)$ on M and $\tilde{M}$ resp. are normalized as in (1.18) then the map

$$\tilde{t}^1 = t^1,\ \tilde{t}^n = c^2 t^n,\ \ \tilde{t}^\alpha = c t^\alpha \text{ for } \alpha \neq 1,\, n \tag{1.59a}$$

$$\tilde{F}(\tilde{t}^1, \dots, \tilde{t}^n) = c^2 F(t^1, \dots, t^n). \tag{1.59b}$$

for a constant $c \neq 0$ is an equivalence. Any equivalence is a superposition of (1.59) and of a linear η-orthogonal transformation of the coordinates $t^1, \dots, t^n$.

Examples of Frobenius manifolds.

Example 1.5. Trivial Frobenius manifold. Let A be a graded Frobenius algebra. That means that some weights q_1, ..., q_n are assigned to the basic vectors e_1, ..., e_n such that

$$c_{\alpha\beta}^\gamma = 0 \text{ for } q_\alpha + q_\beta \neq q_\gamma \tag{1.60a}$$

and also

$$\eta_{\alpha\beta} = 0 \text{ for } q_\alpha + q_\beta \neq d \tag{1.60b}$$

for some d. Here

$$e_\alpha e_\beta = c_{\alpha\beta}^\gamma e_\gamma$$

$$\eta_{\alpha\beta} = < e_\alpha, e_\beta >$$

in the algebra A. Particularly, any Frobenius algebra can be considered as a graded one w.r.t. the trivial grading $q_\alpha = d = 0$.

These formulae define a structure of Frobenius manifold on $M = A$. The corresponding free energy F is a cubic function

$$F(t) = \frac{1}{6} c_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma = \frac{1}{6} < \mathbf{t}^3, e > \tag{1.61}$$

for

$$c_{\alpha\beta\gamma} = \eta_{\alpha\epsilon} c^{\epsilon}_{\alpha\beta},$$

$$\mathbf{t} = t^\alpha e_\alpha, \quad e = e_1 \text{ is the unity.}$$

Here the degrees of the coordinates t^α and of the function F are

$$d_\alpha = 1 - q_\alpha, \quad d_F = 3 - d.$$

For example, the cohomology ring $A = H^*(X)$ of a $2d$-dimensional oriented closed manifold X satisfying

$$H^{2i+1}(X) = 0 \text{ for any } i \tag{1.62}$$

is a graded Frobenius algebra w.r.t. the cup product and the Poincaré duality pairing. The degree of an element $x \in H^{2q}(X)$ equals q.

Remark 1.5. To get rid of the restriction (1.62) one is to generalize the notion of Frobenius manifold to supermanifolds, i.e. to admit anticommuting coordinates t^α. Such a generalization was done by Kontsevich and Manin [92].

Example 1.6. The direct product $M' \times M''$ of two Frobenius manifolds of the dimensions n and m resp. carries a natural structure of a Frobenius manifold if the scaling dimensions satisfy the constraint

$$\frac{d_1'}{d_1''} = \frac{d_F'}{d_F''}. \tag{1.63}$$

If the flat coordinates $t^{1'}, \dots, t^{n'}$, $t^{1''}, \dots, t^{m''}$ are normalized as in (1.18), and $\deg t^{1'} = \deg t^{1''} = 1$, then (1.63) reads $\deg t^{n'} = \deg t^{m''}$. Thus only the case $m > 1,\ n > 1$ is of interest. The prepotential F for the direct product has the form

$$F\left(t^1, \hat t^1, t^{2'}, \dots, t^{n-1'}, t^{2''}, \dots, t^{m-1''}, \hat t^N, t^N\right) =$$

$$= \frac{1}{2} {t^1}^2 t^N + t^1 \hat t^1 \hat t^N + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^{\alpha'} t^{n-\alpha+1'} + \frac{1}{2} t^1 \sum_{\beta=2}^{m-1} t^{\beta''} t^{m-\beta+1''} +$$

$$+ f'\left(t^{2'}, \dots, t^{n-1'}, \frac{1}{2}(t^N + \hat t^N)\right) + f''\left(t^{2''}, \dots, t^{m-1''}, \frac{1}{2}(t^N - \hat t^N)\right) \tag{1.64a}$$

where the functions $f'(t^{2'}, \dots, t^{n'})$ and $f''(t^{2''}, \dots, t^{m''})$ determine the prepotentials of M' and M'' in the form (1.18). Here $N = n + m$,

$$\begin{aligned} t^1 &= \frac{t^{1'} + t^{1''}}{2}, \quad & \hat t^1 &= \frac{t^{1'} - t^{2''}}{2} \\ t^N &= t^{n'} + t^{m''}, \quad & \hat t^N &= t^{n'} - t^{m''}. \end{aligned} \tag{1.64b}$$

Observe that only trivial Frobenius manifolds can be multiplied by a one-dimensional Frobenius manifold.

Example 1.7. [42] M is the space of all polynomials of the form

$$M = \{\lambda(p) = p^{n+1} + a_n p^{n-1} + \ldots + a_1 | a_1, \ldots, a_n \in \mathbf{C}\} \tag{1.65}$$

with a nonstandard affine structure. We identify the tangent plane to M with the space of all polynomials of the degree less than n. The algebra A_λ on $T_\lambda M$ by definition coincides with the algebra of truncated polynomials

$$A_\lambda = \mathbf{C}[p]/(\lambda'(p)) \tag{1.66a}$$

(the prime denotes d/dp). The invariant inner product is defined by the formula

$$< f, g >_\lambda = \operatorname*{res}_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)}. \tag{1.66b}$$

The unity vector field e and the Euler vector field E read

$$e := \frac{\partial}{\partial a_1}, \quad E := \frac{1}{n+1} \sum_i (n-i+1) a_i \frac{\partial}{\partial a_i}. \tag{1.66c}$$

We will see that this is an example of a Frobenius manifold in Lecture 4.

Remark 1.6. The notion of Frobenius manifolfd admits algebraic formalization in terms of the ring of functions on a manifold. More precisely, let R be a commutative associative algebra with a unity over a field k of characteristics $\neq 2$. We are interested in structures of Frobenius algebra over R in the R-module of k-derivations $Der(R)$ (i.e. $u(\kappa) = 0$ for $\kappa \in k, u \in Der(R)$) satisfying

$$\tilde\nabla_u(\lambda)\tilde\nabla_v(\lambda) - \tilde\nabla_v(\lambda)\tilde\nabla_u(\lambda) = \tilde\nabla_{[u,v]}(\lambda) \;\text{ identicaly in } \lambda \tag{1.67a}$$

$$\text{for } \tilde\nabla_u(\lambda)v = \nabla_u v + \lambda u \cdot v, \tag{1.67b}$$

(see below Lemma 3.1)

$$\nabla_u e = 0 \text{ for all } u \in Der(R) \tag{1.67c}$$

where e is the unity of the Frobenius algebra $Der(R)$. Non-degenerateness of the symmetric inner product

$$< \, , \, >: Der(R) \times Der(R) \to R$$

means that it provides an isomorphism $\mathrm{Hom}_R(Der(R), R) \to Der(R)$. I recall that the covariant derivative is a derivation $\nabla_u v \in Der(R)$ defined for any $u, \; v \in Der(R)$ being determined from the equation

$$< \nabla_u v, w >=$$

$$\frac{1}{2}[u < v, w > + v < w, u > - w < u, v > + < [u, v], w > + < [w, u], v > + < [w, v], u >] \tag{1.68}$$

for any $w \in Der(R)$ (here [,] denotes the commutator of derivations).

To reformulate algebraically the scaling invariance (1.50) we need to introduce gradings to the algebras R and $Der(R)$. In the case of algebras of functions the gradings are determined by the assumptions

$$\deg t^\alpha = 1 - q_\alpha, \text{ for } q_\alpha \neq 1, \quad \text{or } \deg e^{t^\alpha} = r_\alpha \text{ for } q_\alpha = 1, \tag{1.69a}$$

$$\deg \partial_\alpha = q_\alpha \tag{1.69b}$$

where the numbers q_α, r_α are defined by the Euler vector field

$$E = \sum_\alpha [(1 - q_\alpha)t^\alpha + r_\alpha]\partial_\alpha. \tag{1.69c}$$

Remark 1.7. From the definition it follows that the topology of a Frobenius manifold is very simple: it is a domain in $\mathbf{C}^k \times \mathbf{C}^{*l}$ for some k, l, $k + l = n$ due to existence of locally Euclidean metric on M. Below (in Appendix B) we will modify the definition introducing twisted Frobenius manifolds. They are glued from Frobenius manifolds as from the building blocks. The topology of twisted Frobenius manifolds can be nontrivial (see also Lecture 5).

Appendix A.
Polynomial solutions of WDVV. Towards classification of Frobenius manifolds with good analytic properties.

Let all the structure constants of a Frobenius manifold be analytic in the point $t = 0$. Then the germ of the Frobenius manifold near the point $t = 0$ can be considered as a deformation of the Frobenius algebra $A_0 := T_{t=0}M$. This is a graded Frobenius algebra with a basis $e_1, \ldots, e_n$ and with the structure constants $c_{\alpha\beta}^{\gamma}(0)$. The degrees of the basic vectors are

$$\deg e_\alpha = q_\alpha$$

where the numbers q_α are defined in (1.21). The algebras T_tM for $t \neq 0$ can be considered thus as *deformations* of the graded Frobenius algebra T_0M. In the physical setting (see Lecture 2 below) T_0M is the primary chiral algebra of the corresponding topological conformal field theory. The algebras T_tM are operator algebras of the perturbed topological field theory. So the problem of classification of analytic deformations of graded Frobenius algebras looks to be also physically motivated. (Probably, analytic deformability in the sense that the graded Frobenius algebra can be the tangent algebra at the origin of an analytic Frobenius manifold imposes a strong constraint on the algebra A_0.)

We consider here the case where all the degrees $\deg t^\alpha$ are *real positive* numbers and not all of them are equal. In the normalization (1.18) that means that $0 < d < 1$.

Problem. To find all the solutions of WDVV being analytic in the origin $t = 0$ with real positive degrees of the flat coordinates.

Notice that for the positive degrees analyticity in the origin and the quasihomogeneity (1.5) implies that the function $F(t)$ is a polynomial in $t^1, \ldots, t^n$. So the problem coincides with the problem of finding of the *polynomial solutions* of WDVV.

For $n = 2$ all the noncubic polynomial solutions have the form (1.24a) where k is an integer and $k \geq 4$. Let us consider here the next case $n = 3$. Here we have a function F of the form (1.26) and

$$\deg t^1 = 1,\ \deg t^2 = 1 - \frac{d}{2},\ \deg t^3 = 1 - d,\ \deg f = 3 - d. \tag{A.1}$$

The function f must satisfy the equation (1.28). If

$$f(x, y) = \sum a_{pq} x^p y^q$$

then the condition of quasihomogeneity reads

$$a_{pq} \neq 0 \text{ only for } p + q - 3 = (\frac{1}{2}p + q - 1)d. \tag{A.2}$$

Hence d must be a rational number. Solving the quasihomogeneity equation (A.2) we obtain the following two possibilities for the function f: 1.

$$f = \sum_k a_k x^{4-2km} y^{kn-1} \tag{A.3a}$$

$$d = \frac{n-2m}{n-m} \tag{A.3b}$$

for some natural numbers $n,\ m$, n is odd, and 2.

$$f = \sum_k a_k x^{4-km} y^{kn-1} \tag{A.4a}$$

$$d = \frac{2(n-m)}{2n-m} \tag{A.4b}$$

for some natural numbers $n,\ m$, m is odd. Since the powers in the expansions of f must be nonnegative, we obtain the following three possibilities for f:

$$f = ax^2y^{n-1} + by^{2n-1}, \quad n \geq 3 \tag{A.5a}$$

$$f = ay^{n-1}, \quad n \geq 5 \tag{A.5b}$$

$$f = ax^3y^{n-1} + bx^2y^{2n-1} + cxy^{3n-1} + dy^{4n-1}, \quad n \geq 2. \tag{A.5c}$$

The inequalities for n in these formulae can be assumed since the case when f is at most cubic polynomial is not of interest. (It is easy to see that for a cubic solution with $n = 3$ necessary $f = 0$.)

Substituting (A.5) to (1.28) we obtain resp.

$$\begin{aligned} a(n-1)(n-2)(n-3) &= 0 \\ -4a^2(n-1)^2 + b(2n-1)(2n-2)(2n-3) &= 0 \end{aligned} \tag{A.6a}$$

$$a(n-1)(n-2)(n-3) = 0 \tag{A.6b}$$

$$\begin{aligned} a(n-1)(n-2)(n-3) &= 0 \\ 18a^2\left((n-1)(n-2) - 2(n-1)^2\right) + b(2n-1)(2n-2)(2n-3) &= 0 \\ c(3n-1)(3n-2)(3n-3) &= 0 \\ -4b^2(2n-1)^2 + 6ac(3n-1)(3n-2) + d(4n-1)(4n-2)(4n-3) &= 0. \end{aligned} \tag{A.6c}$$

It is clear that a must not vanish for a nonzero f. So n must equal 3 in the first case (A.6a), 2 or 3 in the third one, and in the second case (A.6b) there is no non-zero solutions. Solving the system (A.6a) for $n = 3$ and the system (A.6c) for $n = 2$ and $n = 3$ we obtain the following 3 polynomial solutions of WDVV:

$$F = \frac{t_1^2t_3 + t_1t_2^2}{2} + \frac{t_2^2t_3^2}{4} + \frac{t_3^5}{60} \tag{A.7}$$

$$F = \frac{t_1^2t_3 + t_1t_2^2}{2} + \frac{t_2^3t_3}{6} + \frac{t_2^2t_3^3}{6} + \frac{t_3^7}{210} \tag{A.8}$$

$$F = \frac{t_1^2t_3 + t_1t_2^2}{2} + \frac{t_2^3t_3^2}{6} + \frac{t_2^2t_3^5}{20} + \frac{t_3^{11}}{3960}. \tag{A.9}$$

These are unique up to the equivalence noncubic polynomial solutions of WDVV with $n = 3$ with positive degrees of t^α.

For the case $d = 1$ one can easily find a polynomial solution

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + t_2^4. \qquad (A.10)$$

This admits a one-parameter group of equivalences

$$t_3 \mapsto t_3 + const.$$

The corresponding Frobenius manifold is essentially reduced to a two-dimensional one.

Refining above arguments we prove the following

Theorem A.1. *1). All noncubic polynomial solutions of WDVV for $n = 3$ are equivalent to one of (A.7), (A.8), (A.9), (A.10).*

2). Besides these for $n = 3$ there exist solutions analytic at $t = 0$ only for $d = 2\frac{m+1}{m+2}$ (one-parameter family of solutions), $d = 2\frac{m+2}{m+4}$ (one solution) and $d = 2\frac{m+3}{m+6}$ (one solution) for an arbitrary integer m.

Proof. From the ansatz (1.30a) we obtain for the function $f(x, y)$ an expansion

$$f(x, y) = \sum a_k y^{k-1} x^{4+qk}$$

where, we recall that

$$q = -2\frac{1-d}{2-d}.$$

So q must be a rational number. For $q = 0$ we obtain the solution (A.10). For negative

$$q = -\frac{m}{n}$$

we rewrite the expansion for f in the form

$$f = \sum_{k\geq 1} a_k x^{4-mk} y^{kn-1}.$$

So $m \leq 4$. This gives the solutions (A.7) - (A.9). For positive

$$q = \frac{m}{n}$$

substitution of the series

$$f = \sum_{k\geq 1} a_k x^{4+mk} y^{kl-1}$$

into (1.31a) gives

$$(n-1)(n-2)(n-3)a_1 = 0$$

and the recursion relations of the form

$$(kn-1)(kn-2)(kn-3)a_k = \text{polynomial in } a_1, \ldots, a_{k-1}.$$

This proves the second part of the theorem.

The polynomial (A.7) coincides with the prepotential of Example 1.7 with $n=3$. The crucial observation to understand the nature of the other polynomials (A.8) and (A.9) was done by V.I.Arnold. He observed that the degrees $5=4+1$, $7=6+1$, and $11=10+1$ of the polynomials have a simple relation to the Coxeter numbers of the groups of symmetries of the Platonic solids (4, 6, 10 for the tetrahedron, cube and icosahedron resp.). In Lecture 4 I will show how to explain this observation using a hidden symmetry of WDVV (see also Appendix G). I do not know the sense of the solutions in the form of infinite power series described in the second part of the theorem.

We will consider now solutions of WDVV with $n=3$ and $d=1$. The Euler vector field must have the form

$$E = t_1\partial_1 + \frac{1}{2}t_2\partial_2 + r\partial_3. \qquad (A.11)$$

We will assume that $r>0$ (the case $r=0$ will be considered in Appendix C). We will be interested in solutions of the form

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \sum_{k,l\geq 0} a_{kl} t_2^k e^{lt_3}. \qquad (A.12)$$

Theorem A.2. *For $d=1$ and $r>0$ there exist only 3 solutions of WDVV of the form (A.12):*

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{1}{24}t_2^4 + t_2 e^{t_3}, \quad r=\frac{3}{2} \qquad (A.13)$$

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \frac{1}{2}t_2^4 + t_2^2 e^{t_3} - \frac{1}{48}e^{2t_3}, \quad r=1 \qquad (A.14)$$

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{1}{72}t_2^4 + \frac{2}{3}t_2^3 e^{t_3} + \frac{2}{3}t_2^2 e^{2t_3} + \frac{9}{16}e^{4t_3}, \quad r=\frac{1}{2}. \qquad (A.15)$$

The proof is obtained by a direct substitution of the ansatz (1.30b) to the equation (1.31b)

In Lecture 4 it will be shown that the three solutions correspond to certain extensions of the affine Weyl groups of the type A_2, B_2 and G_2 resp.

More generally, we will be interested in solutions of WDVV of the form

$$F(t) = \text{cubic} + \sum_{k_1,k_2,\ldots\geq 0} a_{k_1k_2\ldots}(t'')\exp\left(k_1t'_1 + k_2t'_2 + \ldots\right) \qquad (A.16)$$

where the coordinates t are subdivided in two parts $t=(t',t'')$ in such a way that $\deg t'=0$, $\deg t''\neq 0$, assuming convergence of the series in a neibourghood of $t'=-\infty$, $t''=0$. These will be called *solutions with good analytic properties.* From the first experiments of this Appendix it follows that solutions with good analytic properties are isolated points in the space of all solutions of WDVV (at least, under some additional assumptions about the solution, like the semi-simplicity assumption below). The results of this paper suggest a conjectural correspondence between Frobenius manifolds with good analytic properties and certain reflection groups and their extensions.

Appendix B.
Symmetries of WDVV. Twisted Frobenius manifolds.

By definition *symmetries* of WDVV are the transformations

$$\begin{aligned} t^\alpha &\mapsto \hat t^\alpha, \\ \eta_{\alpha\beta} &\mapsto \hat\eta_{\alpha\beta}, \\ F &\mapsto \hat F \end{aligned} \tag{B.1}$$

preserving the equations. First examples of the symmetries have been introduced above: they are equivalencies of Frobenius manifolds and shifts along vectors belonging to the kernel of the grading operator Q.

Here we describe two types of less trivial symmetries for which the map $t^\alpha \mapsto \hat t^\alpha$ preserves the multiplication of the vector fields.

Type 1. Legendre-type transformation S_κ for a given $\kappa = 1, \ldots, n$

$$\hat t_\alpha = \partial_\alpha \partial_\kappa F(t) \tag{B.2a}$$

$$\frac{\partial^2 \hat F}{\partial \hat t^\alpha \partial \hat t^\beta} = \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} \tag{B.2b}$$

$$\hat\eta_{\alpha\beta} = \eta_{\alpha\beta}. \tag{B.2c}$$

We have

$$\partial_\alpha = \partial_\kappa \cdot \hat\partial_\alpha. \tag{B.3}$$

So the transformation S_κ is invertible where ∂_κ is an invertible element of the Frobenius algebra of vector fields. Note that the unity vector field

$$e = \frac{\partial}{\partial \hat t^\kappa}. \tag{B.2d}$$

The transformation S_1 is the identity; the transformations S_κ commute for different κ.

To describe what happens with the scaling degrees (assuming diagonalizability of the degree operator Q) we shift the degrees putting

$$\mu_\alpha := q_\alpha - \frac{d}{2}, \quad \alpha = 1, \ldots, n. \tag{B.4}$$

Observe that the spectrum consists of the eigenvalues of the operator $-\hat V$ (see (1.51)). The shifted degrees are symmetric w.r.t. zero

$$\mu_\alpha + \mu_{n-\alpha+1} = 0. \tag{B.5}$$

We will call the numbers $\mu_1, \ldots, \mu_n$ *spectrum* of the Frobenius manifold. Knowing the spectrum we can uniquely reconstruct the degrees putting

$$q_\alpha = \mu_\alpha - \mu_1, \quad d = -2\mu_1. \tag{B.6}$$

It is easy to see that the transformations S_κ preserve the spectrum up to permutation of the numbers $\mu_1, \ldots, \mu_n$: for $\kappa \neq \frac{n}{2}$ it interchanges the pair (μ_1, μ_n) with the pair $(\mu_\kappa, \mu_{n-\kappa+1})$. For $\kappa = \frac{n}{2}$ the transformed Frobenius manifold is of the second type (1.22).

To prove that (B.2) determines a symmetry of WDVV we introduce on the Frobenius manifold M a new metric $<\ ,\ >_\kappa$ putting

$$< a, b >_\kappa := < \partial_\kappa^2, a \cdot b > . \tag{B.7}$$

Exercise B.1. Prove that the variables $\hat t^\alpha$ (B.2a)) are the flat coordinates of the metric (B.7). Prove that

$$< \hat\partial_\alpha \cdot \hat\partial_\beta, \hat\partial_\gamma >_\kappa = \hat\partial_\alpha \hat\partial_\beta \hat\partial_\gamma \hat F(\hat t). \tag{B.8}$$

Example B.1. For

$$F = \frac{1}{2} {t^1}^2 t^2 + e^{t^2}$$

$(d = 1)$ the transformation S_2 gives

$$\hat t^1 = e^{t^2}$$
$$\hat t^2 = t^1.$$

Renumbering $\hat t^1 \leftrightarrow \hat t^2$ (due to (B.2d)) we obtain

$$\hat F = \frac{1}{2} (\hat t^1)^2 \hat t^2 + \frac{1}{2} (\hat t^2)^2 \left(\log t^2 - \frac{3}{2} \right). \tag{B.9}$$

This coincides with (1.24b) (now $d = -1$). See also Example 5.5 below.

If there are coincidences between the degrees

$$q_{\kappa_1} = \ldots = q_{\kappa_s} \tag{B.10}$$

then we can construct more general transformation S_c putting

$$\hat t_\alpha = \sum_{i=1}^s c^i \partial_\alpha \partial_{\kappa_i} F(t) \tag{B.11}$$

for arbitrary constants $(c^1, \ldots, c^s) =: c$. This is invertible when the vector field

$$\sum c^i \partial_{\kappa_i}$$

is invertible. The transformed metric on M depends quadratically on c^i

$$< a, b >_c := < \left(\sum c^i \partial_{\kappa_i} \right)^2, a \cdot b > . \tag{B.12}$$

Type 2. The inversion I:

$$\hat{t}^1 = \frac{1}{2}\frac{t_\sigma t^\sigma}{t^n}$$
$$\hat{t}^\alpha = \frac{t^\alpha}{t^n} \text{ for } \alpha \neq 1,\, n \qquad (B.13a)$$
$$\hat{t}^n = -\frac{1}{t^n}$$

(the coordinates are normalized as in (1.18)),

$$\hat{F}(\hat{t}) = (t^n)^{-2}\left[F(t) - \frac{1}{2}t^1 t_\sigma t^\sigma\right] = (\hat{t}^n)^2 F + \frac{1}{2}\hat{t}^1\hat{t}_\sigma\hat{t}^\sigma, \qquad (B.13b)$$
$$\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}.$$

Note that the inversion acts as a conformal transformation of the metric $<\,,\,>$

$$\eta_{\alpha\beta}d\hat{t}^\alpha d\hat{t}^\beta = (t^n)^{-2}\eta_{\alpha\beta}dt^\alpha dt^\beta. \qquad (B.14)$$

The inversion *changes* the spectrum $\mu_1, \ldots, \mu_n$.

Lemma B.1. *If*

$$E(t) = \sum (1 - q_\alpha)t^\alpha \partial_\alpha \qquad (B.15)$$

then after the transform one obtains

$$\hat{E}(\hat{t}) = \sum (1 - \hat{q}_\alpha)\hat{t}^\alpha \hat{\partial}_\alpha \qquad (B.16a)$$

where

$$\hat{\mu}_1 = -1 + \mu_n,\ \hat{\mu}_n = 1 + \mu_1,\ \hat{\mu}_\alpha = \mu_\alpha \text{ for } \alpha \neq 1,\, n. \qquad (B.16b)$$

Particularly,

$$\hat{d} = 2 - d. \qquad (B.16d)$$

If

$$E(t) = \sum (1 - q_\alpha)t^\alpha \partial_\alpha + \sum_{q_\sigma = 1} r^\sigma \partial_\sigma$$

and $d \neq 1$, or $d = 1$ but $r^n = 0$, then

$$\hat{E}(\hat{t}) = \hat{E}^\alpha(\hat{t})\hat{\partial}_\alpha \qquad (B.17a)$$

where

$$\hat{E}^1 = \hat{t}^1 + \sum_{q_\sigma = 1} r^\sigma \hat{t}^{n-\sigma+1}$$
$$\hat{E}^\alpha = (d - q_\alpha)\hat{t}^\alpha \quad \text{for any } \alpha \text{ s.t. } q_\alpha \neq 1 \qquad (B.17b)$$
$$\hat{E}^\sigma = (d-1)\hat{t}^\sigma - r^\sigma \hat{t}^n \quad \text{for any } \sigma \text{ s.t. } q_\sigma = 1$$
$$\hat{E}^n = (d-1)\hat{t}^n.$$

Proof is straightforward.

The transformation of the type 2 looks more misterious (we will see in Lecture 3 that this is a Schlesinger transformation of WDVV in the sense of [135]). We leave to the reader to verify that the inversion preserves the multiplication of vector fields. Hint: use the formulae

$$
\begin{aligned}
\hat c_{\alpha\beta\gamma} &= t_1 c_{\alpha\beta\gamma} - t_\alpha \eta_{\beta\gamma} - t_\beta \eta_{\alpha\gamma} - t_\gamma \eta_{\alpha\beta} + \delta_{\alpha\,1}\eta_{\beta\gamma} + \delta_{\beta\,1}\;\eta_{\alpha\gamma} + \delta_{\gamma\,1}\eta_{\alpha\beta}\\
\hat c_{\alpha\beta n} &= t_1 c_{\alpha\beta\sigma} t^\sigma - \frac{1}{2}\eta_{\alpha\beta} t_\sigma t^\sigma - t_\alpha t_\beta + \delta_{\alpha\,1}\delta_{\beta\,1}\\
\hat c_{\alpha n n} &= t_1 c_{\alpha\lambda\mu} t^\lambda t^\mu - t_\alpha t_\sigma t^\sigma\\
\hat c_{nnn} &= t_1 c_{\lambda\mu\nu} t^\lambda t^\mu t^\nu - \frac{3}{4}(t_\sigma t^\sigma)^2
\end{aligned}
\tag{B.18a}
$$

(here $\alpha, \beta, \gamma \neq 1,\, n$) together with (B.14) to prove that

$$
\hat c_{\alpha\beta\gamma} = (t^n)^{-2} \frac{\partial t^\lambda}{\partial \hat t^\alpha} \frac{\partial t^\mu}{\partial \hat t^\beta} \frac{\partial t^\nu}{\partial \hat t^\gamma} c_{\lambda\mu\nu}. \tag{B.18b}
$$

Exercise B.2. Show that the solution (1.24c) is the I-transform of the solution (1.24b).

Exercise B.3. Prove that the group $SL(2, \mathbf{C})$ acts on the space of solutions of WDVV with $d = 1$ by

$$
\begin{aligned}
t^1 &\mapsto t^1 + \frac{1}{2}\frac{c}{ct^n + d}\sum_{\sigma\neq 1} t_\sigma t^\sigma\\
t^\alpha &\mapsto \frac{t^\alpha}{ct^n + d}\\
t^n &\mapsto \frac{at^n + b}{ct^n + d},\\
&ad - bc = 1.
\end{aligned}
\tag{B.19}
$$

[Hint: consider superpositions of I with the shifts along t^n.]

The inversion is an involutive transformation up to an equivalence

$$
\begin{gathered}
I^2 : (t^1, t^2, \ldots, t^{n-1}, t^n) \mapsto (t^1, -t^2, \ldots, -t^{n-1}, t^n)\\
F \mapsto F.
\end{gathered}
\tag{B.20}
$$

Proposition B.1. *Assuming invertibility of the transformations S_κ, I we can reduce by these transformations any solution of WDVV to a solution with*

$$
0 \leq q_\alpha \leq d \leq 1. \tag{B.21}
$$

Definition B.1. A Frobenius manifold will be called *reduced* if it satisfies the inequalities (B.21).

Particularly, the transformations S_κ are invertible near those points t of a Frobenius manifold M where the algebra T_tM has no nilpotents. In the next Lecture we will obtain complete local classification of such Frobenius manifolds.

Using the above transformations I, S we can glue together a few Frobenius manifolds to obtain a more complicated geometrical object that will be called *twisted Frobenius manifold*. The multiplication of tangent vector fields is globally well-defined on a twisted Frobenius manifold. But the invariant inner product (and therefore, the function F) is defined only locally. On the intersections of the coordinate charts these are to transform according to the formula (B.12) or (B.14). We will construct examples of twisted Frobenius manifolds in Appendices C, J. Twisted Frobenius manifolds could also appear as the moduli spaces of the topological sigma models of the B-type [151] where the flat metric is well-defined only locally.

Appendix C.
WDVV and Chazy equation.
Affine connections on curves with projective structure.

Here we consider three-dimensional Frobenius manifolds with $d = 1$. The degrees of the flat variables are

$$\deg t^1 = 1, \ \deg t^2 = 1/2, \ \deg t^3 = 0, \tag{C.1a}$$

and the Euler vector field

$$E = t^1 \partial_1 + \frac{1}{2} t^2 \partial_2. \tag{C.1b}$$

Let us look for a solution of WDVV being periodic in t^3 with the period 1 (modulo quadratic terms) and analytic in the point $t^1 = t^2 = 0, \ t^3 = i\infty$. The function F must have the form

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16}\gamma(t^3) \tag{C.2}$$

for some unknown 1-periodic function $\gamma = \gamma(\tau)$ analytic at $\tau = i\infty$

$$\gamma(\tau) = \sum_{n\geq 0} a_n q^n, \quad q = e^{2\pi i \tau}. \tag{C.3}$$

The coefficients a_n are determined up to a shift along τ,

$$\tau \mapsto \tau + \tau_0, \quad a_n \mapsto a_n e^{2\pi i n \tau_0}. \tag{C.4}$$

For the function γ we obtain from (1.28)

$$\gamma''' = 6\gamma\gamma'' - 9{\gamma'}^2. \tag{C.5}$$

Exercise C.1. Prove that the equation (C.5) has a unique (modulo the ambiguity (C.4)) nonconstant solution*) of the form (C.3),

$$\gamma(\tau) = \frac{\pi i}{3}\left[1 - 24q - 72q^2 - 96q^3 - 168q^4 - \dots\right], \quad q = e^{2\pi i \tau}. \tag{C.6}$$

The equation (C.5) was considered by J.Chazy [31] as an example of ODE with the general solution having moving natural boundary. It arose as a reduction of the self-dual Yang - Mills equation in [1]. Following [1, 142] I will call (C.5) *Chazy equation.*

---

*) The solution can be reduced to a real one by the change

$$t_3 \mapsto i t_3, \ t_1 \mapsto i^{-1} t_1, \ F \mapsto i^{-1} F.$$

Exercise C.2. Show that the roots $\omega_1(\tau)$, $\omega_2(\tau)$, $\omega_3(\tau)$ of the cubic equation

$$\omega^3 + \frac{3}{2}\gamma(\tau)\omega^2 + \frac{3}{2}\gamma'(\tau)\omega + \frac{1}{4}\gamma''(\tau) = 0 \tag{C.7}$$

satisfy the system

$$\begin{aligned} \dot\omega_1 &= -\omega_1(\omega_2 + \omega_3) + \omega_2\omega_3 \\ \dot\omega_2 &= -\omega_2(\omega_1 + \omega_3) + \omega_1\omega_3 \\ \dot\omega_3 &= -\omega_3(\omega_1 + \omega_2) + \omega_1\omega_2. \end{aligned} \tag{C.8}$$

The system (C.8) was integrated by Halphen [75]. It was rediscovered in the context of the self-dual Einstein equations by Atiyah and Hitchin [11].

The main property [31] of Chazy equation is the invariance w.r.t. the group $SL(2,\mathbf{C})$

$$\tau \mapsto \tilde\tau = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \tag{C.9a}$$

$$\gamma(\tau) \mapsto \tilde\gamma(\tilde\tau) = (c\tau + d)^2\gamma(\tau) + 2c(c\tau + d). \tag{C.9b}$$

The invariance (C.9) follows immediately from the invariance of WDVV w.r.t. the transformation (3.19). Observe that (C.9b) coincides with the transformation law of one-dimensional affine connection w.r.t. the Möbius transformations (C.9a) (cf. [142]).

We make here a digression about one-dimensional affine connections. One-dimensional real or complex manifolds will be considered; in the complex case only holomorphic connections will be of interest. The connection is determined by a function

$$\gamma(\tau) := \Gamma^1_{11}(\tau)$$

(holomorphic in the complex case, and $\Gamma^{\bar 1}_{\bar 1 \bar 1} = \overline{\gamma(\tau)}$) for any given local coordinate τ. The covariant derivative of a k-tensor $f(\tau)\, d\tau^k$ by definition is a $(k+1)$-tensor of the form

$$\nabla f(\tau)\, d\tau^{k+1} := \left(\frac{df}{d\tau} - k\gamma(\tau)f(\tau)\right) d\tau^{k+1}. \tag{C.10}$$

This implies that under a change of coordinate

$$\tau \mapsto \tilde\tau = \tilde\tau(\tau) \tag{C.11a}$$

(holomorphic in the complex case) the connection must transform as follows

$$\tilde\gamma(\tilde\tau) = \frac{1}{d\tilde\tau/d\tau}\gamma(\tau) - \frac{d^2\tilde\tau/d\tau^2}{d\tilde\tau/d\tau}. \tag{C.11b}$$

One-dimensional affine connection has no local invariants: it can be reduced to zero by an appropriate change of coordinate. To find the *flat* local parameter x one is to look for a 1-form $\omega = \phi(\tau)\, d\tau$ such that $\nabla\omega = 0$, i.e.

$$\frac{d\phi}{d\tau} - \gamma\phi = 0 \tag{C.12a}$$

and then put

$$\omega = dx. \tag{C.12b}$$

The covariant derivative of a k-tensor $f\,dx^k$ coincides with the usual derivative w.r.t. the flat coordinate x

$$\nabla f\,dx^{k+1} \equiv \frac{df}{dx}\,dx^{k+1}. \tag{C.13}$$

In arbitrary coordinate τ the covariant derivative can be written in the form

$$\nabla f\,d\tau^{k+1} = \phi^k \frac{d}{d\tau}\left(f\phi^{-k}\right)\,d\tau^{k+1}. \tag{C.14}$$

Let us assume now that there is fixed a projective structure on the one-dimensional manifold. That means that the transition functions (C.11a) now are not arbitrary but they are the Möbius transformations (C.9a). Then the transformation law (C.11b) coincides with (C.9b).

When is it possible to reduce the connection to zero by a Möbius transformation? What is the complete list of differential-geometric invariants of an affine connection on one-dimensional manifold with a projective structure?

The following simple construction gives the answer to the questions.

Proposition C.1.

1. For a one-dimensional connection γ the quadratic differential

$$\Omega\,d\tau^2, \quad \Omega := \frac{d\gamma}{d\tau} - \frac{1}{2}\gamma^2 \tag{C.15}$$

is invariant w.r.t. the Möbius transformations.

2. The connection γ can be reduced to zero by a Möbius transformation iff $\Omega = 0$.

Proof. The verification of the invariance of $\Omega\,d\tau^2$ is straightforward. From this it follows that $\Omega = 0$ when γ is reducible to 0 by a Möbius transformation. Conversely, solving the equation $\Omega = 0$ we obtain

$$\gamma = -\frac{2}{\tau - \tau_0}.$$

After the inversion

$$\tilde{\tau} = \frac{1}{\tau - \tau_0}$$

we obtain $\tilde{\gamma}(\tilde{\tau}) = 0$. Proposition is proved.

Remark C.1. For arbitrary change of coordinate $\tilde{\tau} = \tilde{\tau}(\tau)$ the "curvature" Ω transforms like projective connection

$$\tilde{\Omega} = \left(\frac{d\tau}{d\tilde{\tau}}\right)^2 \Omega + S_{\tilde{\tau}}(\tau) \tag{C.16}$$

where $S_{\tilde{\tau}}(\tau)$ stands for the Schwartzian derivative

$$S_z(w) := \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2. \qquad (C.17)$$

We obtain a map

$$\text{affine connections} \;\to\; \text{projective connections}.$$

This is the appropriate differential-geometric interpretation of the well-known Miura transformation.

Exercise C.3. Let $P = P(\gamma, d\gamma/d\tau, d^2\gamma/d\tau^2, \ldots)$ be a polynomial such that for any affine connection γ the tensor $P\, d\tau^k$ for some k is invariant w.r.t. Möbius transformations. Prove that P can be represented as

$$P = Q(\Omega, \nabla\Omega, \nabla^2\Omega, \ldots) \qquad (C.18)$$

where Q is a graded homogeneous polynomial of the degree k assuming that $\deg \nabla^l\Omega = l+2$.

In other words, the "curvature" Ω and the covariant derivatives of it provide the complete set of differential-geometric invariants of an affine connection on a one-dimensional manifold with a projective structure.

Example C.1. Consider a Sturm - Liouville operator

$$L = -\frac{d^2}{dx^2} + u(x), \quad x \in D, \quad D \subset S^1 \text{ or } D \subset \mathbf{CP}^1. \qquad (C.19)$$

It determines a projective structure on D in the following standard way. Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the differential equation

$$L\,y = 0. \qquad (C.20)$$

We introduce a new local coordinate τ in D putting

$$\tau = \frac{y_2(x)}{y_1(x)}. \qquad (C.21)$$

This specifies a projective structure in D. If D is not simply connected then a continuation of $y_1(x),\ y_2(x)$ along a closed curve gives a linear substitution

$$y_1(x) \mapsto c y_2(x) + d y_1(x)$$

$$y_2(x) \mapsto a y_2(x) + b y_1(x) \qquad (C.22)$$

for some constants $a,\ b,\ c,\ d,\ ad - bc = 1$ (conservation of the Wronskian $y_1 y_2' - y_2 y_1'$). This is a Möbius transformation of the local parameter τ. Another choice of the basis $y_1(x),\ y_2(x)$ produces an equivalent projective structure in D.

We have also a natural affine connection in D. It is uniquely specified by saying that x is the flat coordinate for the connection. What is the "curvature" Ω of this affine connection w.r.t. the projective structure (C.21)? The answer is[†]

$$\Omega d\tau^2 = 2u(x)dx^2 \tag{C.23}$$

(verify it!).

Let us come back to the Chazy equation. It is natural to consider the general class of equations of the form

$$P(\gamma, d\gamma/d\tau, \ldots, d^{k+1}\gamma/d\tau^{k+1}) = 0 \tag{C.24}$$

for a polynomial P invariant w.r.t. the transformations (C.9). Due to (C.18) these can be rewritten in the form

$$Q(\Omega, \nabla\Omega, \ldots, \nabla^k\Omega) = 0 \tag{C.25a}$$

for

$$\Omega = \frac{d\gamma}{d\tau} - \frac{1}{2}\gamma^2, \tag{C.25b}$$

Q is a graded homogeneous polynomial with $\deg \nabla^l\Omega = l + 2$. Putting

$$u := \frac{1}{2}\frac{\Omega d\tau^2}{\omega^2} \tag{C.26}$$

(cf. (C.23)) where $\nabla\omega = 0$, $\omega =: dx$, we can represent (C.25) as

$$Q(2u, 2u', \ldots, 2u^{(k)}) = 0 \tag{C.27}$$

[†] More generally, a second order differential equation

$$p\frac{d^2y}{dt^2} + q\frac{dy}{dt} + ry = 0$$

determines a projective structure

$$\tau := \frac{y_2(t)}{y_1(t)}$$

and an affine structure with the flat coordinate x such that

$$dx = y_1\, dy_2 - y_2\, dy_1 = y_1^2\, d\tau.$$

The "curvature" (C.15) in this case is given by the formula

$$\Omega\, d\tau^2 = -2\frac{r}{p}dt^2.$$

for $u' = du/dx$ etc. Solving (C.27) we can reconstruct $\gamma(\tau)$ via two independent solutions $y_1(x),\ y_2(x)$ of the Sturm - Liouville equation (C.20) normalized by $y_2' y_1 - y_1' y_2 = 1$

$$\tau = \frac{y_2(x)}{y_1(x)}, \quad \gamma = \frac{d(y_1^2)}{dx}. \tag{C.28}$$

Let us consider examples of the equations of the form (C.25). I will consider only the equations linear in the highest derivative $\nabla^k \Omega$.

For $k = 0$ we have only the conditions of flateness. For $k = 1$ there exists only one invariant differential equation $\nabla \Omega = 0$ or

$$\gamma'' - 3\gamma\gamma' + \gamma^3 = 0. \tag{C.29}$$

We have $u(x) = c^2$ (a constant); a particular solution of (C.29) is

$$\gamma = -\frac{2}{\tau}. \tag{C.30}$$

The general solution can be obtained from (C.30) using the invariance (C.9).

For $k = 2$ the equations of our class must have the form

$$\nabla^2 \Omega + c\Omega^2 = 0 \tag{C.31a}$$

for a constant c or more explicitly

$$\gamma''' - 6\gamma\gamma'' + 9\gamma'^2 + (c - 12)\left(\gamma' - \frac{1}{2}\gamma^2\right)^2 = 0. \tag{C.31b}$$

For $c = 12$ this coincides with the Chazy equation (C.5). The corresponding equation (C.27)

$$u'' + 2cu^2 = 0 \tag{C.32}$$

for $c \neq 0$ can be integrated in elliptic functions

$$u(x) = -\frac{3}{c}\wp_0(x) \tag{C.33}$$

where $\wp_0(x)$ is the equianharmonic Weierstrass elliptic function, i.e. the inverse to the elliptic integral

$$x = \int_\infty^{\wp_0} \frac{dz}{2\sqrt{z^3 - 1}}. \tag{C.34}$$

[All the solutions of (C.32) can be obtained from (C.33) by shifts and dilations along x. There is also a particular solution $u = -3/cx^2$ and the orbit of this w.r.t. (C.9).] So the solutions of (C.31) can be expressed as in (C.28) via the solutions of the Lamé equation with the equianharmonic potential

$$y'' + \frac{3}{c}\wp_0(x)y = 0 \tag{C.35}$$

(for $c = 0$ via Airy functions). It can be reduced to the hypergeometric equation

$$t(t-1)\frac{d^2y}{dt^2} + \left(\frac{7}{6}t - \frac{1}{2}\right)\frac{dy}{dt} + \frac{1}{12c}y = 0 \qquad (C.36)$$

by the substitution

$$t = 1 - \wp_0^3(x). \qquad (C.37)$$

From (C.28) we express the solution of (C.31) in the form

$$\tau = \frac{y_2(t)}{y_1(t)}, \quad \gamma = \frac{d\log y_1^2}{d\tau} \qquad (C.38)$$

for two linearly independent solutions $y_1(t)$, $y_2(t)$ of the hypergeometric equation.

Particularly, for the Chazy equation one obtains [31] the hypergeometric equation

$$t(t-1)\frac{d^2y}{dt^2} + \left(\frac{7}{6}t - \frac{1}{2}\right)\frac{dy}{dt} + \frac{1}{144}y = 0. \qquad (C.39)$$

Note that the function $t = t(\tau)$ is the Schwartz triangle function $S(0, \pi/2, \pi/3; \tau)$. So the (projective) monodromy group of (C.39) coincides with the modular group.

Remark C.2. In the theory of the Lamé equation (C.35) the values

$$\frac{3}{c} = -m(m+1) \qquad (C.40)$$

for an integer m are of particular interest [56]. These look not to be discussed from the point of view of the theory of projective structures.

Chazy considered also the equation

$$\gamma''' - 6\gamma\gamma'' + 9\gamma'^2 + \frac{432}{n^2-36}\left(\gamma' - \frac{1}{2}\gamma^2\right)^2 = 0 \qquad (C.41)$$

for an integer $n > 6$. The corresponding Lamé equation

$$y'' - m(m+1)\wp_0(x)y = 0 \qquad (C.42)$$

has

$$m = \frac{3}{n} - \frac{1}{2}. \qquad (C.43)$$

Particularly, for the equation (C.5) $m = -\frac{1}{2}$. The solutions of (C.41), according to Chazy, can be expressed via the Schwartz triangle function $S(\pi/n, \pi/2, \pi/3; \tau)$. This can be seen from (C.36).

Exercise C.4.

1. Show that the equation of the class (C.25) of the order $k = 3$ can be integrated via solutions of the Lamé equation $y'' + A\wp(x)y = 0$ with arbitrary Weierstrass elliptic potential.

2. Show that for $k = 4$ γ can be expressed via the solutions of the equation (C.20) with the potential $u(x)$ satisfying

$$u^{IV} + au^3 + buu'' + cu'^4 = 0 \qquad (C.44)$$

for arbitrary constants a, b, c. Observe that for $a = -b = 10$, $c = -5$ the equation (C.44) is a particular case of the equation determining the genus two algebraic-geometrical (i.e. "two gap") potentials of the Sturm - Liouville operator [56].

Let me explain now the geometrical meaning of the solution (C.6) of the Chazy equation (C.5). The underlined complex one-dimensional manifold M here will be the modular curve

$$M := \{Im\tau > 0\} / SL(2, \mathbf{Z}). \qquad (C.45)$$

(This is not a manifold but an orbifold. So I will drop away the "bad" points $\tau = i\infty$, $\tau = e^{2\pi i/3}$, $\tau = i$ and the $SL(2, \mathbf{Z})$-images of them.) A construction of a natural affine connection on M essentially can be found in the paper [67] of Frobenius and Stickelberger. They described an elegant approach to the problem of differentiating of elliptic functions w.r.t. their periods. I recall here the basic idea of this not very wellknown paper because of its very close relations to the subject of the present lectures.

Let us consider a lattice on the complex plane

$$L = \{2m\omega + 2n\omega' | m, n \in \mathbf{Z}\} \qquad (C.46)$$

with the basis 2ω, $2\omega'$ such that

$$Im\left(\tau = \frac{\omega'}{\omega}\right) > 0. \qquad (C.47)$$

Another basis

$$\omega',\ \omega \mapsto \tilde{\omega}' = a\omega' + b\omega,\ \tilde{\omega} = c\omega' + d\omega, \qquad (C.48)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \qquad (C.49)$$

determines the same lattice.

Let $\mathcal{L}$ be the set of all lattices. I will drop away (as above) the orbifold points of $\mathcal{L}$ corresponding to the lattices with additional symmetry. So $\mathcal{L}$ is a two-dimensional manifold.

By E_L we denote the complex torus (elliptic curve)

$$E_L := \mathbf{C}/L. \qquad (C.50)$$

We obtain a natural fiber bundle

$$\mathcal{M} = \begin{array}{c} \downarrow \quad E_L \\ \mathcal{L} \end{array}. \qquad (C.51)$$

The space of this fiber bundle will be called *universal torus*. (Avoid confusion with universal elliptic curve: the latter is two-dimensional while our universal torus is three-dimensional. The points of the universal torus corresponding to proportional lattices give isomorphic elliptic curves.) Meromorphic functions on $\mathcal{M}$ will be called *invariant elliptic functions*. They can be represented as

$$f = f(z; \omega, \omega') \tag{C.52a}$$

with f satisfying the properties

$$f(z + 2m\omega + 2n\omega'; \omega, \omega') = f(z; \omega, \omega'), \tag{C.52b}$$

$$f(z; c\omega' + d\omega,\, a\omega' + b\omega) = f(z; \omega, \omega') \tag{C.52c}$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}). \tag{C.52d}$$

An example is the Weierstrass elliptic function

$$\wp \equiv \wp(z; \omega, \omega') = \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(z - 2m\omega - 2n\omega')^2} - \frac{1}{(2m\omega + 2n\omega')^2} \right). \tag{C.53}$$

It satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{C.54}$$

with

$$g_2 \equiv g_2(\omega, \omega') = 60 \sum_{m^2+n^2 \neq 0} \frac{1}{(2m\omega + 2n\omega')^4}, \tag{C.55}$$

$$g_3 \equiv g_3(\omega, \omega') = 140 \sum_{m^2+n^2 \neq 0} \frac{1}{(2m\omega + 2n\omega')^6}. \tag{C.56}$$

Frobenius and Stickelberger found two vector fields on the universal torus $\mathcal{M}$. The first one is the obvious Euler vector field

$$\omega \frac{\partial}{\partial \omega} + \omega' \frac{\partial}{\partial \omega'} + z \frac{\partial}{\partial z}. \tag{C.57}$$

In other words, if f is an invariant elliptic function then so is

$$\omega \frac{\partial f}{\partial \omega} + \omega' \frac{\partial f}{\partial \omega'} + z \frac{\partial f}{\partial z}.$$

(There is even more simple example of a vector field on the universal torus: $\partial/\partial z$.) To construct the second vector field we need the Weierstrass ζ-function

$$\zeta \equiv \zeta(z; \omega, \omega') = \frac{1}{z} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{z - 2m\omega - 2n\omega'} + \frac{1}{2m\omega + 2n\omega'} + \frac{z}{(2m\omega + 2n\omega')^2} \right), \tag{C.58}$$

$$\frac{d\zeta}{dz} = -\wp. \tag{C.59}$$

The ζ-function depends on the lattice L (but not on the particular choice of the basis $\omega,\ \omega'$) but it is not an invariant elliptic function in the above sense since

$$\zeta(z + 2m\omega + 2n\omega'; \omega, \omega') = \zeta(z; \omega, \omega') + 2m\eta + 2n\eta' \tag{C.60a}$$

where

$$\eta \equiv \eta(\omega, \omega') := \zeta(\omega; \omega, \omega'), \tag{C.60b}$$

$$\eta' \equiv \eta'(\omega, \omega') := \zeta(\omega'; \omega, \omega'). \tag{C.60c}$$

The change (C.48) of the basis in the lattice acts on $\eta,\ \eta'$ as

$$\tilde{\eta}' = a\eta' + b\eta,\ \ \tilde{\eta} = c\eta' + d\eta. \tag{C.61}$$

Lemma C.1. [67] *If f is an invariant elliptic function then so is*

$$\eta\frac{\partial f}{\partial \omega} + \eta'\frac{\partial f}{\partial \omega'} + \zeta\frac{\partial f}{\partial z}. \tag{C.62}$$

Proof is in a simple calculation using (C.60) and (C.61).

Exercise.

1). For $f = \wp(z; \omega, \omega')$ obtain [67, formula 11.]

$$\eta\frac{\partial \wp}{\partial \omega} + \eta'\frac{\partial \wp}{\partial \omega'} + \zeta\frac{\partial \wp}{\partial z} = -2\wp^2 + \frac{1}{3}g_2. \tag{C.63}$$

2). For $f = \zeta(z; \omega, \omega')$ (warning: this is not an elliptic function!) obtain [67, formula 29.]

$$\eta\frac{\partial \zeta}{\partial \omega} + \eta'\frac{\partial \zeta}{\partial \omega'} + \zeta\frac{\partial \zeta}{\partial z} = \frac{1}{2}\wp' - \frac{1}{12}g_2 z. \tag{C.64}$$

Consider now the particular class of invariant elliptic functions not depending on z.

Corollary C.1. *If $f = f(\omega, \omega')$ is a homogeneous function on the lattice of the weight $(-2k)$,*

$$f(c\omega, c\omega') = c^{-2k} f(\omega, \omega') \tag{C.65}$$

then

$$\eta\frac{\partial f}{\partial \omega} + \eta'\frac{\partial f}{\partial \omega'} \tag{C.66}$$

is a homogeneous function of the lattice of the degree $(-2k - 2)$.

Exercise C.5. Using (C.63) and (C.64) prove that

$$\eta\frac{\partial g_2}{\partial \omega} + \eta'\frac{\partial g_2}{\partial \omega'} = -6g_3 \tag{C.67}$$

$$\eta \frac{\partial g_3}{\partial \omega} + \eta' \frac{\partial g_3}{\partial \omega'} = -\frac{g_2^2}{3} \tag{C.68}$$

[67, formula 12.] and

$$\eta \frac{\partial \eta}{\partial \omega} + \eta' \frac{\partial \eta}{\partial \omega'} = -\frac{1}{12} g_2 \omega \tag{C.69}$$

[67, formula 31.].

Any homogeneous function $f(\omega, \omega')$ on $\mathcal{L}$ of the weight $(-2k)$ determines a k-tensor

$$\hat{f}(\tau) d\tau^k \tag{C.70a}$$

on the modular curve M where

$$f(\omega, \omega') = \omega^{-2k} \hat{f}(\tau), \quad \tau = \frac{\omega'}{\omega}. \tag{C.70b}$$

In the terminology of the theory of automorphic functions $\hat{f}$ is an automorphic form of the modular group of the weight $2k$. [Also some assumptions about behaviour of $\hat{f}$ in the orbifold points are needed in the definition of an automorphic form; we refer the reader to a textbook in automorphic functions (e.g., [87]) for the details.] Due to Corollary we obtain a map

$$k - \text{tensors on } M \to (k+1) - \text{tensors on } M, \tag{C.71a}$$

$$\hat{f}(\tau) \mapsto \nabla \hat{f}(\tau) := -\frac{2}{\pi i} \omega^{2k+2} \left(\eta \frac{\partial f}{\partial \omega} + \eta' \frac{\partial f}{\partial \omega'} \right). \tag{C.71b}$$

(Equivalently: an automorphic form of the weight $2k$ maps to an automorphic form of the weight $2k+2$.) This is the affine connection on M we need. We call it *FS-connection.*

Explicitly:

$$\nabla \hat{f} = -\frac{2}{\pi i} \left(\eta \frac{\partial}{\partial \omega} + \eta' \frac{\partial}{\partial \omega'} \right) \left[\omega^{-2k} \hat{f}\left(\frac{\omega'}{\omega} \right) \right]$$

$$= -\frac{2}{\pi i} \left[(-\eta \omega' + \eta' \omega) \frac{d\hat{f}}{d\tau} - 2k\omega\eta \hat{f} \right] = \frac{d\hat{f}}{d\tau} - k\gamma \hat{f} \tag{C.72a}$$

(I have used the Legendre identity $\eta\omega' - \eta'\omega = \pi i/2$) where

$$\gamma \equiv \gamma(\tau) := -\frac{4}{\pi i} \omega \eta(\omega, \omega'). \tag{C.72b}$$

The FS-connection was rediscovered in the theory of automorphic forms by Rankin [126] (see also [87, page 123]). From [107, page 389] we obtain

$$\gamma(\tau) = \frac{1}{3\pi i} \frac{\theta_1'''(0;\tau)}{\theta_1'(0;\tau)}. \tag{C.72c}$$

From (C.60) it follows the representation of $\gamma(\tau)$ via the normalized Eisenstein series $E_2(\tau)$ (this is not an automorphic form!)

$$\gamma(\tau) = \frac{i\pi}{3} E_2(\tau) \tag{C.72d}$$

$$E_2(\tau) = 1 + \frac{3}{\pi^2} \sum_{m\neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau + n)^2} = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{C.73}$$

Here $\sigma(n)$ stands for the sum of all the divisors of n.

Proposition C.2. *The FS-connection on the modular curve satisfies the Chazy equation (C.5).*

Proof (cf. [142]). Put

$$\hat{g}_2 \equiv \hat{g}_2(\tau) = \omega^4 g_2(\omega, \omega'), \quad \hat{g}_3 = \hat{g}_3(\tau) = \omega^6 g_3(\omega, \omega'). \tag{C.74}$$

From (C.69) we obtain that

$$\Omega \equiv \gamma' - \frac{1}{2}\gamma^2 = \frac{2}{3(\pi i)^2} \hat{g}_2. \tag{C.75}$$

Substituting to (C.67), (C.68) we obtain

$$\nabla^2 \Omega + 12 \Omega^2 = 0.$$

Proposition is proved.

From (C.72d), (C.73) we conclude that the solution (C.6) specified by the analyticity at $\tau = i\infty$ coincides with the FS-connection.

Exercise C.6. Derive from (C.5) the following recursion relation for the sums of divisors of natural numbers

$$\sigma(n) = \frac{12}{n^2(n-1)} \sum_{k=1}^{n-1} k(3n - 5k)\sigma(k)\sigma(n-k). \tag{C.76}$$

To construct the flat coordinate for the FS-connection we observe that [67] for the discriminant

$$\Delta \equiv \Delta(\omega, \omega') = g_2^3 - 27 g_3^2 \tag{C.77}$$

we have from (C.67), (C.68)

$$\eta \frac{\partial \Delta}{\partial \omega} + \eta' \frac{\partial \Delta}{\partial \omega'} = 0. \tag{C.78}$$

So

$$\nabla \hat{\Delta}(\tau) = 0 \tag{C.79}$$

where $\hat{\Delta}(\tau)$ is a 6-tensor

$$\hat{\Delta}(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{C.80}$$

The sixth root of $(2\pi)^{-12}\hat{\Delta}(\tau)d\tau^6$ gives the covariantly constant 1-form dx

$$dx := \eta^4(\tau)d\tau \tag{C.81}$$

where $\eta(\tau)$ is the Dedekind eta-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n) \tag{C.82}$$

(avoid confusions with the function $\eta = \zeta(\omega; \omega, \omega')$!). We obtain particularly that the FS covariant derivative of a k-tensor $\hat{f}(\tau)$ can be written as

$$\nabla \hat{f} = \eta^{4k}(\tau) \frac{d}{d\tau} \left[\frac{\hat{f}}{\eta^{4k}(\tau)} \right]. \tag{C.82}$$

Another consequence is the following formula for the FS-connection

$$\gamma(\tau) = \frac{1}{6}\frac{d}{d\tau} \log \hat{\Delta}(\tau) = 4\frac{d}{d\tau} \log \eta(\tau) = 8\pi i \left(\frac{1}{24} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right). \tag{C.83}$$

Remark C.3. Substituting (C.84) in the Chazy equation we obtain a 4-th order differential equation for the modular discriminant. It is a consequence of the third order equation of Jacobi [83, S. 103]

$$\left[12\psi^3 \frac{d^2\psi}{dz^2} \right]^3 - 27 \left[\frac{1}{8}\psi^4 \frac{d^3(\psi^2)}{dz^2} \right]^2 = 1, \tag{C.85a}$$

$$\psi = \eta^{-2}(\tau), \quad z = 2\pi i \tau. \tag{C.85b}$$

Notice also the paper [80] of Hurwitz where it is shown that any holomorphic automorphic form or a meromorphic automorphic function associated with a group arising from Riemann surfaces of algebraic functions satisfies certain algebraic differential equation of the third order.

Consider now the Frobenius structure on the space

$$\hat{\mathcal{M}} := \{t^1, t^2, t^3 | Im t^3 > 0\} \tag{C.86}$$

specified by the FS solution (C.72) of the Chazy equation. So

$$\begin{aligned} F &= \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 - \frac{\pi i}{2}(t^2)^4 \left(\frac{1}{24} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right) \\ &= \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 - \frac{\pi i}{2}(t^2)^4 \left(\frac{1}{24} - \sum_{n=1}^{\infty} \sigma(n) q^n \right) \end{aligned} \tag{C.87}$$

where $q = \exp 2\pi i t^3$. Here we have $\tilde{\gamma} = \gamma$, i.e. the solution $\gamma(\tau)$ obeys the transformation rule

$$\gamma\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \gamma(\tau) + 2c(c\tau + d), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}). \tag{C.88}$$

The formulae (B.19) for integer a, b, c, d determine a realisation of the group $SL(2, \mathbf{Z})$ as a group of symmetries of the Frobenius manifold (C.87). Factorizing $\hat{\mathcal{M}}$ over the transformations (B.19) we obtain a first example of twisted Frobenius manifold in the sense of Appendix B.

The invariant metric is a section of a line bundle over the manifold. This is the pull-back of the tangent bundle of the modular curve under the natural projection

$$(t^1, t^2, t^3) \mapsto t^3.$$

Indeed, the object

$$\left((dt^2)^2 + 2dt^1 dt^3\right) \otimes \frac{\partial}{\partial \tau} \tag{C.89}$$

is invariant w.r.t. the transformations (B.19) (this follows from (B.14)).

Exercise C.7. Show that the formulae

$$\begin{aligned} t^1 &= -\frac{1}{2\pi i}\left[\wp(z; \omega, \omega') + \omega^{-1}\eta(\omega; \omega, \omega')\right] \\ t^2 &= \frac{\sqrt{2}}{\omega} \\ t^3 &= \tau = \omega'/\omega \end{aligned} \tag{C.90}$$

establish an isomorphism of the twisted Frobenius manifold (C.87) with the universal torus $\mathcal{M}$.

In Appendix J I will explain the relation of this example to geometry of complex crystallographic group.

Remark C.4. The triple correlators $c_{\alpha\beta\gamma}(t)$ can be represented like a "sum over instanton corrections" [157] in topological sigma models (see the next lecture). For example,

$$c_{333} = 4\pi^4 (t^2)^4 \sum_{n \geq 1} n^3 A(n) \frac{q^n}{1 - q^n} \tag{C.91}$$

where

$$A(n) = n^{-3} \prod [p_i^{k_i}(p_i^3 + p_i^2 + p_i + 1) - (p_i^2 + p_i + 1)] \tag{C.92a}$$

for

$$n = \prod_i p_i^{k_i} \tag{C.92b}$$

being the factorization of n in the product of powers of different primes $p_1, p_2, \ldots$.

Lecture 2.
Topological conformal field theories
and their moduli.

A quantum field theory (QFT) on a D-dimensional manifold Σ consists of:

1). a family of local fields $\phi_\alpha(x),\ x \in \Sigma$ (functions or sections of a fiber bundle over Σ). A metric $g_{ij}(x)$ on Σ usualy is one of the fields (the gravity field).

2). A Lagrangian $L = L(\phi, \phi_x, ...)$. Classical field theory is determined by the Euler – Lagrange equations

$$\frac{\delta S}{\delta \phi_\alpha(x)} = 0, \quad S[\phi] = \int_\Sigma L(\phi, \phi_x, ...). \tag{2.1}$$

As a rule, the metric $g_{ij}(x)$ on Σ is involved explicitly in the Lagrangian even if it is not a dynamical variable.

3). Procedure of quantization usualy is based on construction of an appropriate path integration measure $[d\phi]$. The partition function is a result of the path integration over the space of all fields $\phi(x)$

$$Z_\Sigma = \int [d\phi] e^{-S[\phi]}. \tag{2.2}$$

Correlation functions (non normalized) are defined by a similar path integral

$$< \phi_\alpha(x)\phi_\beta(y)\dots >_\Sigma = \int [d\phi]\phi_\alpha(x)\phi_\beta(y)\dots e^{-S[\phi]}. \tag{2.3}$$

Since the path integration measure is almost never well-defined (and also taking into account that different Lagrangians could give equivalent QFTs) an old idea of QFT is to construct a self-consistent QFT by solving a system of differential equations for correlation functions. These equations were scrutinized in 2D conformal field theories where D=2 and Lagrangians are invariant with respect to conformal transformations

$$\delta g_{ij}(x) = \epsilon g_{ij}(x), \quad \delta S = 0.$$

This theory is still far from being completed.

Here I will consider another class of solvable 2-dimensional QFT: *topological field theories*. These theories admit *topological invariance*: they are invariant with respect to arbitrary change of the metric $g_{ij}(x)$ on the 2-dimensional surface Σ

$$\delta g_{ij}(x) = \text{arbitrary}, \quad \delta S = 0. \tag{2.4}$$

On the quantum level that means that the partition function Z_Σ depends only on topology of Σ. All the correlation functions also are topological creatures: they depend only on the labels of operators and on topology of Σ but not on the positions of the operators

$$< \phi_\alpha(x)\phi_\beta(y)\dots >_\Sigma \equiv < \phi_\alpha\phi_\beta\cdots >_g \tag{2.5}$$

where g is the genus of Σ. The simplest example is 2D gravity with the Hilbert – Einstein action

$$S = \int R\sqrt{g}d^2x = \text{Euler characteristic of } \Sigma. \tag{2.6}$$

There are two ways of quantization of this functional. The first one is based on an appropriate discrete version of the model ($\Sigma \to$ polihedron). This way leads to considering matrix integrals of the form [23]

$$Z_N(t) = \int_{X^*=X} \exp\{-\mathrm{tr}(X^2 + t_1X^4 + t_2X^6 + \ldots)\}dX \tag{2.7}$$

where the integral should be taken over the space of all $N \times N$ Hermitean matrices X. Here t_1, t_2 ... are called coupling constants. A solution of 2D gravity is based on the observation that after an appropriate limiting procedure $N \to \infty$ (and a renormalization of t) the limiting partition function coincides with τ-function of a particular solution of the KdV-hierarchy [24, 46, 73].

Another approach called *topological 2D gravity* is based on an appropriate supersymmetric extension of the Hilbert – Einstein Lagrangian [155 - 157]. This reduces the path integral over the space of all metrics $g_{ij}(x)$ on a surface Σ of the given genus g to an integral over the finite-dimensional space of conformal classes of these metrics, i.e. over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Correlation functions of the model are expressed via intersection numbers of certain cycles on the moduli space [157, 41]

$$\sigma_p \leftrightarrow c_p \in H_*(\mathcal{M}_g), \quad p = 0, 1, \ldots \tag{2.8a}$$

$$< \sigma_{p_1}\sigma_{p_2}\ldots >_g = \#(c_{p_1} \cap c_{p_2} \cap \ldots) \tag{2.8b}$$

(here the subscript g means correlators on a surface of genus g). This approach is often called *cohomological field theory.*

More explicitly, let g, s be integers satisfying the conditions

$$g \geq 0, \quad s > 0, \quad 2 - 2g - s < 0. \tag{2.9}$$

Let

$$\mathcal{M}_{g,s} = \{(\Sigma, x_1, \ldots, x_s)\} \tag{2.10}$$

be the moduli space of smooth algebraic curves Σ of genus g with s ordered distinct marked points $x_1, \ldots, x_s$ (the inequalities (2.9) provide that the curve with the marked points is *stable*, i.e. it admits no infinitesimal automorphisms). By $\overline{\mathcal{M}}_{g,s}$ we will denote the Deligne – Mumford compactification of $\mathcal{M}_{g,s}$. Singular curves with double points obtained by a degeneration of Σ keeping the marked points off the singularities are to be added to compactify $\mathcal{M}_{g,s}$. Any of the components of $\Sigma \setminus$ (singularities) with the marked and the singular points on it is required to be stable. Natural line bundles $L_1, \ldots, L_s$ over $\overline{\mathcal{M}}_{g,s}$ are defined. By definition,

$$\text{fiber of } L_i|_{(\Sigma, x_1, \ldots, x_s)} = T^*_{x_i}\Sigma. \tag{2.11}$$

The Chern classes $c_1(L_i) \in H^*(\overline{\mathcal{M}}_{g,s})$ of the line bundles and their products are *Mumford - Morita - Miller classes* of the moduli space [112]. The genus g correlators of the topological gravity are defined via the intersection numbers of these cycles

$$< \sigma_{p_1} \dots \sigma_{p_s} >_g := \prod_{i=1}^{s} (2p_i + 1)!! \int_{\overline{\mathcal{M}}_{g,s}} c_1^{p_1}(L_1) \wedge \dots \wedge c_1^{p_s}(L_s). \tag{2.12}$$

These numbers could be nonzero only if

$$\sum (p_i - 1) = 3g - 3. \tag{2.13}$$

These are nonnegative rational numbers but not integers since $\overline{\mathcal{M}}_{g,s}$ is not a manifold but an orbifold.

It was conjectured by Witten that the both approaches to 2D quantum gravity should give the same results. This conjecture was proved by Kontsevich [89 - 90] (another proof was obtained by Witten [160]). He showed that the generating function

$$\begin{aligned} F(t) &= \sum_{g,n} \sum_{p_1 < \dots < p_n} \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{T_{p_1}^{k_1} \dots T_{p_n}^{k_n}}{k_1! \dots k_n!} < \sigma_{p_1}^{k_1} \dots \sigma_{p_n}^{k_n} >_g \\ &= \sum_{g=0}^{\infty} \left\langle \exp \sum_{p=0}^{\infty} T_p \sigma_p \right\rangle_g \end{aligned} \tag{2.14}$$

(the free energy of 2D gravity) is logarythm of τ-function of a solution of the KdV hierarchy where $T_0 = x$ is the spatial variable of the hierarchy, T_1, T_2, ... are the times (this was the original form of the Witten's conjecture). The τ-function is specified by the string equation (see eq. (6.54b) below for genus zero). Warning: the matrix gravity and the topological one correspond to two different τ-functions of KdV (in the terminology of Witten these are *different phases* of 2D gravity).

Other examples of 2D TFT's (see below) proved out to have important mathematical applications, probably being the best tool for treating sophisticated topological objects. For some of these 2D TFT's a description in terms of integrable hierarchies was conjectured.

This gives rise to the following

Problem. To find a rigorous mathematical foundation of 2D topological field theory. More concretely, to elaborate a system of axioms providing the description (if any) of 2D TFT's in terms of integrable hierarchies of KdV type.

A first step on the way to the solution of the problem was done by Atiyah [10] (for any dimension D) in the spirit of G.Segal's axiomatization of conformal field theory. He proposed simple axioms specifying properties of correlators of the fields in the *matter sector* of a 2D topological field theory. In the matter sector the set of local fields $\phi_1(x), \dots, \phi_n(x)$ (the so-called *primary fields* of the model) does not contain the metric on Σ. (Afterwards one should integrate over the space of metrics. This gives rise to a procedure of *coupling to topological gravity* that will be described below in Lecture 6. In the above example of topological gravity the matter sector consists only of the identity operator.) Then the correlators of the fields $\phi_1(x)$, ..., $\phi_n(x)$ obey very simple algebraic axioms. According to these axioms the matter sector of a 2D TFT is specified by:

1. The space of the local physical states A. I will consider only finite-dimensional spaces of the states

$$\dim A = n < \infty.$$

2. An assignment

$$(\Sigma, \partial\Sigma) \mapsto v_{(\Sigma,\partial\Sigma)} \in A_{(\Sigma,\partial\Sigma)} \tag{2.15}$$

for any oriented 2-surface Σ with an oriented boundary $\partial\Sigma$ that depends only on the topology of the pair $(\Sigma, \partial\Sigma)$ *). Here the linear space $A_{(\Sigma,\partial\Sigma)}$ is defined as follows:

$$\begin{aligned} A_{(\Sigma,\partial\Sigma)} &= \mathbf{C} \text{ if } \partial\Sigma = \emptyset \\ &= A_1 \otimes \ldots \otimes A_k \end{aligned} \tag{2.16}$$

if the boundary $\partial\Sigma$ consists of k components $C_1, \ldots, C_k$ (oriented cycles) and

$$A_i := \begin{cases} A & \text{if the orientation of } C_i \text{ is coherent to the orientation of } \Sigma \\ A^* & \text{(the dual space) otherwise.} \end{cases} \tag{2.17}$$

Drawing the pictures I will assume that the surfaces are oriented via the external normal vector; so only the orientation of the boundary will be shown explicitly.

The assignment (2.15) is assumed to satisfy the following three axioms.

1. *Normalization*:

---

*) We can modify this axiom assuming that the assignment (2.15) is covariant w.r.t. some representation in $A_{(\Sigma,\partial\Sigma)}$ of the mapping class group $(\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma)$. The simplest generalisation of such a type is that, where the space of physical states is $\mathbf{Z_2}$-graded

$$A = A_{even} \oplus A_{odd}.$$

A homeomorphism $(\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma)$ permuting the co-oriented components $C_1, \ldots, C_k$ of $\partial\Sigma$

$$C_1, \ldots, C_k \to C_{i_1}, \ldots, C_{i_k}$$

acts trivially on A_{even} but it multiplies the vectors of A_{odd} by the sign of the permutation $(i_1, \ldots, i_k)$. In this lectures we will not consider such a generalisation.

1. *Normalization*:

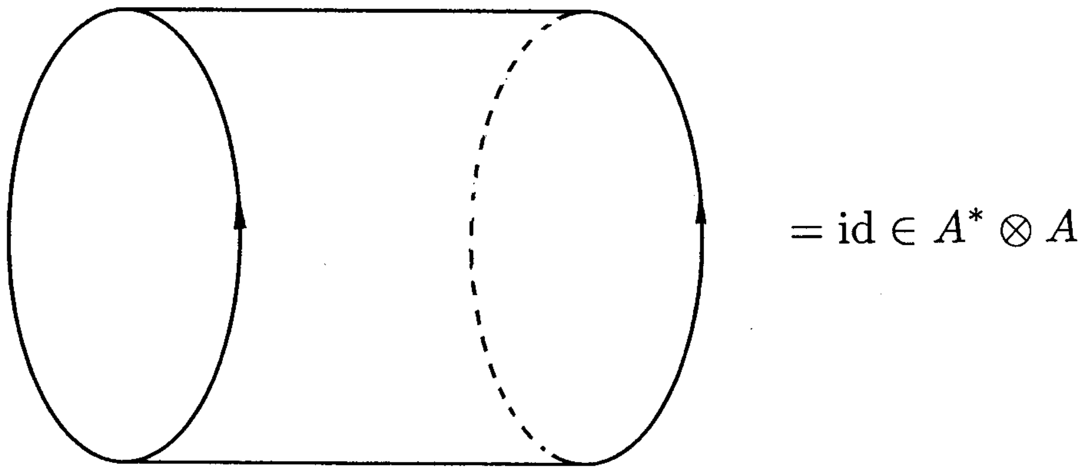


Figure 1

2. *Multiplicativity*: if

$$(\Sigma, \partial\Sigma) = (\Sigma_1, \partial\Sigma_1) \cup (\Sigma_2, \partial\Sigma_2) \tag{2.18a}$$

(disjoint union) then

$$v_{(\Sigma,\partial\Sigma)} = v_{(\Sigma_1,\partial\Sigma_1)} \otimes v_{(\Sigma_2,\partial\Sigma_2)} \in A_{(\Sigma,\partial\Sigma)} = A_{(\Sigma_1,\partial\Sigma_1)} \otimes A_{(\Sigma_2,\partial\Sigma_2)}. \tag{2.18b}$$

3. *Factorization.* To formulate this axiom I recall the operation of contraction defined in tensor products like (2.16), (2.17). By definition, ij-contraction

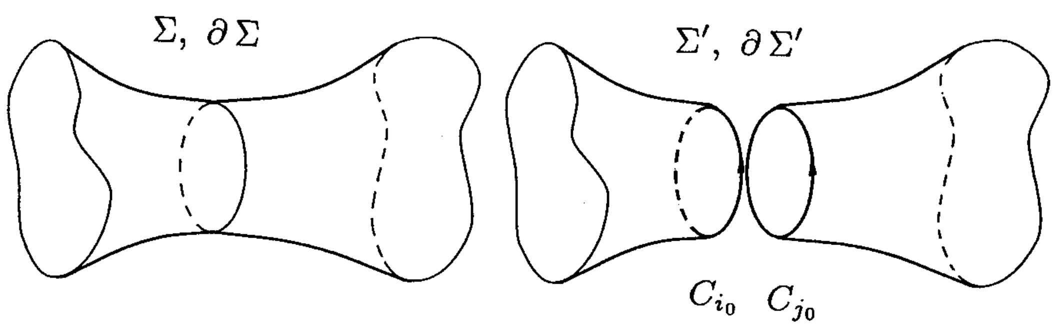
$$A_1 \otimes \ldots \otimes A_k \to A_1 \otimes \ldots \otimes \hat{A}_i \otimes \ldots \otimes \hat{A}_j \otimes \ldots \otimes A_k \tag{2.19}$$

(the i-th and the j-th factors are omitted in the r.h.s.) is defined when A_i and A_j are dual one to another using the standard pairing

$$A^* \otimes A \to \mathbf{C}$$

of the i-th and j-th factors and the identity on the other factors.

Let $(\Sigma, \partial\Sigma)$ and $(\Sigma', \partial\Sigma')$ coincide outside of a ball; inside the ball the two have the



form

Figure 2

(I draw a cycle on the neck of Σ to emphasize that it is obtained from Σ' by gluing together the cycles C_{i_0} and C_{j_0}.) Then we require that

$$v_{(\Sigma,\partial\Sigma)} = i_0 j_0\text{−contraction of } v_{(\Sigma',\partial\Sigma')}. \tag{2.20}$$

Particularly let us redenote by $v_{g,s}$ the vector

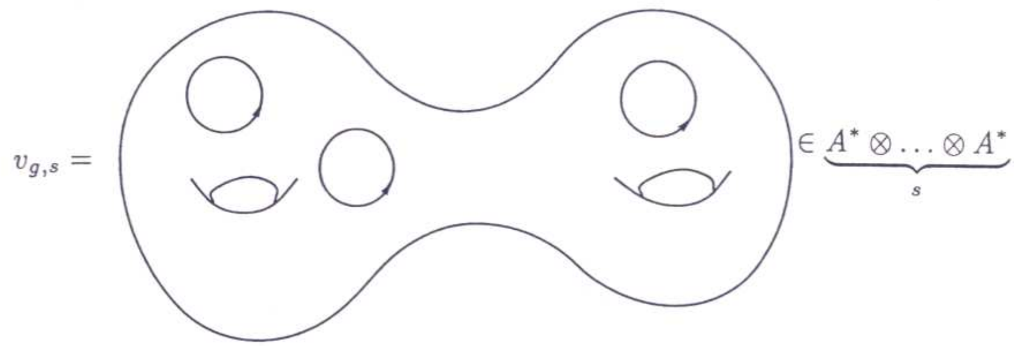


Figure 3. On the picture $g = 2$, $s = 3$.

This is a symmetric polylinear function on the space of the states. Choosing a basis

$$\phi_1, \ldots, \phi_n \in A \tag{2.21}$$

we obtain the components of the polylinear function

$$v_{g,s}\left(\phi_{\alpha_1} \otimes \ldots \phi_{\alpha_s}\right) =:< \phi_{\alpha_1} \ldots \phi_{\alpha_s} >_g \tag{2.22}$$

that by definition are called the genus g correlators of the fields $\phi_{\alpha_1}, \ldots, \phi_{\alpha_s}$.

We will prove, following [40], that the space of the states A carries a natural structure of a Frobenius algebra. All the correlators can be expressed in a pure algebraic way in terms of this algebra.

Let

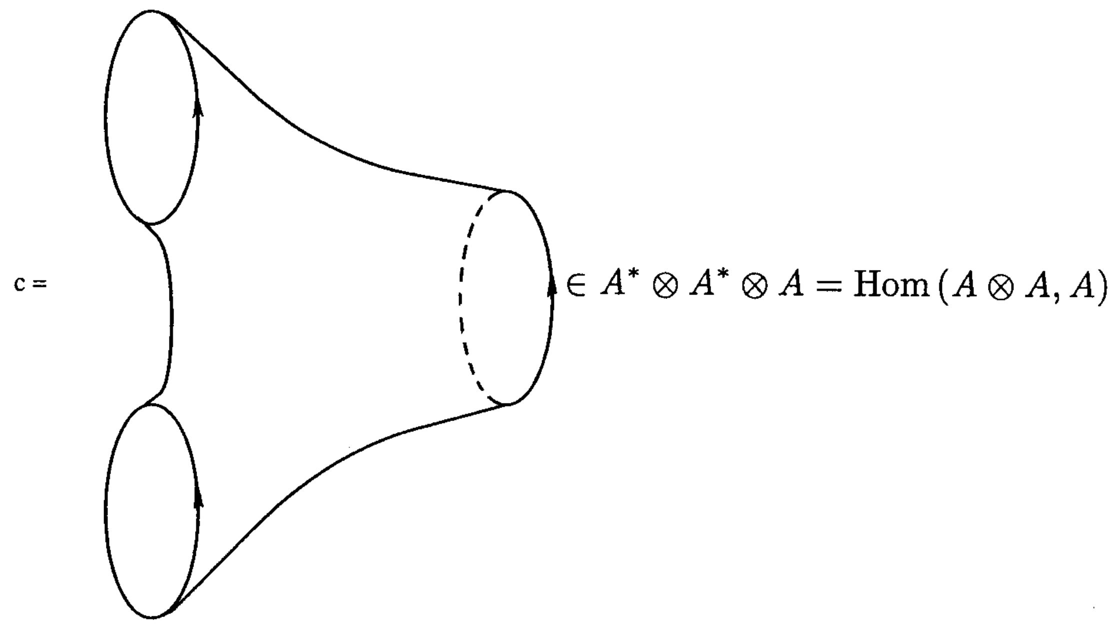


Figure 4

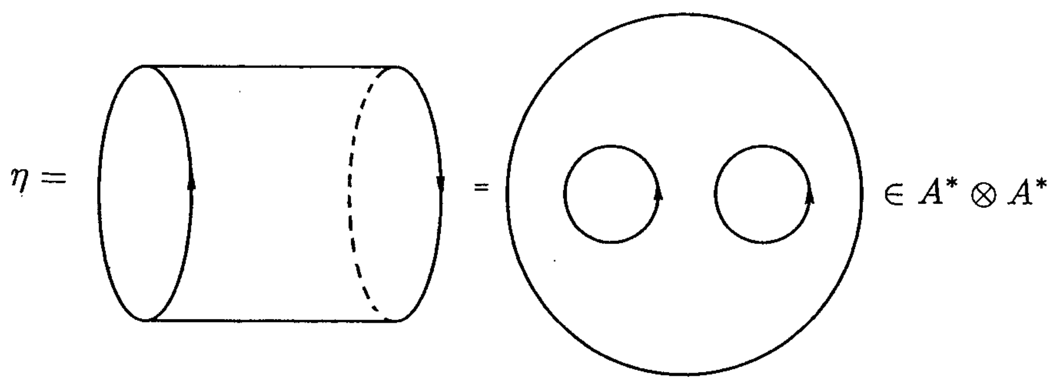


Figure 5. Bilinear form $< \, , \, >$ on A.

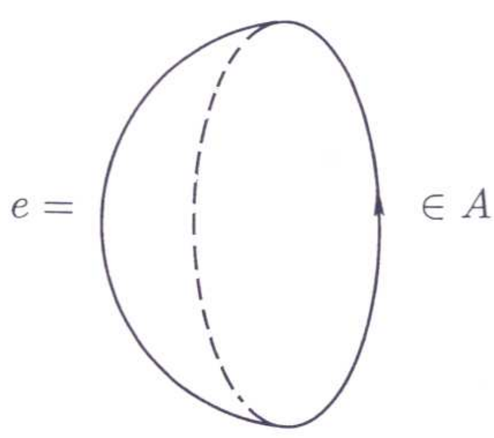


Figure 6

Theorem 2.1.

1. *The tensors c, η specify on A a structure of a Frobenius algebra with the unity e.*
2. *Let*

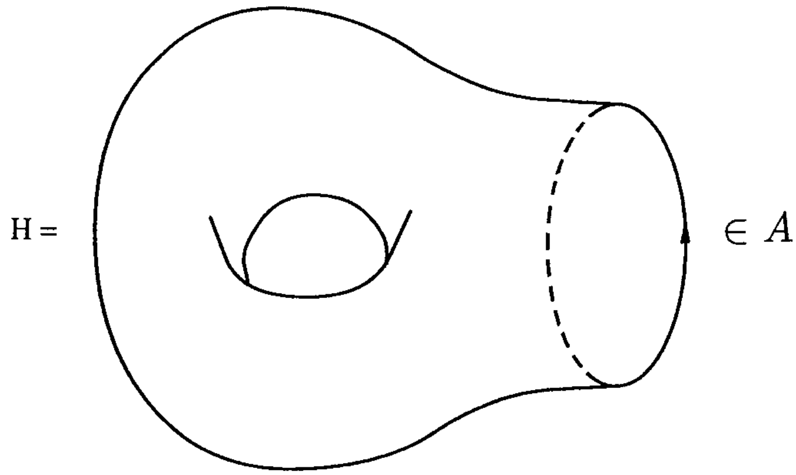


Figure 7

Then

$$< \phi_{\alpha_1} \ldots \phi_{\alpha_k} >_g = < \phi_{\alpha_1} \cdot \ldots \cdot \phi_{\alpha_k}, H^g > \tag{2.23}$$

(in the r.h.s. $\cdot$ means the product in the algebra A).

Proof. Commutativity of the multiplication is obvious since we can interchange the legs of the pants on Fig. 4 by a homeomorphism. Similarly, we obtain the symmetry of the inner product $< , >$. Associativity follows from Fig. 8

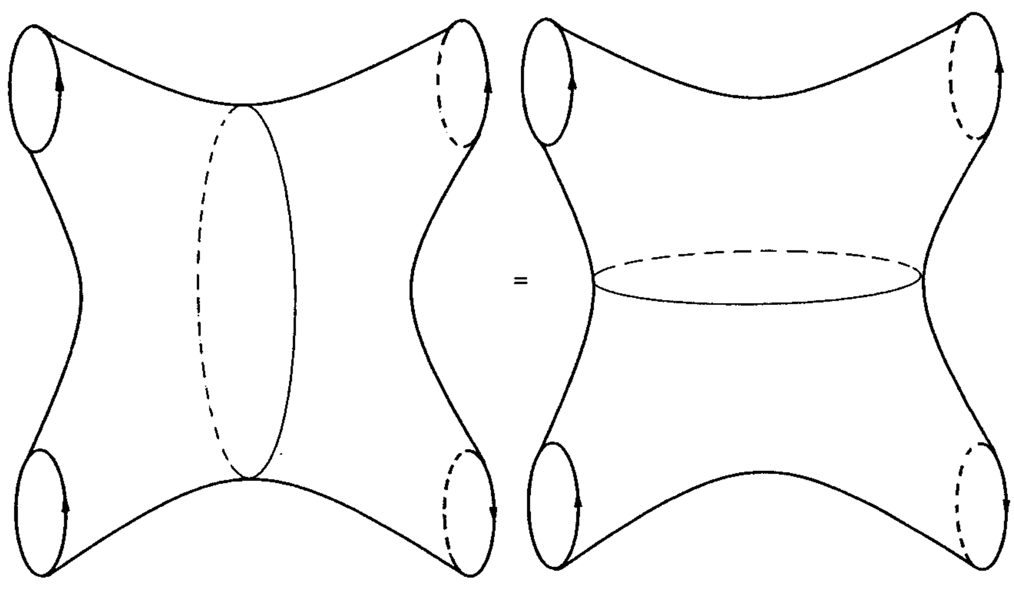


Figure 8

Particularly, the k-product is determined by the k-leg pants

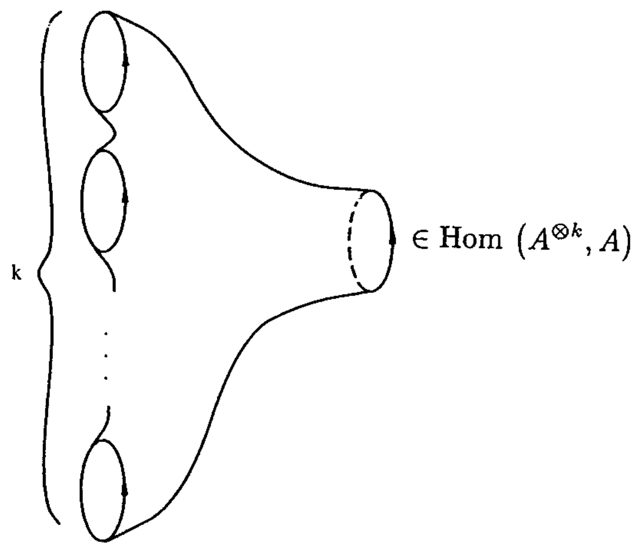


Figure 9

Unity:

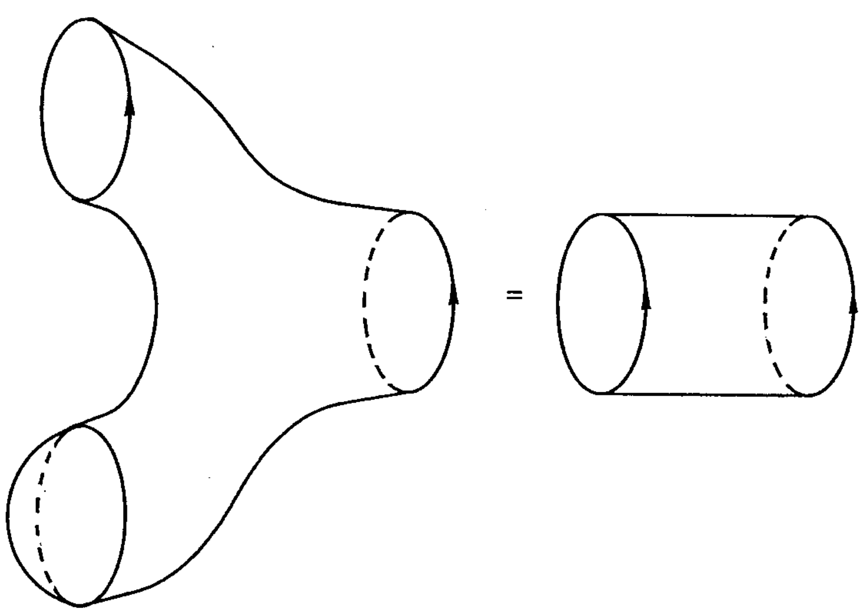


Figure 10

Nondegenerateness of η. We put

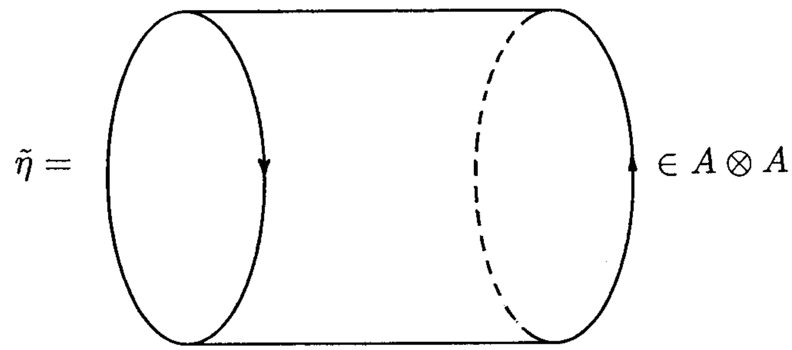


Figure 11

and prove that $\tilde{\eta} = \eta^{-1}$. This follows from Fig. 12

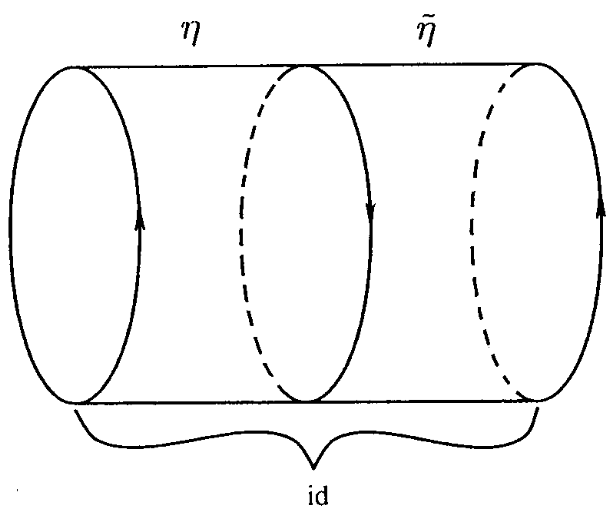


Figure 12

Compatibility of the multiplication with the inner product is proved on the next picture:

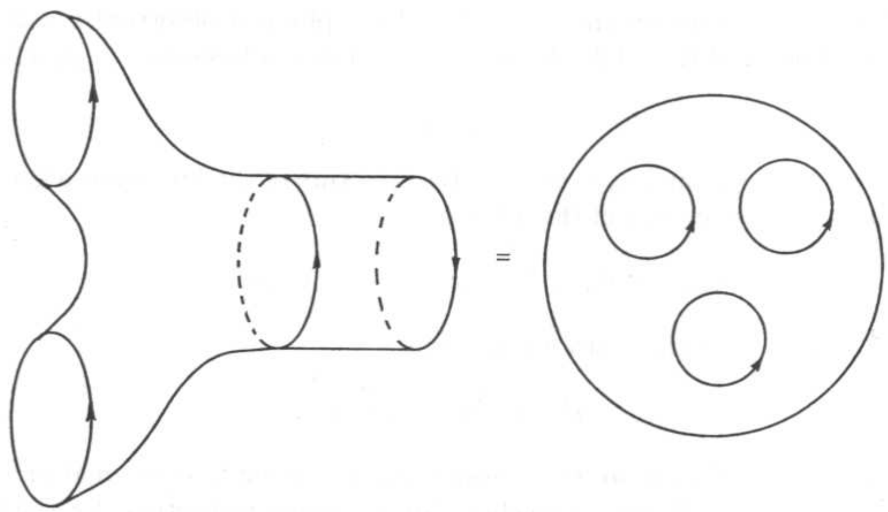


Figure 13

The first part of the theorem is proved.

The proof of the second part is given on the following picture:

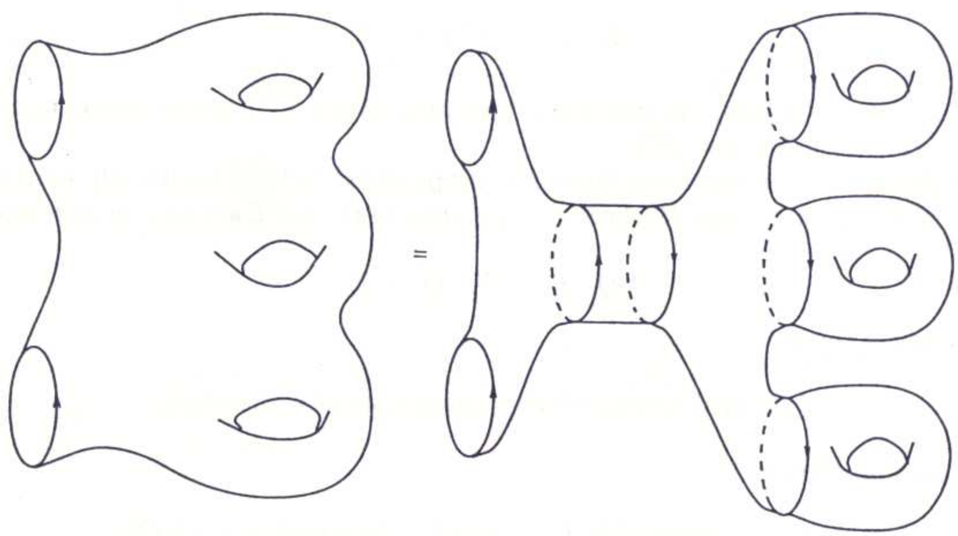


Figure 14

Theorem is proved.

Remark 2.1. If the space of physical states is $\mathbf{Z_2}$-graded (see the footnote on page 48) then we obtain a $\mathbf{Z_2}$-graded Frobenius algebra. Such a generalization was considered by Kontsevich and Manin in [92].

The Frobenius algebra on the space A of local physical observables will be called *primary chiral algebra* of the TFT. We always will choose a basis $\phi_1, \ldots, \phi_n$ of A in such a way that

$$\phi_1 = 1. \tag{2.24}$$

Note that the tensors $\eta_{\alpha\beta}$ and $c_{\alpha\beta\gamma}$ defining the structure of the Frobenius algebra are the following genus zero correlators of the fields ϕ_α

$$\eta_{\alpha\beta} =< \phi_\alpha \phi_\beta >_0, \quad c_{\alpha\beta\gamma} =< \phi_\alpha \phi_\beta \phi_\gamma >_0 . \tag{2.25}$$

The handle operator H is the vector of the form

$$H = \eta^{\alpha\beta} \phi_\alpha \cdot \phi_\beta \in A. \tag{2.26}$$

Summarizing, we can reformulate the Atiyah's axioms saying that the matter sector of a 2D TFT is encoded by a Frobenius algebra. No additional restrictions for the Frobenius algebra can be read out of the axioms.

On this way

Topologicaly invariant Lagrangian $\rightarrow$ correlators of local physical fields

we lose too much relevant information. To capture more information on a topological Lagrangian we will consider a topological field theory together with its deformations preserving topological invariance

$$L \rightarrow L + \sum t^\alpha L_\alpha^{(pert)} \tag{2.27}$$

(t^α are coupling constants). To construct these moduli of a TFT we are to say more words about the construction of a TFT.

A realization of the topological invariance is provided by QFT with a nilpotent symmetry. We have a Hilbert space $\mathcal{H}$ where the operators of the QFT act and an endomorphism (symmetry)

$$Q : \mathcal{H} \rightarrow \mathcal{H}, \quad Q^2 = 0. \tag{2.28}$$

In the classical theory

$$\{\text{physical observables}\} = \{\text{invariants of symmetry}\}.$$

In the quantum theory

$$\{\text{physical observables}\} = \{\text{operators commuting with } Q\}.$$

I will denote by $\{Q, \phi\}$ the commutator/anticommutator of Q with the operator ϕ (depending on the statistics of ϕ).

Lemma 2.1. $\{Q, \{Q, \cdot\}\} = 0$.
Proof follows from $Q^2 = 0$ and from the Jacobi identity.

Hence the operators of the form

$$\phi = \{Q, \psi\} \tag{2.29}$$

are always physical. However, they do not contribute to the correlators

$$< \{Q, \psi\} \phi_1 \phi_2 \ldots >= 0 \tag{2.30}$$

if ϕ_1, ϕ_2 ... are physical fields. So the space of physical states can be identified with the *cohomology* of the operator Q

$$A = \mathrm{Ker} Q / \mathrm{Im} Q. \tag{2.31}$$

The operators in A are called *primary states.*

The topological symmetry will follow if we succeed to construct operators $\phi_\alpha^{(1)}$, $\phi_\alpha^{(2)}$ for any primary field $\phi_\alpha = \phi_\alpha(x)$ such that

$$d\phi_\alpha(x) = \{Q, \phi_\alpha^{(1)}\}, \quad d\phi_\alpha^{(1)}(x) = \{Q, \phi_\alpha^{(2)}\}. \tag{2.32}$$

(We assume here that the fields $\phi_\alpha(x)$ are scalar functions of $x \in \Sigma$. So $\phi_\alpha^{(1)}(x)$ and $\phi_\alpha^{(2)}(x)$ will be 1-forms and 2-forms on Σ resp.) Indeed,

$$d_x < \phi_\alpha(x) \phi_\beta(y) \ldots >=< \{Q, \phi_\alpha^{(1)}(x)\} \phi_\beta(y) \ldots >= 0. \tag{2.33}$$

The operators $\phi_\alpha^{(1)}$ and $\phi_\alpha^{(2)}$ can be constructed for a wide class of QFT obtained by a procedure of *twisting* [100] from a $N = 2$ supersymmetric quantum field theory (see [41]). Particularly, in this case the primary chiral algebra is a *graded* Frobenius algebra in the sense of Lecture 1. The degrees q_α of the fields ϕ_α are the corresponding eigenvalues of the $U(1)$-charge of the $N = 2$ algebra; d is just the label of the $N = 2$ algebra (it is called *dimension**) since in the case of topological sigma-models it coincides with the complex dimension of the target space). The class of TFT's obtained by the twisting procedure from $N = 2$ superconformal QFT is called *topological conformal field theories* (TCFT).

From

$$\{Q, \oint_C \phi_\alpha^{(1)}\} = \oint_C d\phi_\alpha = 0 \tag{2.34}$$

we see that $\oint_C \phi_\alpha^{(1)}$ is a physical observable for any closed cycle C on Σ. Due to (2.30) this operator depends only on the homology class of the cycle.

Similarly, we obtain that

$$\int\int_\Sigma \phi_\alpha^{(2)} \tag{2.35}$$

is also a physical observable. (Both the new types of observables are non-local!)

Using the operators (2.35) we can construct a very important class of perturbations of the TCFT modifying the action as follows

$$S \mapsto \tilde{S}(t) := S - \sum_{\alpha=1}^{n} t^\alpha \int\int_\Sigma \phi_\alpha^{(2)} \tag{2.36}$$

*) In the physical literature it is sometimes denoted by $d = \hat{c} = c/3$.

where the parameters $t = (t^1, \dots t^n)$ are called *coupling constants.* The perturbed correlators will be functions of t

$$< \phi_\alpha(x)\phi_\beta(y)\dots > (t) := \int [d\phi]\phi_\alpha(x)\phi_\beta(y)\dots e^{-\tilde{S}(t)}. \tag{2.37}$$

Theorem 2.2. [42]

1. *The perturbation (2.36) preserves the topological invariance.*
2. *The perturbed primary chiral algebra A_t satisfies the WDVV equations.*

Due to this theorem the construction (2.36) determines a *canonical moduli space* of dimension n of a TCFT with n primaries. And this moduli space carries a structure of Frobenius manifold.

I will not reproduce here the proof of this (physical) theorem (it looks like the statement holds true under more general assumptions than those were used in the proof [42] - see below). It would be interesting to derive the theorem directly from Segal's-type axioms (see in [150]) of TCFT.

The basic idea of my further considerations is to add the statement of this theorem as *a new axiom* of TFT. In other words, we will axiomatize not an isolated TCFT but the TCFT together with its canonical moduli space (2.36). Let me repeat that the axioms of TCFT now read:

The canonical moduli space of a TCFT is a Frobenius manifold.

The results of Appendices A, C above show that the axiom together with certain analiticity assumptions of the primary free energy could give rise to a reasonable classification of TCFT. Particularly, the formula (A.7) gives the free energy of the A_3 topological minimal model (see below). More general relation of Frobenius manifolds to discrete groups will be established in Appendix G. In lecture 6 we will show that the axioms of coupling (at tree-level) to topological gravity of Dijkgraaf and Witten can be derived from geometry of Frobenius manifolds. The description of Zamolodchikov metric (the $t\,t^*$ equations of Cecotti and Vafa [28]) is an additional differential-geometric structure on the Frobenius manifold [51].

For the above example of topological gravity the matter sector is rather trivial: it consists only of the unity operator. The corresponding Frobenius manifold (the moduli space) is one-dimensional,

$$F = \frac{1}{6}(t^1)^3.$$

All the nontrivial fields σ_p for $p > 0$ in the topological gravity come from the integration over the space of metrics (coupling to topological gravity that we will discuss in Lecture 6).

We construct now other examples of TCFT describing their matter sectors. I will skip to describe the Lagrangians of these TCFT giving only the "answer": the description of the primary correlators in topological terms.

Example 2.1. Witten's algebraic-geometrical description [158] of the A_n topological minimal models [42] (due to K.Li [100] this is just the topological counterpart of the n-matrix model). We will construct some coverings over the moduli spaces $\mathcal{M}_{g,s}$. Let us fix numbers $\alpha = (\alpha_1, \dots, \alpha_s)$ from 1 to n such that

$$n(2g-2) + \sum_{i=1}^{s} (\alpha_i - 1) = (n+1)l \tag{2.38}$$

for some integer l. Consider a line bundle

$$\mathcal{L} \to \Sigma \tag{2.39}$$

of the degree l such that

$$\mathcal{L}^{\otimes(n+1)} = K_\Sigma^n \bigotimes_i O(x_i)^{\alpha_i - 1}. \tag{2.40}$$

Here K_Σ is the canonical line bundle of the curve Σ (of the genus g). The sections of the line bundle in the r.h.s. are n-tensors on Σ having poles only at the marked points x_i of the orders less than α_i.

We have $(n+1)^{2g}$ choices of the line bundle $\mathcal{L}$. Put

$$\mathcal{M}'_{g,s}(\alpha) := \{(\Sigma, x_1, \dots, x_s, \mathcal{L})\}. \tag{2.41}$$

This is a $(n+1)^{2g}$-sheeted covering over the moduli space of stable algebraic curves. An important point [158] is that the covering can be extended onto the compactification $\overline{\mathcal{M}_{g,s}}$. Riemann – Roch implies that, generically

$$\dim H^0(\Sigma, \mathcal{L}) = l + 1 - g = d(g-1) + \sum_{i=1}^{s} q_{\alpha_i} =: N(\alpha) \tag{2.42}$$

where we have introduced the notations

$$d := \frac{n-1}{n+1}, \quad q_\alpha := \frac{\alpha - 1}{n+1}. \tag{2.43}$$

Let us consider now the vector bundle

$$\begin{array}{cc} \downarrow & V(\alpha) \\ \mathcal{M}'_{g,s}(\alpha) & \end{array} \tag{2.44}$$

where

$$V(\alpha) := H^0(\Sigma, \mathcal{L}). \tag{2.45}$$

Strictly speaking, this is not a vector bundle since the dimension (2.42) can jump on the curves where $H^1(\Sigma, \mathcal{L}) \neq 0$. However, the top Chern class $c_N(V(\alpha))$, $N = N(\alpha)$ of the bundle is well-defined (see [158] for more detail explanation). We define the primary correlators by the formula

$$< \phi_{\alpha_1} \dots \phi_{\alpha_s} >_g := (n+1)^{-g} \int_{\overline{\mathcal{M}'_{g,s}}(\alpha)} c_N\left(V(\alpha)\right) \tag{2.46}$$

(the $\overline{\mathcal{M}'_{g,s}}$ is an appropriate compactification of $\mathcal{M}'_{g,s}$). These are nonzero only if

$$\sum_{i=1}^{s} (q_{\alpha_i} - 1) = (3-d)(g-1). \tag{2.47}$$

The generating function of the genus zero primary correlators

$$F(t) := < \exp \sum_{\alpha=1}^{n} t^\alpha \phi_\alpha >_0 \tag{2.48}$$

due to (2.47) is a quasihomogeneous polynomial of the degree $3-d$ where the degrees of the coupling constants t^α equal $1-q_\alpha$. One can verify directly that $F(t)$ satisfies WDVV [158]. It turns out that the Frobenius manifold (2.48) coincides with the Frobenius manifold of polynomials of Example 1.7!

One can describe in algebraic-geometrical terms also the result of "coupling to topological gravity" (whatever it means) of the above matter sector. One should take into consideration an analogue of Mumford – Morita – Miller classes (see above in the construction of topological gravity) $c_1(L_i)$ (I recall that the fiber of the line bundle L_i is the cotangent line $T^*_{x_i}\Sigma$). After coupling to topological gravity of the matter sector (2.46) we will obtain an infinite number of fields $\sigma_p(\phi_\alpha)$, $p = 0, 1, \dots$ (also denoted by $\phi_{\alpha,p}$ in Lecture 6). The fields $\sigma_0(\phi_\alpha)$ can be identified with the primaries ϕ_α. For $p > 0$ the fields $\sigma_p(\phi_\alpha)$ are called *gravitational descendants* of ϕ_α. Their correlators are defined by the following intersection number

$$< \sigma_{p_1}(\phi_{\alpha_1}) \dots \sigma_{p_s}(\phi_{\alpha_s}) >_g :=$$

$$= \frac{\left(\frac{\alpha_1}{n+1}\right)_{p_1} \dots \left(\frac{\alpha_s}{n+1}\right)_{p_s}}{(n+1)^{g-p_1-\dots-p_s}} \int_{\overline{\mathcal{M}'_{g,s}}(\alpha)} c_1^{p_1}(L_1) \wedge \dots \wedge c_1^{p_s}(L_s) \wedge c_N(V(\alpha)). \tag{2.49}$$

Here we introduce the notation

$$(r)_p := r(r+1)\dots(r+p-1), \quad (r)_0 := 1. \tag{2.50}$$

The correlator is nonzero only if

$$\sum_{i=1}^{s} (p_i + q_{\alpha_i} - 1) = (3-d)(g-1). \tag{2.51}$$

The generating function of the correlators

$$\mathcal{F}(T) := \sum_g \left\langle \exp \sum_{p=0}^{\infty} \sum_{\alpha=1}^{n} T_{\alpha,p} \sigma_p(\phi_\alpha) \right\rangle_g \tag{2.52}$$

is conjectured by Witten [158] to satisfy the n-th generalized KdV (or Gelfand - Dickey) hierarchy. Here T is the infinite vector of the indeterminates $T_{\alpha,p}$, $\alpha = 1, \dots, n$, $p = 0, 1, \dots$. We will come back to the discussion of the conjecture in Lecture 6.

Example 2.2. Topological sigma-models [155] (A-models in the terminology of Witten [151]). Let X be a compact Kähler manifold of (complex) dimension d with non-positive canonical class. The fields in the matter sector of the TFT will be in 1-1-correspondence with the cohomologies $H^*(X, \mathbf{C})$. I will describe only the genus zero correlators of the fields. For simplicity I will consider only the case when the odd-dimensional cohomologies of X vanish (otherwise one should consider $\mathbf{Z_2}$-graded Frobenius manifolds [92]).

The two-point correlator of two cocycles ϕ_α, $\phi_\beta \in H^*(X, \mathbf{C})$ coincides with the intersection number

$$< \phi_\alpha, \phi_\beta >= \int_X \phi_\alpha \wedge \phi_\beta. \tag{2.53}$$

The definition of multipoint correlators is given [151] in terms of the intersection theory on the moduli spaces of "instantons", i.e. holomorphic maps of the Riemann sphere CP^1 to X.

Let

$$\psi : CP^1 \to X \tag{2.54}$$

be a holomorphic map of a given homotopical type $[\psi]$. Also some points x, y, ... are to be fixed on CP^1. Let $\mathcal{M}[\psi]$ be the moduli space of all such maps for the given homotopical type $[\psi]$. We consider the "universal instanton": the natural map

$$\Psi : CP^1 \times \mathcal{M}[\psi] \to X. \tag{2.55}$$

We define the primary correlators putting

$$< \phi_\alpha \phi_\beta \dots >_0 := \int_{\mathcal{M}[\psi]} \Psi^*(\phi_\alpha)|_{x \times \mathcal{M}[\psi]} \wedge \Psi^*(\phi_\beta)|_{y \times \mathcal{M}[\psi]} \wedge \dots. \tag{2.56}$$

This TFT can be obtained by twisting of a N=2 superconformal theory if X is a Calabi - Yau manifold, i..e. if the canonical class of X vanishes. So one could expect to describe this class of TFT's by Frobenius manifolds only for Calabi - Yau X. However, we still obtain a Frobenius manifold for more general Kähler manifolds X (though it has been proved rigorously only for some particular classes of Kähler manifolds [111, 127]). But the scaling invariance (1.7) must be modified to (1.12).

To define a generating function we are to be more careful in choosing of a basis in $H^{1,1}(X, \mathbf{C})$. We choose this basis $\phi_{\alpha_1}, \dots, \phi_{\alpha_k}$ of integer Kähler forms $\in H^{1,1}(X, \mathbf{C}) \cap H^2(X, \mathbf{Z})$. By definition, integrals of the form

$$d_i \equiv d_i[\psi] := \int_{CP^1} \psi^*(\phi_{\alpha_i}), \quad i = 1, \dots, k \tag{2.57}$$

are all nonnegative. They are homotopy invariants of the map (2.54). The generating function

$$F(t) := \langle \sum_{[\psi]} \exp \sum_{\phi_\alpha \in H^*(X)} t^\alpha \phi_\alpha \rangle_0 \tag{2.58}$$

will be a formal series in $e^{-t^{\alpha_i}}$, $i = 1, \ldots, k$ and a formal power series in other coupling constants. So the Frobenius manifold coincides with the cohomology space

$$M = \oplus H^{2i}(X, \mathbf{C}). \tag{2.59}$$

The free energy $F(t)$ is $2\pi i$-periodic in t^{α_i} for $\phi_{\alpha_i} \in H^{1,1}(X, \mathbf{C}) \cap H^2(X, \mathbf{Z})$. In the limit

$$t^{\alpha_i} \to +\infty \text{ for } \phi_{\alpha_i} \in H^{1,1}(X, \mathbf{C}) \cap H^2(X, \mathbf{Z}),\ t^\alpha \to 0 \text{ for other } \phi_\alpha \tag{2.60}$$

the multiplication on the Frobenius manifold coincides with the multiplication in the cohomologies. In other words, the cubic part of the corresponding free energy is determined by the graded Frobenius algebra $H^*(X)$ in the form (1.61). Explicitly, the free energy has the structure of (formal) Fourier series

$$F(t, \tilde{t}) = \text{cubic part} + \sum_{k_i,\, [\psi]} N_{[\psi]}(k_1, \ldots, k_m) \frac{(\tilde{t}^{\beta_1})^{k_1} \ldots (\tilde{t}^{\beta_m})^{k_m}}{k_1! \ldots k_m!} e^{-t^{\alpha_1} d_1 - \ldots - t^{\alpha_k} d_k}. \tag{2.61}$$

Here the coupling constants $t^{\alpha_1}, \ldots t^{\alpha_k}$ correspond to a basis of Kähler forms as above, the coupling constants $\tilde{t}^{\beta_j}$ correspond to a basis in the rest part of cohomologies

$$\tilde{t}^{\beta_j} \leftrightarrow \tilde{\phi}_{\beta_j} \in \oplus_{k>1} H^{2k}(X, \mathbf{Z}) (\text{modulo torsion}). \tag{2.62}$$

The coefficients $N_{[\psi]}(k_1, \ldots, k_m)$ are defined as the Gromov - Witten invariants [72, 157] of X*). By definition, they count the numbers of rational maps (2.54) of the homotopy type $0 \neq [\psi] = (d_1, \ldots, d_k)$ intersecting with the Poincaré-dual cycles

$$\# \left\{ \psi(CP^1) \cap \mathcal{D}(\tilde{\phi}_{\beta_j}) \right\} = k_j,\ j = 1, \ldots, m. \tag{2.63}$$

(We put $N_{[\psi]}(k_1, \ldots, k_m) = 0$ if the set of maps satisfying (2.63) is not discrete.)

The Euler vector field E has the form

$$E = \sum (1 - q_\alpha) t^\alpha \partial_\alpha - \sum_{i=1}^k r_{\alpha_i} \partial_{\alpha_i} \tag{2.64}$$

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*) One needs to make some technical genericity assumptions about the maps ψ in order to prove WDVV for the free energy (2.61). One possibility is to perturb the complex structure on X and to consider pseudoholomorphic curves ψ. This was done (under some restrictions on X) in [111, 127]. Another scheme was proposed recently by Kontsevich [91].

where q_α is the *complex* dimension of the cocycle ϕ_α and the integers r_{α_i} are defined by the formula

$$c_1(X) = \sum_{i=1}^{k} r_{\alpha_i} \phi_{\alpha_i}. \tag{2.65}$$

Particularly, suppressing all the couplings but those corresponding to the Kähler forms $\in H^{1,1}(X, \mathbf{C}) \cap H^2(X, \mathbf{Z})$ we reduce the perturbed primary chiral ring (2.61) to the *quantum cohomology ring* of C.Vafa [146]. For a Calabi - Yau manifold X (where $c_1(X) = 0$) this is the only noncubic part of the Frobenius structure. The quantum multiplication on Calabi - Yau manifolds is still defined on $H^*(X)$. It has a structure of a graded algebra with the same gradings as in usual cohomology algebra, but this structure depends on the parameters $t^{\alpha_1}, \dots, t^{\alpha_k}$.

An equivalent reformulation of the multiplication (depending on the coupling constants $t^{\alpha_1}, \dots, t^{\alpha_k}$) in the quantum cohomology ring reads as follows [146]

$$< \phi_\alpha \cdot \phi_\beta, \phi_\gamma >:= \sum_{[\psi]} e^{-t^{\alpha_1} d_1 - \dots - t^{\alpha_k} d_k} \tag{2.66}$$

where the summation is taken over all the homotopy classes of the maps

$$\psi : CP^1 \to X \ \text{ such that } \psi(0) \in \mathcal{D}(\phi_\alpha), \ \psi(1) \in \mathcal{D}(\phi_\beta), \ \psi(\infty) \in \mathcal{D}(\phi_\gamma) \tag{2.67}$$

(here $\mathcal{D}$ is the Poincaré duality operator $\mathcal{D} : H^i(X) \to H_{d-i}(X)$). By definition we put zero in the r.h.s. of (2.66) for those classes $[\psi]$ when the set of maps satisfying (2.67) is not discrete. In the limit

$$t^{\alpha_i} \to +\infty \tag{2.68}$$

the quantum cohomology ring coincides with $H^*(X)$.

The most elementary example is the quantum cohomology ring of CP^d

$$\mathbf{C}[x]/(x^{d+1} = e^{-t}). \tag{2.69}$$

Here $t = t_2$ corresponds to the Kähler class of the standard metric on CP^d.

The trivial case is that $d = 1$ (complex projective line). We obtain two coupling parameters $t_1 \leftrightarrow e_1 \in H^0(CP^1, \mathbf{Z})$, $t_2 \leftrightarrow -e_2 \in H^2(CP^1, \mathbf{Z})$ (I change the sign for convenience). The Frobenius structure on $H^*(CP^1)$ is completely determined by the quantum multiplication (2.66). So

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + e^{t_2}. \tag{2.70}$$

The Euler vector field has the form, according to (2.64)

$$E = t_1 \partial_1 + 2\partial_2. \tag{2.71}$$

In the case of the projective plane CP^2 ($d = 2$) we choose again the basic elements e_1, e_2, e_3 in H^0, H^2, H^4 resp. The classical cohomology ring has the form

$$e_2^2 = e_3 \tag{2.72a}$$

$$e_2 e_3 = 0. \tag{2.72b}$$

In the quantum cohomology ring instead of (2.72b) we have

$$e_2 e_3 = e^{t_2} e_1. \tag{2.73}$$

So we must have

$$\partial_2 \partial_3^2 F|_{t_1=t_3=0} = e^{t_2}. \tag{2.74}$$

Hence the Frobenius structure on $H^*(CP^2)$ must have the free energy of the form

$$F(t_1, t_2, t_3) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + \frac{1}{2} t_3^2 e^{t_2} + o(t_3^2\, e^{t_2}). \tag{2.75}$$

The Euler vector field for the Frobenius manifold reads

$$E = t_1 \partial_1 + 3\partial_2 - t_3 \partial_3. \tag{2.76}$$

So we obtain finally the following structure of the free energy

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + t_3^{-1} \phi(t_2 + 3 \log t_3) \tag{2.77}$$

where the function $\phi = \phi(x)$ is a solution of the differential equation

$$9\phi''' - 18\phi'' + 11\phi' - 2\phi = \phi''\phi''' - \frac{2}{3}\phi'\phi''' + \frac{1}{3}\phi''^2 \tag{2.78}$$

of the form

$$\phi = \sum_{k \geq 1} A_k e^{kx}. \tag{2.79}$$

Due to (2.74) one must have

$$A_1 = \frac{1}{2}. \tag{2.80}$$

For the coefficients of the Fourier series (2.79) we obtain from (2.78) the recursion relations for $n > 1$

$$A_n = \frac{1}{(n-1)(9n^2 - 9n + 2)} \sum_{k+l=n,\ 0<k,l} kl \left[\left(n + \frac{2}{3} \right) kl - \frac{2}{3}(k^2 + l^2) \right] A_k A_l. \tag{2.81}$$

So the normalization (2.80) uniquely specifies the solution (2.79) (observation of [92]). The coefficients A_k are identified by Kontsevich and Manin [92] as

$$A_k = \frac{N_k}{(3k-1)!}$$

where N_k is the number of rational curves of the degree k on CP^2 passing through generic $3k - 1$ points.

Let us prove convergence of the series (2.79)*).

Lemma 2.2. *A_k are positive numbers satisfying*

$$A_k \le \frac{3}{5k^4}\left(\frac{5}{6}\right)^k \tag{2.83}$$

for any k.

Proof. Positivity of A_n follows from positivity of the coefficients in (2.81). The inequality (2.83) holds true for $k = 1,\ 2$ ($A_2 = 1/120$). We continue the proof by induction in k. For $k \ge 3$ we have

$$(n-1)(9n^2 - 9n + 2) \ge 3n^3. \tag{2.84}$$

Assuming the inequality (2.83) to be valid for any $k < n$, we obtain from (2.83), (2.84)

$$A_n \le \frac{3}{50n^3}\cdot\left(\frac{5}{6}\right)^n \sum_{k+l=n}\left[\frac{\left(n+\frac{2}{3}\right)}{k^2\, l^2} + \frac{2}{3}\left(\frac{1}{k\, l^3} + \frac{1}{k^3\, l}\right)\right]$$

$$= \frac{3}{50n^3}\left(\frac{5}{6}\right)^n \left[\frac{4(n+1)}{n^3}\sum_1^{n-1}\frac{1}{k} + \frac{2n+\frac{8}{3}}{n^2}\sum_1^{n-1}\frac{1}{k^2} + \frac{4}{3n}\sum_1^{n-1}\frac{1}{k^3}\right] < \frac{3}{5n^4}\left(\frac{5}{6}\right)^n$$

where we use the following elementary inequalities

$$\frac{4(n+1)}{n^3}\sum_1^{n-1}\frac{1}{k} < \frac{3}{n}$$

$$\frac{2n+\frac{8}{3}}{n^2}\sum_1^{n-1}\frac{1}{k^2} < \frac{5}{n}$$

$$\frac{4}{3n}\sum_1^{n-1}\frac{1}{k^3} < \frac{2}{n}.$$

Lemma is proved.

Corollary 2.1. *The function $\phi(x)$ is analytic in the domain*

$$\operatorname{Re} x < \log\frac{6}{5}. \tag{2.85}$$

It would be interesting to give more neat description of the analytic properties of the function $\phi(x)$. An analytic expression for this function in a closed form is still not available,

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*) As I learned very recently from D.Morrison, a general scheme of proving convergence of the series for the free energy in topological sigma-models was recently proposed by J.Kollár.

although there are some very interesting formulae in the recent preprint of Kontsevich [91]. The asymptotics of the numbers N_k for big k was found in a very recent preprint [39] of Di Francesco and Itzykson.

More complicated example is the quantum cohomology ring of the quintic in CP^4 (the simplest example of a Calabi - Yau three-fold). Here $\dim H^{1,1}(X) = 1$. We denote by ϕ the basic element in $H^{1,1}(X)$ (the Kähler class) and by t the corresponding coupling constant. The only nontrivial term in the quantum cohomology ring is

$$< \phi \cdot \phi, \phi >= 5 + \sum_{n=1}^{\infty} A(n) n^3 \frac{q^n}{1-q^n}, \quad q = e^{-t} \tag{2.86}$$

where $A(n)$ is the number of rational *curves* in X of the degree n. The function (2.86) has been found in [26] in the setting of mirror conjecture. Quantum cohomologies for other manifolds were calculated in [9, 15, 30, 71, 86]. In [71] quantum cohomologies of flag varieties were found. A remarkable description of them in terms of the ring of functions on a particular Lagrangian manifold of the Toda system was discovered. Also a relation of quantum cohomologies and Floer cohomologies [129, 124] was elucidated in these papers. A general approach to calculating numbers of higher genera curves in Calabi - Yau varieties was found in [17].

Example 2.3. Topological sigma-models of B-type [151]. Let X be a compact Calabi - Yau manifold (we consider only 3-folds, i.e. $d = \dim_{\mathbf{C}} X = 3$). The correlation functions in the model are expressed in terms of periods of some differential forms on X. The structure of the Frobenius manifold $M = M(X)$ is decribed only on the sublocus $M_0(X)$ of complex structures on X. I recall that $\dim M_0(X) = \dim H^{2,1}(X)$ while the dimension of M is equal to the dimension of the full cohomology space of X. The bilinear form $\eta_{\alpha\beta}$ coincides with the intersection number

$$< \phi', \phi'' >= \int_X \phi' \wedge \phi''. \tag{2.87}$$

To define the trilinear form on the tangent space to $M_0(X)$ we fix a holomorphic 3-form

$$\Omega \in H^{3,0}(X) \tag{2.88}$$

and normalize it by the condition

$$\oint_{\gamma_0} \Omega = 1 \tag{2.89}$$

(I recall that $\dim H^{3,0}(X) = 1$ for a Calabi - Yau 3-fold) for an appropriate cycle $\gamma_0 \in H_3(X, \mathbf{Z})$. Then the trilinear form reads

$$< \partial, \partial', \partial'' >:= \int_X \partial\partial'\partial''\Omega \wedge \Omega \tag{2.90}$$

for three tangent vector fields ∂, ∂', ∂'' on $M_0(X)$ (I refer the reader to [114] regarding the details of the construction of holomorphic vector fields on $M_0(X)$ using technique of variations of Hodge structures). From the definition it follows that the Frobenius structure (2.90) is well-defined only locally. Globally we would expect to obtain a twisted Frobenius manifold in the sense of Appendix B because of the ambiguity in the choice of the normalizing cycle γ_0.

The mirror conjecture claims that for any Calabi - Yau manifold X there exists another Calabi - Yau manifold $\hat{X}$ such that the Frobenius structure of A-type determined by the quantum multiplication on the cohomologies of X is locally isomorphic to the Frobenius structure of the B-type defined by the periods of $\hat{X}$. See [146, 151] concerning motivations of the conjecture and [8, 26, 30, 79, 115] for consideration of the particular examples of mirror dual manifolds X and $\hat{X}$.

Example 2.4. Topological Landau - Ginsburg (LG) models [145]. The bosonic part of the LG-action S has the form

$$S = \int d^2 z \left(|\frac{\partial p}{\partial z}|^2 + |\lambda'(p)|^2 \right) \tag{2.91}$$

where the holomorphic function $\lambda(p)$ is called *superpotential* and S is considered as a functional of the holomorphic *superfield* $p = p(z)$. The classical states are thus in the one-to-one correspondence with the critical points of $\lambda(p)$

$$p(z) \equiv p_i, \quad \lambda'(p_i) = 0, \quad i = 1, \ldots, n. \tag{2.92}$$

Quantum correlations can be computed [145, 29] in terms of solitons propagating between the classical vacua (2.92). If the critical points of the superpotential are non-degenerate and the critical values are pairwise distinct then masses of the solitons are proportional to the differences of the critical values. In this case we obtain a *massive* TFT.

The moduli space of a LG theory can be realized as a family of LG models with an appropriately deformed superpotential

$$\lambda = \lambda(p; t^1, \ldots, t^n). \tag{2.93}$$

The Frobenius structure on the space of parameters is given by the following formulae [145, 29]

$$< \partial, \partial' >_\lambda = \sum \operatorname*{res}_{\lambda'=0} \frac{\partial(\lambda dp)\partial'(\lambda dp)}{d\lambda(p)} \tag{2.94a}$$

$$< \partial, \partial', \partial'' >_\lambda = \sum \operatorname*{res}_{\lambda'=0} \frac{\partial(\lambda dp)\partial'(\lambda dp)\partial''(\lambda dp)}{d\lambda(p)dp.} \tag{2.94b}$$

By definition, in these formulae the vector fields ∂, ∂', ∂'' on the space of parameters act trivially on p.

Particularly, for the superpotential

$$\lambda(p) = p^{n+1}$$

the deformed superpotential coincides with the generic polynomial of the degree $n+1$ of the form (1.65). The Frobenius structure (2.94) coincides with the one of Example 1.7 (see Lecture 4 below).

We will not discuss here interesting relations between topological sigma-models and topological LG models (see [161]).

Lecture 3.

Spaces of isomonodromy deformations as Frobenius manifolds.

Let us consider a linear differential operator of the form

$$\Lambda = \frac{d}{dz} - U - \frac{1}{z} V \tag{3.1}$$

where U and V are two complex $n \times n$ z-independent matrices, U is a diagonal matrix with pairwise distinct diagonal entries, and the matrix V is skew-symmetric. The solutions of the differential equation with rational coefficients

$$\Lambda \psi = 0 \tag{3.2}$$

are analytic multivalued vector functions in $z \in \mathbf{C} \setminus 0$. The monodromy of these solutions will be called monodromy of the differential operator Λ (we will give below the precise definition of the monodromy).

Let $\mathcal{M}(\Lambda)$ be the space of all operators of the form (3.1) with a given monodromy. As it follows from the general theory of isomonodromy deformations [82, 109, 113, 135] the diagonal entries $u_1, \ldots, u_n$ of the matrix U can serve as local coordinates near generic point of $\mathcal{M}(\Lambda)$ (thus a function $V = V(u)$ locally is well defined). We define a Frobenius structure on $\mathcal{M}(\Lambda)$ by the multiplication

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i, \quad \partial_i := \frac{\partial}{\partial u_i}$$

the quadratic form

$$< \, , \, >:= \sum_{i=1}^{n} \psi_i^2(u) du_i^2$$

where $\psi = (\psi_1(u), \ldots, \psi_n(u))^T$ is an eigenvector of $V(u)$ (it can be normalized in such a way that the metric $< \, , \, >$ is flat), the unity

$$e := \sum_{i=1}^{n} \partial_i$$

and the Euler vector field

$$E := \sum_{i=1}^{n} u_i \partial_i .$$

Observe that the Frobenius algebra on the tangent planes to $\mathcal{M}(\Lambda)$ is semisimple in any point u (we will say that such a Frobenius manifold satisfies semisimplicity condition).

The main result of this Lecture is the following

Main Theorem. *The above formulae determine a Frobenius structure on the space $\mathcal{M}(\Lambda)$ of all the operators of the form (3.1) with a given monodromy. Choice of another* *eigenvector ψ changes the Frobenius structure by a transformation of the form (B.2). Conversely, any Frobenius manifold satisfying semisimplisity assumption locally can be obtained by such a construction.*

The following two creatures are the principal playing characters on a Frobenius manifold M.

1. Deformed Euclidean connection

$$\tilde{\nabla}_u(z)v := \nabla_u v + zu \cdot v. \tag{3.3}$$

It is a symmetric connection depending on the parameter z. Here u, v are two vector fields on M, $\nabla_u v$ is the Levi-Civita connection for the invariant metric $< \, , \, >$.

Lemma 3.1. *The connection $\tilde{\nabla}(z)$ is flat identically in z* iff *the algebra $c_{\alpha\beta}^{\gamma}(t)$ is associative and a function $F(t)$ locally exists such that*

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t). \tag{3.4}$$

Proof. In the flat coordinates vanishing of the curvature of the deformed connection reads

$$[\tilde{\nabla}_\alpha(z), \tilde{\nabla}_\beta(z)]^\epsilon = [z(\partial_\beta c_{\alpha\gamma}^\epsilon - \partial_\alpha c_{\beta\gamma}^\epsilon) + z^2(c_{\alpha\gamma}^\sigma c_{\beta\sigma}^\epsilon - c_{\beta\gamma}^\sigma c_{\alpha\sigma}^\epsilon)] = 0.$$

Vanishing of the coefficients of the quadratic polynomial in z together with the symmetry of the tensor $c_{\alpha\beta\gamma} := \eta_{\gamma\epsilon} c_{\alpha\beta}^\epsilon$ is equivalent to the statements of the lemma.

Remark 3.1. Vanishing of the curvature is equivalent to the compatibility of the linear system

$$\partial_\alpha \xi_\beta = z c_{\alpha\beta}^\gamma(t) \xi_\gamma. \tag{3.5a}$$

This gives a "Lax pair" for the associativity equations (1.14) (z plays the role of the spectral parameter). Note that

$$\partial_1 \xi_\beta = z \xi_\beta. \tag{3.5b}$$

This coincides with the normalization (1.1). For any given z the system has n-dimensional space of solutions. The solutions are closely related to the flat coordinates of the deformed connection $\tilde{\nabla}(z)$, i.e to the independent functions $\tilde{t}^1(t,z)$, ..., $\tilde{t}^n(t,z)$ such that in these new coordinates the deformed covariant derivatives coincide with partial derivatives

$$\tilde{\nabla}_\alpha(z) = \frac{\partial}{\partial \tilde{t}^\alpha}, \; \tilde{t}^\alpha = \tilde{t}^\alpha(t,z). \tag{3.6}$$

Exercise 3.1. Prove that 1) any solution of the system (3.5) is the gradient of some function

$$\xi_\alpha = \partial_\alpha \tilde{t}; \tag{3.7}$$

2) if ξ_α^1, ..., ξ_α^n is a fundamental system of solutions of the system (3.5) for a given z then the corresponding functions $\tilde{t}^1$, ..., $\tilde{t}^n$ are flat coordinates for the deformed connection $\tilde{\nabla}(z)$.

Exercise 3.2. Derive from (3.5) the following Lax pair for the Chazy equation (C.5)

$$[\partial_z + U, \partial_x + V] = 0 \tag{3.8}$$

where the matrices U and V have the form

$$U = \begin{pmatrix} 0 & -1 & 0 \\ \frac{3}{4}z^2\gamma' & \frac{3}{2}z\gamma & -1 \\ \frac{1}{4}z^3\gamma'' & \frac{3}{4}z^2\gamma' & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & -1 \\ \frac{1}{4}z^3\gamma'' & \frac{3}{4}z^2\gamma' & 0 \\ \frac{1}{16}z^4\gamma''' & \frac{1}{4}z^3\gamma'' & 0 \end{pmatrix}. \tag{3.9}$$

Observe that the more strong commutativity condition

$$[U, V] = 0 \tag{3.10}$$

holds true for the matrices U, V. It is easy to see that one can put arbitrary three unknown functions in (3.9) instead of γ', γ'', γ'''. Then the commutativity (3.8) together with (3.10) is still equivalent to the Chazy equation.

The commutation representation (3.8), (3.10) looks to be intermediate one between Lax pairs with a derivative w.r.t. the spectral parameter z (those being typical in the theory of isomonodromic deformations) and "integrable algebraic systems" of [55], i.e. the equations of commutativity of matrices depending on the spectral parameter.

A Lax pair for the Chazy equation was obtained also in [1]. But instead of finite dimensional matrices some differential operators with partial derivatives are involved. This gives no possibility to apply the machinery of the theory of integrable systems.

I learned recently from S.Chakravarty that he has found another finite-dimensional Lax pair for Chazy equation. This looks similar to the Lax pair of the Painlevé-VI equation with a nontrivial dependence of the poles on the both dependent and independent variables.

The further step is to consider WDVV as the scaling reduction of the equations of associativity (1.14) as of an integrable system. The standard machinery of integration of scaling reductions of integrable systems [64, 82, 135] suggests to add to (3.5) a differential equation in the spectral parameter z for the auxiliary function $\xi = \xi(t, z)$.

Proposition 3.1. *WDVV is equivalent to compatibility of the system of equations (3.5) together with the equation*

$$z\partial_z\xi_\alpha = zE^\gamma(t)c_{\gamma\alpha}^\beta(t)\xi_\beta + Q_\alpha^\gamma\xi_\gamma \tag{3.11}$$

where $Q_\alpha^\gamma = \nabla_\alpha E^\gamma$.

Proof. Due to (1.50b) the system (3.5) is invariant w.r.t. the group of rescalings (1.43) together with the transformations $z \mapsto kz$. So the system (3.5) for the covector ξ is compatible with the equation

$$x\partial_z\xi = \mathcal{L}_E\xi.$$

On the solutions of (3.5) the last equation can be rewritten in the form (3.11). Proposition is proved.

The compatibility of (3.5) and (3.11) can be reformulated [92] as vanishing of the curvature of the connection on $M \times CP^1$ (the coordinates are (t, z)) given by the operators (3.5) and (3.11). For tangent vectors v on $M \times CP^1$ parallel to M the z-component $\tilde{\nabla}_z$ of the connection acts by the formula

$$\tilde{\nabla}_z v = \partial_z v + E \cdot v + \frac{1}{z} Q v. \tag{3.12}$$

Along CP^1 the connection acts as $\tilde{\nabla}_z = \partial_z$, $\tilde{\nabla}_\alpha(z) = \partial_\alpha$.

The further step of the theory (also being standard [64, 82, 135]) is to parametrize the solutions of WDVV by the monodromy data of the operator (3.11) with rational coefficients. We are not able to do this in general. The difficulty is to describe quantitavely the monodromy of the operator. The main problem is to choose a trivialization in the space of solutions of the equation (3.11) for big z. This problem looks not to be purely technical: one can see that WDVV does not satisfy the Painlevé property of absence of movable critical points in the t-coordinates (see, e.g., the discussion of the analytic properties of solutions of Chazy equation in Appendix C above). Another point is that, the monodromy of the equation (3.11) can be trivial for a nontrivial solution of WDVV (e.g., for the nilpotent solutions (1.38)).

Our main strategy will be to find an appropriate coordinate system on M and to do a gauge transform of the operator (3.11) providing applicability of the isomonodromy deformations technique to WDVV. This can be done under semi-simplicity assumptions imposed onto M (see below).

Another important playing character is a new metric on a Frobenius manifold. It is convenient to define it as a metric on the cotangent bundle T^*M i.e. as an inner product of 1-forms. For two 1-forms ω_1 and ω_2 we put

$$(\omega_1, \omega_2)^* := i_E(\omega_1 \cdot \omega_2) \tag{3.13}$$

(I label the metric by * to stress that this is an inner product on T^*M). Here i_E is the operator of contraction of a 1-form with the vector field E; we multiply two 1-forms using the operation of multiplication of tangent vectors on the Frobenius manifold and the duality between tangent and cotangent spaces established by the invariant inner product.

Exercise 3.3. Prove that the inner product (u, v) of two vector fields w.r.t. the new metric is related to the old inner product $< u, v >$ by the equation

$$(E \cdot u, v) = < u, v > . \tag{3.14}$$

Thus the new metric on the *tangent* bundle is welldefined in the points t of M where $E(t)$ is an invertible element of the algebra T_tM.

In the flat coordinates t^α the metric $(\ ,\ )^*$ has the components

$$g^{\alpha\beta}(t) := (dt^\alpha, dt^\beta)^* = E^\epsilon(t) c_\epsilon^{\alpha\beta}(t) \tag{3.15}$$

where

$$c_\epsilon^{\alpha\beta}(t) := \eta^{\alpha\sigma} c_{\sigma\epsilon}^{\beta}(t). \tag{3.16}$$

If the degree operator is diagonalizable then

$$g^{\alpha\beta}(t) = (d + 1 - q_\alpha - q_\beta) F^{\alpha\beta}(t) + A^{\alpha\beta} \tag{3.17}$$

where

$$F^{\alpha\beta}(t) := \eta^{\alpha\lambda}\eta^{\beta\mu} \frac{\partial^2 F(t)}{\partial t^\lambda \partial t^\mu} \tag{3.18}$$

(I recall that we normalise the degrees d_α in such a way that $d_1 = 1$) and the matrix $A_{\alpha\beta} = \eta_{\alpha\alpha'}\eta_{\beta\beta'}A^{\alpha'\beta'}$ is defined in (1.9).

Lemma 3.2. *The metric (3.13) does not degenerate identically near the t^1-axis for sufficiently small $t^1 \neq 0$.*

Proof. We have

$$c_1^{\alpha\beta}(t) \equiv \eta^{\alpha\beta}.$$

So for small t^2, ..., t^n

$$g^{\alpha\beta}(t) \simeq t^1 c_1^{\alpha\beta} + A^{\alpha\beta} = t^1\eta^{\alpha\beta} + A^{\alpha\beta}.$$

This cannot be degenerate identically in t^1. Lemma is proved.

It turns out that the new metric also is flat. In fact I will prove a more strong statement, that any linear combination of the metrics $(\, , \,)^*$ and $< \, , \, >^*$ is a flat metric (everywhere when being nondegenerate). To formulate the precise statement I recall some formulae of Riemannian geometry.

Let $(\, , \,)^*$ be a symmetric nondegenerate bilinear form on the cotangent bundle T^*M to a manifold M. In a local coordinate system x^1, ..., x^n the metric is given by its components

$$g^{ij}(x) := (dx^i, dx^j)^* \tag{3.19}$$

where (g^{ij}) is an invertible symmetric matrix. The inverse matrix $(g_{ij}) := (g^{ij})^{-1}$ specifies a metric on the manifold i.e. a nondegenerate inner product on the tangent bundle TM

$$(\partial_i, \partial_j) := g_{ij}(x) \tag{3.20}$$

$$\partial_i := \frac{\partial}{\partial x^i}. \tag{3.21}$$

The *Levi-Civita connection* ∇_k for the metric is uniquely specified by the conditions

$$\nabla_k g_{ij} := \partial_k g_{ij} - \Gamma_{ki}^s g_{sj} - \Gamma_{kj}^s g_{is} = 0 \tag{3.22a}$$

or, equivalently,

$$\nabla_k g^{ij} := \partial_k g^{ij} + \Gamma_{ks}^i g^{sj} + \Gamma_{ks}^j g^{is} = 0 \tag{3.22b}$$

and

$$\Gamma_{ij}^k = \Gamma_{ji}^k. \tag{3.23}$$

(I recall that summation over twice repeated indices here and below is assumed. We will keep the symbol of summation over more than twice repeated indices.) Here the coefficients Γ^k_{ij} of the connection (the Christoffel symbols) can be expressed via the metric and its derivatives as

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks} \left(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij} \right). \tag{3.24}$$

For us it will be more convenient to work with the *contravariant components* of the connection

$$\Gamma^{ij}_k := (dx^i, \nabla_k dx^j)^* = -g^{is} \Gamma^j_{sk}. \tag{3.25}$$

The equations (3.22) and (3.23) for the contravariant components read

$$\partial_k g^{ij} = \Gamma^{ij}_k + \Gamma^{ji}_k \tag{3.26}$$

$$g^{is} \Gamma^{jk}_s = g^{js} \Gamma^{ik}_s. \tag{3.27}$$

It is also convenient to introduce operators

$$\nabla^i = g^{is} \nabla_s \tag{3.28a}$$

$$\nabla^i \xi_k = g^{is} \partial_s \xi_k + \Gamma^{is}_k \xi_s. \tag{3.28b}$$

For brevity we will call the operators ∇^i and the corresponding coefficients Γ^{ij}_k *contravariant connection.*

The *curvature tensor* R^k_{slt} of the metric measures noncommutativity of the operators ∇_i or, equivalently ∇^i

$$(\nabla_s \nabla_l - \nabla_l \nabla_s) \xi_t = -R^k_{slt} \xi_k \tag{3.29a}$$

where

$$R^k_{slt} = \partial_s \Gamma^k_{lt} - \partial_l \Gamma^k_{st} + \Gamma^k_{sr} \Gamma^r_{lt} - \Gamma^k_{lr} \Gamma^r_{st}. \tag{3.29b}$$

We say that the metric is *flat* if the curvature of it vanishes. For a flat metric local *flat coordinates* p^1, ..., p^n exist such that in these coordinates the metric is constant and the components of the Levi-Civita connection vanish. Conversely, if a system of flat coordinates for a metric exists then the metric is flat. The flat coordinates are determined uniquely up to an affine transformation with constant coefficients. They can be found from the following system

$$\nabla^i \partial_j p = g^{is} \partial_s \partial_j p + \Gamma^{is}_j \partial_s p = 0, \ \ i, j = 1, ..., n. \tag{3.30}$$

If we choose the flat coordinates orthonormalized

$$(dp^a, dp^b)^* = \delta^{ab} \tag{3.31}$$

then for the components of the metric and of the Levi-Civita connection the following formulae hold

$$g^{ij} = \frac{\partial x^i}{\partial p^a} \frac{\partial x^j}{\partial p^a} \tag{3.32a}$$

$$\Gamma^{ij}_k dx^k = \frac{\partial x^i}{\partial p^a} \frac{\partial^2 x^j}{\partial p^a \partial p^b} dp^b. \tag{3.32b}$$

All these facts are standard in geometry (see, e.g., [59]). We need to represent the formula (3.29b) for the curvature tensor in a slightly modified form (cf. [57, formula (2.18)]).

Lemma 3.3. *For the curvature of a metric the following formula holds*

$$R^{ijk}_l := g^{is} g^{jt} R^k_{slt} = g^{is} \left(\partial_l \Gamma^{jk}_s - \partial_s \Gamma^{jk}_l \right) + \Gamma^{ij}_s \Gamma^{sk}_l - \Gamma^{ik}_s \Gamma^{sj}_l. \tag{3.33}$$

Proof. Multiplying the formula (3.29b) by $g^{is} g^{jt}$ and using (3.25) and (3.26) we obtain (3.33). The lemma is proved.

Let us consider now a manifold supplied with two nonproportional metrics $(\ ,\ )^*_1$ and $(\ ,\ )^*_2$. In a coordinate system they are given by their components g^{ij}_1 and g^{ij}_2 resp. I will denote by Γ^{ij}_{1k} and Γ^{ij}_{2k} the corresponding Levi-Civita connections ∇^i_1 and ∇^i_2. Note that the difference

$$\Delta^{ijk} = g^{is}_2 \Gamma^{jk}_{1s} - g^{is}_1 \Gamma^{jk}_{2s} \tag{3.34}$$

is a tensor on the manifold.

Definition 3.1. We say that the two metrics form a *flat pencil* if:

1. The metric

$$g^{ij} = g^{ij}_1 + \lambda g^{ij}_2 \tag{3.35a}$$

is flat for arbitrary λ and

2. The Levi-Civita connection for the metric (3.35a) has the form

$$\Gamma^{ij}_k = \Gamma^{ij}_{1k} + \lambda \Gamma^{ij}_{2k}. \tag{3.35b}$$

I will describe in more details the conditions for two metrics to form a flat pencil in Appendix D below (it turns out that these conditions are very close to the axioms of Frobenius manifolds).

Let us consider the metrics $(\ ,\ )^*$ and $<\ ,\ >^*$ on a Frobenius manifold M (the second metric is induced on T^*M by the invariant metric $<\ ,\ >$). We will assume that the Euler vector field E in the flat coordinates has the form

$$E = \sum_\alpha [(1 - q_\alpha) t^\alpha + r_\alpha] \partial_\alpha.$$

Proposition 3.2. *The metrics $(\ ,\ )^*$ and $<\ ,\ >^*$ on a Frobenius manifold form a flat pencil.*

Lemma 3.4. *In the flat coordinates the contravariant components of the Levi-Civita connection for the metric $(\ ,\ )^*$ have the form*

$$\Gamma^{\alpha\beta}_\gamma = \left(\frac{d+1}{2} - q_\beta \right) c^{\alpha\beta}_\gamma. \tag{3.36}$$

Proof. Substituting (3.36) to (3.26), (3.27) and (3.33) we obtain identities. Lemma is proved.

Proof of proposition. We repeat the calculation of the lemma for the same connection and for the metric $g^{\alpha\beta} + \lambda\eta^{\alpha\beta}$. The equations (3.26) and (3.27) hold true identically in λ. Now substitute the connection into the formula (3.33) for the curvature of the metric $g^{\alpha\beta} + \lambda\eta^{\alpha\beta}$. We again obtain identity. Proposition is proved.

Definition 3.2. The metric $(\, , \,)^*$ of the form (3.13) will be called *intersection form of the Frobenius manifold*.

We borrow this name from the singularity theory [5, 6]. The motivation becomes clear from the consideration of the Example 1.7. In this example the Frobenius manifold coincides with the universal unfolding of the simple singularity of the A_n-type [5]. The metric (3.13) for the example coincides with the intersection form of (even-dimensional) vanishing cycles of the singularity [3, 6, 70] as an inner product on the cotangent bundle to the universal unfolding space (we identify [6] the tangent bundle to the universal unfolding with the middle homology fibering using the differential of the period mapping). It turns out that the intersection form of odd-dimensional vanishing cycles (assuming that the base of the bundle is even-dimensional) coincides with the skew-symmetric form $< \hat V \, . \, , \, . >^*$ where the operator $\hat V$ was defined in (1.51). This can be derived from the results of Givental [69] (see below (3.46)).

Example 3.1. For the trivial Frobenius manifold corresponding to a graded Frobenius algebra $A = \{c_{\alpha\beta}^{\gamma}, \ \eta_{\alpha\beta}, \ q_\alpha, \ d\}$ (see Example 1 of Lecture 1) the intersection form is a linear metric on the dual space A^*

$$g^{\alpha\beta} = \sum (1 - q_\epsilon) t^\epsilon c_\epsilon^{\alpha\beta}, \tag{3.37}$$

for

$$c_\epsilon^{\alpha\beta} = \eta^{\alpha\sigma} c_{\sigma\epsilon}^{\beta}, \quad (\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}.$$

From the above considerations it follows that the Christoffel coefficients for this flat metric are

$$\Gamma_\gamma^{\alpha\beta} = \left(\frac{d+1}{2} - q_\beta \right) c_\gamma^{\alpha\beta}. \tag{3.38}$$

Flat linear metrics with constant contravariant Christoffel coefficients were first studied by S.Novikov and A.Balinsky [12] due to their close relations to vector analogues of the Virasoro algebra. We will come back to this example in Lecture 6 (see also Appendix G below).

Remark 3.2. Knowing the intersection form of a Frobenius manifold and the Euler and the unity vector fields E and e resp. we can uniquely reconstruct the Frobenius structure if $d + 1 \neq q_\alpha + q_\beta$ for any $1 \leq \alpha, \, \beta \leq n$. Indeed, we can put

$$< \, , \, >^* := \mathcal{L}_e (\, , \,)^* \tag{3.39}$$

(the Lie derivative along e). Then we can choose the coordinates t^α taking the flat coordinates for the metric $<\ ,\ >^*$ and choosing them homogeneous for E. Putting

$$g^{\alpha\beta} := (dt^\alpha, dt^\beta)^* \tag{3.40}$$

$$\deg g^{\alpha\beta} := \frac{\mathcal{L}_E g^{\alpha\beta}}{g^{\alpha\beta}} \tag{3.41}$$

we can find the function F from the equations

$$g^{\alpha\beta} = \deg g^{\alpha\beta} F^{\alpha\beta} \tag{3.42}$$

($F^{\alpha\beta}$ are the contravariant components of the Hessian of F, see formula (3.17)). This observation will be very important in the constructions of the next lecture.

Exercise 3.4. Let ω will be the 1-form on a Frobenius manifold defined by

$$\omega(\ .\ ) = <e,\ .\ > . \tag{3.43}$$

Show that the formula

$$\left\{x^k, x^l\right\} := \frac{1}{2}\mathcal{L}_e\left[\left(dx^l, dx^i\right)\partial_i\left(dx^k,\omega\right) - \left(dx^k, dx^i\right)\partial_i\left(dx^l,\omega\right)\right] \tag{3.44}$$

defines a Poisson bracket on the Frobenius manifold. [Hint: prove that the bracket is constant in the flat coordinates for the metric $<\ ,\ >$,

$$\left\{t^\alpha, t^\beta\right\} = -\frac{1}{2}(q_\alpha - q_\beta)\eta^{\alpha\beta}.] \tag{3.45}$$

Observe that the tensor of the Poisson bracket has the form

$$\{\ .\ ,\ .\ \} = -\frac{1}{2} < \hat{V}\ .\ ,\ .\ >^* . \tag{3.46}$$

For the case of the Frobenius manifold of Example 1.7 the Poisson structure coincides with the skew-symmetric intersection form on the universal unfolding of the A_n singularity (see [65]). The formula (3.44) for this case was obtained by Givental [65, Corollary 3].

We add now the following *assumption of semisimplicity* on the Frobenius manifold M. We say that a point $t \in M$ is *semisimple* if the Frobenius algebra T_tM is semisimple (i.e. it has no nilpotents). It is clear that semisimplicity is an open property of the point. The assumption of semisimplicity for a Frobenius manifold M means that a generic point of M is semisimple. In physical context this corresponds to massive perturbations of TCFT [29]. So we will also call M satisfying the semisimplicity assumption *massive* Frobenius manifold.

Main lemma. *In a neighborhood of a semisimple point local coordinates u^1, ..., u^n exist such that*

$$\partial_i \cdot \partial_j = \delta_{ij}\,\partial_i, \quad \partial_i = \frac{\partial}{\partial u^i}. \tag{3.47}$$

Proof. In a neighborhood of a semisimple point t *vector fields* ∂_1, ..., ∂_n exist such that $\partial_i \cdot \partial_j = \delta_{ij}\, \partial_i$ (idempotents of the algebra T_tM). We need to prove that these vector fields commute pairwise. Let

$$[\partial_i, \partial_j] =: f_{ij}^k\, \partial_k. \tag{3.48}$$

We rewrite the condition of flatness of the deformed connection $\tilde\nabla(z)$ in the basis ∂_1, ..., ∂_n. I recall that the curvature operator for a connection ∇ is defined by

$$R(X,Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{3.49}$$

for any three vector fields X, Y, Z. We define the coefficients of the Euclidean connection on the Frobenius manifold in the basis ∂_1, ..., ∂_n by the formula (see [59], sect. 30.1)

$$\nabla_{\partial_i}\partial_j =: \Gamma_{ij}^k \partial_k. \tag{3.50}$$

Vanishing of the curvature of $\tilde\nabla(z)$ in the terms linear in z reads

$$\Gamma_{kj}^l \delta_i^l + \Gamma_{ki}^l \delta_{kj} - \Gamma_{ki}^l \delta_j^l - \Gamma_{kj}^l \delta_{ki} = f_{ij}^l \delta_k^l \tag{3.51}$$

(no summation over the repeated indices in this formula!). For $l = k$ this gives $f_{ij}^k = 0$. Lemma is proved.

Observe that neither the scaling invariance (1.9) nor constancy of the unity $e = \partial/\partial t^1$ have been used in the proof of Main Lemma.

Remark 3.3. The main lemma can be reformulated in terms of *algebraic symmetries* of a massive Frobenius manifold. We say that a diffeomorphism $f : M \to M$ of a Frobenius manifold is algebraic symmetry if it preserves the multiplication law of vector fields:

$$f_*(u \cdot v) = f_*(u) \cdot f_*(v) \tag{3.52}$$

(here f_* is the induced linear map $f_* : T_xM \to T_{f(x)}M$).

It is easy to see that algebraic symmetries of a Frobenius manifold form a finite-dimensional Lie group $G(M)$. The generators of action of $G(M)$ on M (i.e. the representation of the Lie algebra of $G(M)$ in the Lie algebra of vector fields on M) are the vector fields w such that

$$[w, u \cdot v] = [w, u] \cdot v + [w, v] \cdot u \tag{3.53}$$

for any vector fields u, v.

Note that the group $G(M)$ always is nontrivial: it contains the one-parameter subgroup of shifts along the coordinate t^1. The generator of this subgroup coincides with the unity vector field e.

Main lemma′. *The connect component of the identity in the group $G(M)$ of algebraic symmetries of a n-dimensional massive Frobenius manifold is a n-dimensional commutative Lie group that acts localy transitively on M.*

I will call the local coordinates u^1, ..., u^n on a massive Frobenius manifold *canonical coordinates*. They can be found as independent solutions of the system of PDE

$$\partial_\gamma u\, c^\gamma_{\alpha\beta}(t) = \partial_\alpha u\, \partial_\beta u \tag{3.54}$$

or, equivalently, the 1-form du must be a homomorphism of the algebras

$$du : T_t M \to \mathbf{C}. \tag{3.55}$$

The canonical coordinates are determined uniquely up to shifts and permutations.

We solve now explicitly this system of PDE using the scaling invariance (1.9).

Proposition 3.3. *In a neighborhood of a semisimple point all the roots $u^1(t)$, ..., $u^n(t)$ of the characteristic equation*

$$\det(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}) = 0 \tag{3.56}$$

are simple. They are canonical coordinates in this neighborhood. Conversely, if the roots of the characteristic equation are simple in a point t then t is a semisimple point on the Frobenius manifold and $u^1(t)$, ..., $u^n(t)$ are canonical coordinates in the neighbourhood of the point.

Here $g^{\alpha\beta}(t)$ is the intersection form (3.15) of the Frobenius manifold.

Lemma 3.5. *Canonical coordinates in a neighborhood of a semisimple point can be chosen in such a way that the Euler vector field E have the form*

$$E = \sum_i u^i \partial_i. \tag{3.57}$$

Proof. Rescalings generated by E act on the idempotents ∂_i as $\partial_i \mapsto k^{-1}\partial_i$. So an appropriate shift of u^i provides $u^i \mapsto k u^i$. Lemma is proved.

Lemma 3.6. *The invariant inner product $<\ ,\ >$ is diagonal in the canonical coordinates*

$$< \partial_i, \partial_j >= \eta_{ii}(u)\, \delta_{ij} \tag{3.58}$$

for some nonzero functions $\eta_{11}(u)$, ..., $\eta_{nn}(u)$. The unity vector field e in the canonical coordinates has the form

$$e = \sum_i \partial_i. \tag{3.59}$$

The proof is obvious (cf. (1.41), (1.42)).

Proof of proposition. In the canonical coordinates of Main Lemma we have

$$du^i \cdot du^j = \eta_{ii}^{-1} du^i\, \delta_{ij}. \tag{3.60}$$

So the intersection form reads

$$g^{ij}(u) = u^i \eta_{ii}^{-1} \delta_{ij}. \tag{3.61}$$

The characteristic equation (3.56) reads

$$\prod_i (u - u^i) = 0.$$

This proves the first part of the proposition.

To prove the second part we consider the linear operators $U = (U^\alpha_\beta(t))$ on T_tM where

$$U^\alpha_\beta(t) := g^{\alpha\epsilon}(t)\eta_{\epsilon\beta}. \tag{3.62}$$

From (3.14) it folows that U is the operator of multiplication by the Euler vector field E. The characteristic equation for this operator coincides with (3.56). So under the assumptions of the proposition the operator of multiplication by E in the point t is a semisimple one. This implies the semisimplicity of all the algebra T_tM because of the commutativity of the algebra.

Proposition is proved.

Using canonical coordinates we reduce the problem of local classification of massive Frobenius manifolds to an integrable system of ODE. To obtain such a system we will study the properties of the invariant metric in the canonical coordinates. I recall that this metric has diagonal form in the canonical coordinates. In other words, u^1, ..., u^n are curvilinear orthogonal coordinates in the (locally) Euclidean space with the (complex) Euclidean metric $<\ ,\ >$. The familiar object in the geometry of curvilinear orthogonal coordinates is the *rotation coefficients*

$$\gamma_{ij}(u) := \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}}, \quad i \neq j \tag{3.63}$$

(locally we can fix some branches of $\sqrt{\eta_{ii}(u)}$). They determine the law of rotation with transport along the u-axes of the natural orthonormal frame related to the orthogonal system of coordinates (see [35]).

Lemma 3.7. *The coefficients $\eta_{ii}(u)$ of the invariant metric have the form*

$$\eta_{ii}(u) = \partial_i t_1(u), \quad i = 1, \dots, n. \tag{3.64}$$

Proof. According to (1.39) the invariant inner product $<\ ,\ >$ has the form

$$< a, b > = < e, a \cdot b > \equiv \omega(a \cdot b) \tag{3.65}$$

for any two vector fields a, b where the 1-form ω is

$$\omega(\,.\,) := < e, . >, \quad e = \frac{\partial}{\partial t^1}. \tag{3.66}$$

Hence $\omega = dt_1$. Lemma is proved.

We summarize the properties of the invariant metric in the canonical coordinates in the following

Proposition 3.4. *The rotation coefficients (3.63) of the invariant metric are symmetric*

$$\gamma_{ij}(u) = \gamma_{ji}(u). \tag{3.67}$$

The metric is invariant w.r.t. the diagonal translations

$$\sum_k \partial_k \eta_{ii}(u) = 0, \quad i = 1, \ldots, n. \tag{3.68}$$

The functions $\eta_{ii}(u)$ and $\gamma_{ij}(u)$ are homogeneous functions of the canonical coordinates of the degrees $-d$ and -1 resp.

Proof. The symmetry (3.67) follows from (3.64):

$$\gamma_{ij}(u) = \frac{1}{2}\frac{\partial_i \partial_j t_1(u)}{\sqrt{\partial_i t_1(u) \partial_j t_1(u)}}. \tag{3.69}$$

To prove (3.68) we use (3.64) and covariant constancy of the vector field

$$e = \sum_i \partial_i.$$

This reads

$$\sum_{k=1}^n \Gamma_{ik}^j = 0.$$

For $i = j$ using the Christoffel formulae (3.24) we obtain (3.68). The homogeneity follows from (1.50). Proposition is proved.

Corollary 3.1. *The rotation coefficients (3.63) satisfy the following system of equations*

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i, j, k \text{ are distinct} \tag{3.70a}$$

$$\sum_{k=1}^n \partial_k \gamma_{ij} = 0 \tag{3.70b}$$

$$\sum_{k=1}^n u^k \partial_k \gamma_{ij} = -\gamma_{ij}. \tag{3.70c}$$

Proof. The equations (3.70a) and (3.70b) for a symmetric off-diagonal matrix $\gamma_{ij}(u)$ coincide with the equations of flatness of the diagonal metric obtained for the metrics of the form (3.64) by Darboux and Egoroff [35]. The equation (3.70c) follows from homogeneity.

We have shown that any massive Frobenius manifold determines a scaling invariant (3.70c) solution of the *Darboux - Egoroff system* (3.70a,b). We show now that, conversely, any solution of the system (3.70) under some genericity assumptions determines locally a massive Frobenius manifold.

Let

$$\Gamma(u) := (\gamma_{ij}(u))$$

be a solution of (3.70).

Lemma 3.8. *The linear system*

$$\partial_k \psi_i = \gamma_{ik} \psi_k, \quad i \neq k \tag{3.71a}$$

$$\sum_{k=1}^{n} \partial_k \psi_i = 0, \quad i = 1, \ldots, n \tag{3.71b}$$

for an auxiliary vector-function $\psi = (\psi_1(u), \ldots, \psi_n(u))^T$ *has* n*-dimensional space of solutions.*

Proof. Compatibility of the system (3.71) follows from the Darboux - Egoroff system (3.70). Lemma is proved.

Let us show that, under certain genericity assumptions a basis of homogeneous in u solutions of (3.71) can be chosen. We introduce the $n \times n$-matrix

$$V(u) := [\Gamma(u), U] \tag{3.72}$$

where

$$U := \text{diag}(u^1, \ldots, u^n), \tag{3.73}$$

[,] stands for matrix commutator.

Lemma 3.9. *The matrix* $V(u)$ *satisfies the following system of equations*

$$\begin{aligned} \partial_k V(u) &= [V(u), [E_k, \Gamma]], \ k = 1, \ldots, n \\ V(u) &= [\Gamma(u), U] \end{aligned} \tag{3.74}$$

where E_k *are the matrix unities*

$$(E_k)_{ij} = \delta_{ik} \delta_{kj}. \tag{3.75}$$

Conversely, all the differential equations (3.70) follow from (3.74).

Proof. From (3.70) we obtain

$$\partial_i \gamma_{ij} = \frac{1}{u^i - u^j} \left(\sum_{k \neq i,j} (u^j - u^k) \gamma_{ik} \gamma_{kj} - \gamma_{ij} \right). \tag{3.76}$$

The equation (3.74) follows from (3.70a) and (3.76). Lemma is proved.

Corollary 3.2.

1). The matrix $V(u)$ *acts on the space of solutions of the linear system (3.71).*

2). Eigenvalues of $V(u)$ *do not depend on* u*.*

3). A solution $\psi(u)$ of the system (3.71) is a homogeneous function of u

$$\psi(cu) = c^{\mu}\psi(u) \tag{3.77}$$

iff *$\psi(u)$ is an eigenvector of the matrix $V(u)$*

$$V(u)\psi(u) = \mu\psi(u). \tag{3.78}$$

Proof. We rewrite first the linear system (3.71) in the matrix form. This reads

$$\partial_k\psi = -[E_k, \Gamma]\psi. \tag{3.79}$$

From (3.74) and (3.79) it follows immediately the first statement of the lemma. Indeed, if ψ is a solution of (3.79) then

$$\partial_k(V\psi) = (V[E_k,\Gamma] - [E_k,\Gamma]V)\psi - V[E_k,\Gamma]\psi = -[E_k,\Gamma]V\psi.$$

The second statement of the lemma also follows from (3.74). The third statement is obvious since

$$\sum u^i\partial_i\psi = V\psi$$

(this follows from (3.79)). Lemma is proved.

Remark 3.4. We will show below that a spectral parameter can be inserted in the linear system (3.71). This will give a way to integrate the system (3.70) using the isomonodromy deformations technique.

We denote by μ_1, ..., μ_n the eigenvalues of the matrix $V(u)$. Due to skew-symmetry of $V(u)$ these can be ordered in such a way that

$$\mu_\alpha + \mu_{n-\alpha+1} = 0. \tag{3.80}$$

Proposition 3.5. *For a massive Frobenius manifold corresponding to a scaling invariant solution of WDVV the matrix $V(u)$ is diagonalizable. Its eigenvectors $\psi_\alpha = (\psi_{1\alpha}(u), \ldots, \psi_{n\alpha}(u))^T$ satisfying (3.71) are*

$$\psi_{i\alpha}(u) = \frac{\partial_i t_\alpha(u)}{\sqrt{\eta_{ii}(u)}}, \quad i,\, \alpha = 1, \ldots, n. \tag{3.81}$$

The corresponding eigenvalues are

$$\mu_\alpha = q_\alpha - \frac{d}{2} \tag{3.82}$$

(the spectrum of the Frobenius manifold in the sense of Appendix C). Conversely, let $V(u)$ be any diagonalizable solution of the system (3.74) and $\psi_\alpha = (\psi_{i\alpha}(u))$ be the solutions of (3.71) satisfying

$$V\psi_\alpha = \mu_\alpha\psi_\alpha. \tag{3.83a}$$

If the eigenvalues μ_α of the matrix V are simple then the functions $\psi_\alpha = (\psi_{i\alpha})$ can be found in quadratures of some rational differential forms on the V-space

$$\psi_\alpha = e^{-\sigma_\alpha}\phi_\alpha \tag{3.83b}$$

$$\sigma_\alpha = \int \sum_i \phi^T_{n-\alpha+1}\left(\partial_i + [E_i, \Gamma]\right)\phi_\alpha \, du^i \tag{3.83c}$$

where ϕ_α are eigenvectors of V with the eigenvalues μ_α normalized by the condition

$$\phi^T_\alpha \phi_\beta = \delta_{\alpha+\beta,n+1}. \tag{3.83d}$$

Then the formulae

$$\eta_{\alpha\beta} = \sum_i \psi_{i\alpha}\psi_{i\beta} \tag{3.84a}$$

$$\partial_i t_\alpha = \psi_{i1}\psi_{i\alpha} \tag{3.84b}$$

$$c_{\alpha\beta\gamma} = \sum_i \frac{\psi_{i\alpha}\psi_{i\beta}\psi_\gamma}{\psi_{i1}} \tag{3.84c}$$

determine locally a massive Frobenius manifold with the scaling dimensions

$$q_\alpha = \mu_\alpha - \mu_1, \quad d = -2\mu_1. \tag{3.84d}$$

Proof is straightforward.

Note that we obtain a Frobenius manifold of the second type (1.22) if the marked vector ψ_{i1} belongs to the kernel of V.

Remark 3.5. The construction of Proposition works also for Frobenius manifolds with nondiagonalizable matrices $\nabla_\alpha E^\beta$. They correspond to nondiagonalizable matrices $V(u)$. The reason of appearing of linear nonhomogeneous terms in the Euler vector field (when some of q_α is equal to 1) is more subtle. We will discuss it in terms of monodromy data below.

Remark 3.6. The change of the coordinates $(u^1, \ldots, u^n) \mapsto (t^1, \ldots, t^n)$ is not invertible in the points where one of the components of the vector-function $\psi_{i1}(u)$ vanishes.

Exercise 3.5. Prove the formula

$$V_{ij}(u) = \sum_{\alpha,\,\beta} \eta^{\alpha\beta}\mu_\alpha \psi_{i\alpha}(u)\psi_{j\beta}(u) \tag{3.85}$$

for the matrix $V(u)$ where $\psi_{i\alpha}(u)$ are given by the formula (3.81).

Due to (3.85) one can represent (3.71) as a closed system of differential equations for the vector functions $\psi_{i1}(u)$, ..., $\psi_{in}(u)$ putting

$$\gamma_{ij}(u) = \frac{V_{ij}(u)}{u_j - u_i}.$$

This is an alternative representation of the equations (3.70)

$$\partial_i\psi_{j\beta}(u) = \sum_{1\leq\alpha\leq\frac{n}{2}} \mu_\alpha \frac{\psi_{i\alpha}\psi_{j\,n-\alpha+1} - \psi_{j\alpha}\psi_{i\,n-\alpha+1}}{u_j - u_i}\psi_{i\beta}(u),\ i\neq j$$

$$\sum_i \partial_i\psi_{i\beta} = 0$$

for any $\beta = 1, \ldots, n$.

Remark 3.7. Forgetting the scaling invariance (1.9) one can still reduce, in the semisimple case the associativity equations (1.14) with the normalization (1.1) to the Darboux - Egoroff system (3.70a,b). The reduction is done by the formula (3.69). This is still an integrable system [47] (the linear system (3.118a,b) below gives a commutation representation for (3.70a,b)). General solution of the sysstem (3.70a,b) depends on $n(n-1)/2$ arbitrary functions of one variable. Conversely, any solution of the Darboux - Egoroff system generates an n-dimensional family of solutions of the equations (1.14), (1.1) by the formulae (3.84a-c). In these formulae $\psi_{i\alpha}(u)$, $\alpha = 1, \ldots, n$ must be arbitrary linearly independent solutions of the system (3.71).

In a similar way one can treat (still in the semisimple case) the associativity equation (1.14) *without* the normalization (1.1). In this case $(\eta^{\alpha\beta})$ in (1.14) is a given symmetric nondegenerate matrix (not related to the derivatives of $F(t)$). To construct solutions of (1.14) starting from a solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ of the Darboux - Egoroff system (3.70a,b) one needs to fix a solution $\psi_{i0}(u)$ of the linear subsystem (3.71a) and a basis $\psi_{i1}(u), \ldots, \psi_{in}(u)$ of solutions of the full linear system (3.71). Then the formulae (3.84a) and

$$t^\alpha = \eta^{\alpha\beta}t_\beta,\quad \partial_i t_\alpha = \psi_{i\alpha}\psi_{i0},\quad \partial_\alpha\partial_\beta\partial_\gamma F = \sum_i \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i0}}$$

determine a solution $F = F(t)$ of the equation (1.14).

Observe that the choice of the solution $\psi_{i0}(u)$ of (3.71a) depends on n arbitrary functions of one variable. The solution satisfies the semisimplicity condition. We leave as an exercise for the reader to prove that, conversely any solution of the associativity equation (1.14) satisfying the semisimplicity condition can be constructed in such a way. [Hint: due to Exercise 1.2 a semisimple algebra with the structure constants

$$c_{\alpha\beta}^\gamma(t) = \eta^{\gamma\epsilon}\partial_\epsilon\partial_\alpha\partial_\beta F(t)$$

has a unity element $e = (e^\alpha(t))$. Prove that the 1-form

$$\omega_\alpha(t) = \eta_{\alpha\beta}e^\beta(t)$$

is closed. So locally

$$\omega_\alpha(t) = \partial_\alpha v(t)$$

for some function $v(t)$. Put

$$\psi_{i0} := \sqrt{\partial_i v}$$

as a function of the canonical coordinates u to obtain the needed solution $\psi_{i0}(u)$ of (3.71a).]

From the construction it follows that a solution $\gamma_{ij}(u)$ of the system (3.70) determines n essentially different (up to an equivalence) solutions of WDVV. This comes from the freedom in the choice of the solution ψ_{i1} in the formulae (3.84). We will see now that these ambiguity is described by the transformations (B.2) (or (B.11), in the case of coincidences between the eigenvalues of V).

Definition 3.3. A 1-form σ on a massive Frobenius manifold M is called *admissible* if the new invariant metric

$$< a, b >_\sigma := \sigma(a \cdot b) \tag{3.86}$$

together with the old multiplication law of tangent vectors and with the old unity e and the old Euler vector field E determines on M a structure of Frobenius manifold with the same rotation coefficients $\gamma_{ij}(u)$.

For example, the 1-form

$$\sigma = dt_1$$

is an admissible one: it determines on M the given Frobenius structure.

Proposition 3.6. *All the admissible forms on a massive Frobenius manifold are*

$$\sigma_c(\ \cdot\ ) := \langle \left(\sum_k c^k \partial_{\kappa_k} \right)^2 ,\ \cdot\ \rangle \tag{3.87a}$$

for arbitary constants c^k and

$$\deg t^{\kappa_1} = \deg t^{\kappa_2} = \ldots . \tag{3.87b}$$

The form σ_c can be written also as follows

$$\sigma_c = \sum_{i,j} c^i c^j F_{\kappa_i \kappa_j \alpha} dt^\alpha . \tag{3.87c}$$

Proof. Flat coordinates $t^{\alpha'}$ for a Egoroff metric

$$< \ , \ >' = \sum_i \eta'_{ii}(u) du^i \tag{3.88}$$

with the given rotation coefficients $\gamma_{ij}(u)$ are determined by the system

$$\begin{aligned} \partial_i \psi'_{j\alpha} &= \gamma_{ij}(u) \psi'_{i\alpha}, \quad i \neq j \\ \sum_{i=1}^n \partial_i \psi'_{j\alpha} &= 0 \\ \partial_i t'_\alpha &= \sqrt{\eta'_{ii}(u)} \psi'_{i\alpha}. \end{aligned} \tag{3.89}$$

Particularly,

$$\psi'_{i1}(u) = \sqrt{\eta'_{ii}(u)}.$$

Also $\psi'_{i\alpha}(u)$ must be homogeneous functions of u. From (3.89) we conclude, as in Corollary 3.2 that they must be eigenvectors of the matrix $V(u)$. So we must have

$$\sqrt{\eta'_{ii}(u)} = \sum_k c^k \psi_{i\,\kappa_k}(u).$$

This gives (3.87). Reversing the calculations we obtain that the metric (3.87) is admissible. Proposition is proved.

From Propositions 3.5 and 3.6 we obtain

Corollary 3.3. *There exists a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Massive Frobenius manifolds} \\ \text{modulo transformations (B.11)} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{solutions of the system (3.74)} \\ \text{with diagonalizable } V(u) \end{array} \right\} \qquad (3.90)$$

Remark 3.8. Solutions of WDVV equations without semisimplicity assumption depend on functional parameters. Indeed, for nilpotent algebras the associativity conditions are very weak (eventually empty, see for example the solution (1.38)). However, it is possible to describe a closure of the class of massive Frobenius manifolds as the set of all Frobenius manifolds with n-dimensional commutative group of algebraic symmetries. Let A be a fixed n-dimensional Frobenius algebra with structure constants c_{ij}^k and an invariant inner nondegenerate inner product $\epsilon = (\epsilon_{ij})$. Let us introduce matrices

$$C_i = (c_{ij}^k). \qquad (3.91)$$

An analogue of the Darboux – Egoroff system (3.70) for an operator-valued function

$$\gamma(u) : A \to A,\ \gamma = (\gamma_i^j(u)),\ u = (u^1, \ldots, u^n) \qquad (3.92)$$

(an analogue of the rotation coefficients) where the operator γ is symmetric with respect to ϵ,

$$\epsilon\gamma = \gamma^{\mathrm{T}}\epsilon \qquad (3.93)$$

has the form

$$[C_i, \partial_j\gamma] - [C_j, \partial_i\gamma] + [[C_i, \gamma], [C_j, \gamma]] = 0,\ i, j = 1, \ldots, n, \qquad (3.94)$$

$\partial_i = \partial/\partial u^i$. This is an integrable system with the Lax representation

$$\partial_i\Psi = \Psi(zC_i + [C_i, \gamma]),\ i = 1, \ldots, n. \qquad (3.95)$$

It is convenient to consider $\Psi = (\psi_1(u), \ldots, \psi_n(u))$ as a function with values in the dual space A^*. Note that A^* also is a Frobenius algebra with the structure constants $c_k^{ij} = c_{ks}^i \epsilon^{sj}$ and the invariant inner product $< \, , \, >_*$ determined by $(\epsilon^{ij}) = (\epsilon_{ij})^{-1}$.

Let $\Psi_\alpha(u)$, $\alpha = 1, \dots ,n$ be a basis of solutions of (3.95) for $z = 0$

$$\partial_i \Psi_\alpha = \Psi_\alpha [C_i, \gamma],\ \alpha = 1, ..., n \tag{3.96a}$$

such that the vector $\Psi_1(u)$ is invertible in A^*. We put

$$\eta_{\alpha\beta} = < \Psi_\alpha(u), \Psi_\beta(u) >_* \tag{3.96b}$$

$$\mathrm{grad}_u t_\alpha = \Psi_\alpha(u) \cdot \Psi_1(u) \tag{3.96c}$$

$$c_{\alpha\beta\gamma}(t(u)) = \frac{\Psi_\alpha(u) \cdot \Psi_\beta(u) \cdot \Psi_\gamma(u)}{\Psi_1(u)}. \tag{3.96d}$$

Theorem 3.1. *Formulae (3.96) for arbitrary Frobenius algebra A localy parametrize all Frobenius manifolds with n-dimensional commutative group of algebraic symmetries.*

Considering u as a vector in A and $\Psi_1^2 = \Psi_1 \cdot \Psi_1$ as a linear function on A one obtains the following analogue of Egoroff metrics (on A)

$$ds^2 = \Psi_1^2 (du \cdot du). \tag{3.97}$$

Examples. We start with the simplest example $n = 2$. The equations (3.70) are linear in this case. They can be solved as

$$\gamma_{12}(u) = \gamma_{21}(u) = \frac{i\mu}{u^1 - u^2} \tag{3.98}$$

where $\pm\mu$ are the eigenvalues of the matrix $V(u)$ being constant in this case. The basis $\psi_{i\alpha}(u)$ of solutions of the system (3.71) has the form

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} r^\mu \\ ir^\mu \end{pmatrix},\quad \psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} r^{-\mu} \\ -ir^{-\mu} \end{pmatrix},\quad r = u^1 - u^2 \tag{3.99}$$

(we omit the inessential normalization constant). For $\mu \neq -1/2$ the flat coordinates are

$$t^1 = \frac{u^1 + u^2}{2},\quad t^2 = \frac{r^{2\mu+1}}{2(2\mu + 1)}. \tag{3.100}$$

We have

$$q_1 = 0,\ q_2 = -2\mu = d.$$

For $\mu \neq \pm 1/2,\ -3/2$ the function F has the form

$$F = \frac{1}{2}(t^1)^2 t^2 + c(t^2)^k$$

for

$$k = (2\mu + 3)/(2\mu + 1),\quad c = \frac{(1 + 2\mu)^3}{2(1 - 2\mu)(2\mu + 3)} [2(2\mu + 1)]^{-4\mu/(2\mu+1)}.$$

For $\mu = 1/2$ the function F has the form

$$F = \frac{1}{2}(t^1)^2 t^2 + \frac{(t^2)^2}{8}\left(\log t^2 - \frac{3}{2}\right).$$

For $\mu = -3/2$ we have

$$F = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{3 \cdot 2^7}\log t^2.$$

For $\mu = -1/2$ the flat coordinates are

$$t^1 = \frac{u^1 + u^2}{2}, \quad t^2 = \frac{1}{2}\log r. \tag{3.101}$$

The function F is

$$F = \frac{1}{2}(t^1)^2 t^2 + 2^{-6} e^{4t^2}.$$

For $n \geq 3$ the system (3.70) is non-linear. I will rewrite the first part of it, i.e. the Darboux - Egoroff equations (3.70a,b) in a more recognizable (for the theory of integrable systems) shape. Let us restrict the functions $\gamma_{ij}(u)$ onto the plane

$$u^i = a_i x + b_i t, \quad i = 1, \ldots, n$$

where the vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, and $(1, 1, \ldots, 1)$ are linearly independent. After the substitution we obtain the following matrix form of the system (3.70a,b)

$$\partial_t [A, \Gamma] - \partial_x [B, \Gamma] + [[A, \Gamma], [B, \Gamma]] = 0 \tag{3.102a}$$

where

$$A = \mathrm{diag}(a_1, \ldots, a_n), \quad B = \mathrm{diag}(b_1, \ldots, b_n), \quad \Gamma = (\gamma_{ij}) \tag{3.102b}$$

[,] stands for the commutator of $n \times n$ matrices. This is a particular reduction of the wellknown n-wave system [119] (let us forget at the moment that all the matrices in (3.102) are complex but not real or hermitean or etc.). 'Reduction' means that the matrices $[A, \Gamma]$, $[B, \Gamma]$ involved in (3.102) are skew-symmetric but not generic. I recall [56] that particularly, when the x-dependence drops down from (3.102), the system reduces to the equations of free rotations of a n-dimensional solid (the so-called *Euler - Arnold top* on the Lie algebra $so(n)$)

$$V_t = [\Omega, V], \tag{3.103a}$$

$$V = [A, \Gamma], \quad \Omega = [B, \Gamma], \tag{3.103b}$$

$$\Omega, \ V \in so(n), \quad \Omega = ad_B ad_A^{-1} V. \tag{3.104}$$

This is a hamiltonian system on $so(n)$ with the standard Lie - Poisson bracket [56] and with the quadratic Hamiltonian

$$H = -\frac{1}{2}\,\mathrm{tr}\,\Omega V = \frac{1}{2}\sum_{i<j}\frac{b_i - b_j}{a_i - a_j} V_{ij}^2 \tag{3.105}$$

(we identify $so(n)$ with the dual space using the Killing foorm).

The equation (3.70c) determines a scaling reduction of the n-wave system (3.102). It turns out that this has still the form of the Euler – Arnold top on $so(n)$ but the Hamiltonian depends explicitly on time.

Proposition 3.7. *The dependence(3.74) of the matrix $V(u)$ on u^i is determined by a hamiltonian system on $so(n)$ with a time-dependent quadratic Hamiltonian*

$$H_i = \frac{1}{2}\sum_{j\neq i} \frac{V_{ij}^2}{u^i - u^j}, \tag{3.106}$$

$$\partial_i V = \{V, H_i\} \equiv \left[V, ad_{E_i} ad_U^{-1} V\right] \tag{3.107}$$

for any $i = 1, \ldots, n$.

Proof follows from (3.74), (3.103).

The variables $u^1, \ldots, u^n$ play the role of the times for the pairwise commuting hamiltonian systems (3.106). From (3.107) one obtains

$$\begin{aligned} \sum_{i=1}^n \partial_i V = 0 \\ \sum_{i=1}^n u^i \partial_i V = 0. \end{aligned} \tag{3.108}$$

So there are only $n-2$ independent parameters among the “times” $u^1, \ldots, u^n$.

The systems (3.74) have “geometrical integrals” (i.e. the Casimirs of the Poisson bracket on $so(n)$). These are the Ad-invariant polynomials on $so(n)$. They are the symmetric combinations of the eigenvalues of the matrix $V(u)$. I recall that, due to Corollary 3.2 the values of these geometrical integrals are expressed via the scaling dimensions of the Frobenius manifold.

It turns out that the non-autonomic tops (3.106) are still integrable. But the integrability of them is more complicated than those for the equations (3.103): they can be integrated by the method of isomonodromic deformations (see below).

Example 3.2. $n = 3$. Take the Hamiltonian

$$H := (u_2 - u_1) H_3 = \frac{1}{2}\left[\frac{\Omega_1^2}{s-1} + \frac{\Omega_2^2}{s}\right] \tag{3.109}$$

where we put

$$\Omega_k(s) := -V_{ij}(u) \tag{3.110}$$

$$s = \frac{u^3 - u^1}{u^2 - u^1}, \tag{3.111}$$

and $(i,\ j,\ k)$ is an even permutation of $(1,\ 2,\ 3)$. The Poisson bracket on $so(3)$ has the form

$$\{\Omega_1, \Omega_2\} = \Omega_3,\ \{\Omega_2, \Omega_3\} = \Omega_1,\ \{\Omega_3, \Omega_1\} = \Omega_2. \tag{3.112}$$

The corresponding hamiltonian system reads

$$\begin{aligned}\frac{d\Omega_1}{ds} &= \frac{1}{s}\Omega_2\Omega_3\\ \frac{d\Omega_2}{ds} &= -\frac{1}{s-1}\Omega_1\Omega_3\\ \frac{d\Omega_3}{ds} &= \frac{1}{s(s-1)}\Omega_1\Omega_2.\end{aligned} \tag{3.113}$$

The system has an obvious first integral (the Casimir of (3.112))

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = -\mu^2. \tag{3.114}$$

The value of the Casimir can be expressed via the scaling dimension d. Indeed, the matrix $\Omega(u)$ (3.110) has the form

$$\Omega(u) = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}. \tag{3.115}$$

The eigenvalues of this matrix are 0 and $\pm R$ where R is defined in (3.114). From (3.84) we know that the eigenvalues are related to the scaling dimensions. For $n = 3$ the scaling dimensions in the problem can be expressed via one parameter d as

$$q_1 = 0,\ q_2 = \frac{d}{2},\ q_3 = d. \tag{3.116}$$

From this we obtain that

$$\mu = -\frac{d}{2}$$

(minus is chosen for convenience).

Proposition 3.8. *WDVV equations for $n = 3$ with the scaling dimensions (3.116) are equivalent to the following Painlevé-VI equation*

$$\begin{aligned} y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-s}\right)(y')^2 - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{y-s}\right)y' \\ + \frac{y(y-1)(y-s)}{s^2(s-1)^2}\left(\frac{(1+d)^2}{2} + \frac{s(s-1)}{2(y-s)^2}\right). \end{aligned} \tag{3.117}$$

Proof can be obtained using [65, 66]. In Appendix E we give another proof that will enable us to construct solutions of the Painlevé-VI starting from particular solutions of WDVV

For $n > 3$ the system (3.74) can be considered as a high-order analogue of the Painlevé-VI. To show this we introduce a commutation representation [47] of the Darboux - Egoroff system (3.70).

Lemma 3.10. *The equations (3.70a,b) are equivalent to the compatibility conditions of the following linear system of differential equations depending on the spectral parameter* z *for an auxiliary function* $\psi = (\psi_1(u,z), \dots, \psi_n(u,z))$

$$\partial_k \psi_i = \gamma_{ik}\psi_k, \quad i \neq k \tag{3.118a}$$

$$\sum_{k=1}^n \partial_k \psi_i = z\psi_i, \quad i = 1, \dots, n. \tag{3.118b}$$

Proof is in a straightforward calculation.

Exercise. Show that the formula

$$z^{\frac{d}{2}-1} \frac{\partial_i \tilde{t}(t(u), z)}{\sqrt{\eta_{ii}(u)}} = \psi_i(u,z) \tag{3.119}$$

establishes one-to-one correspondence between solutions of the linear systems (3.118) and (3.5).

We can say thus that (3.118) gives a gauge transformation of the "Lax pair" of WDVV to the "Lax pair" of the Darboux - Egoroff system (3.70a,b).

The equation (3.70c) specifies the similarity reduction of the system (3.70a,b). It is clear that this is equivalent to a system of ODE of the order $n(n-1)/2$. Indeed, for a given Cauchy data $\gamma_{ij}(u_0)$ we can uniquely find the solution $\gamma_{ij}(u)$ of the system (3.70). We will consider now this ODE system for arbitrary n in more details.

We introduce first a very useful differential operator in z with rational coefficients

$$\Lambda = \partial_z - U - \frac{1}{z}V(u). \tag{3.120}$$

Here

$$U = \mathrm{diag}(u^1, \dots, u^n) \tag{3.121}$$

the matrix $V(u)$ is defined by (3.72).

Proposition 3.9. *Equations (3.70) (or, equivalently, equations (3.74)) are equivalent to the compatibility conditions of the linear problem (3.118) with the differential equation in* z

$$\Lambda\psi = 0. \tag{3.122}$$

Proof is straightforward (cf. the proof of Corollary 3.2).

Observe that the equations (3.118), (3.122) are obtained by transformation (3.119) from the equations for the flat coordinates $\tilde{t}(t,z)$ of the perturbed connection $\left(\tilde{\nabla}_\alpha(z), \tilde{\nabla}_z\right)$ of the formm (3.3), (3.12).

Solutions $\psi(u,z) = (\psi_1(u,z), \dots, \psi_n(u,z))^T$ of the equation (3.122) for a fixed u are multivalued analytic functions in $\mathbf{C} \setminus z = 0 \cup z = \infty$. The multivaluedness is described by *monodromy* of the operator Λ. It turns out the parameters of monodromy of the operator Λ with the coefficients depending on u as on the parameters *do not depend on* u. So they are first integrals of the system (3.74). We will see that a part of the first integrals are the eigenvalues of the matrix $V(u)$ (I recall that these are expressed via the degrees of the variables t^α). But for $n \geq 3$ other integrals are not algebraic in $V(u)$.

Let us describe the monodromy of the operator in more details. We will fix some vector $u = (u^1, \ldots, u^n)$ with the only condition $u^i - u^j \neq 0$ for $i \neq j$. For the moment we will concentrate ourselves on the z-dependence of the solution of the differential equation (3.122) taking aside the dependence of it on u. The equation (3.122) has two singularities in the z-sphere $\mathbf{C} \cup \infty$: the regular singularity at $z = 0$ and the irregular one at $z = \infty$. The monodromy around the origin is defined as an invertible matrix M_0 such that

$$\Psi\left(z\, e^{2\pi i}\right) = \Psi(z)\, M_0.$$

Here $\Psi(z)$ is an invertible matrix solution of the equation (3.122) (the fundamental matrix). The monodromy matrix M_0 depends on the choice of the fundamental matrix $\Psi(z)$. Thus it is determined up to a similarity transformation

$$M_0 \mapsto T^{-1} M_0 T$$

with a nondegenerate T. The eigenvalues of M_0 are determined uniquelly by the eigenvalues $\mu_1, \ldots, \mu_n$ of the matrix V

$$\mathrm{eigen}\,(M_0) = e^{2\pi i \mu_1}, \ldots, e^{2\pi i \mu_n}. \tag{3.123}$$

Particularly, if the matrix V is diagonalizable and none of the differences $\mu_\alpha - \mu_\beta$ for $\alpha \neq \beta$ is an integer then a fundamental matrix $\Psi_0(z) = \left(\psi^0_{j\alpha}(z)\right)$ of (3.122) can be constructed such that

$$\psi^0_{j\alpha}(z) = z^{\mu_\alpha} \psi_{j\alpha}(1 + o(z)), \quad z \to 0. \tag{3.124}$$

Here α is the label of the solution of (3.122); the vectors $\psi_{j\alpha}$ for any $\alpha = 1, \ldots, n$ are the eigenvectors of the matrix V with the eigenvalues μ_α resp. The monodromy of the matrix around $z = 0$ is a diagonal matrix

$$\Psi^0(ze^{2\pi i}) = \Psi^0(z)\mathrm{diag}(e^{2\pi i \mu_1}, \ldots, e^{2\pi i \mu_n}). \tag{3.125}$$

Monodromy at $z = \infty$ (irregular singularity of the operator Λ) is specified by a $n \times n$ *Stokes matrix*. I recall here the definition of Stokes matrices adapted to the operators of the form (3.120).

One of the definitions of the Stokes matrix of the operator (3.120) is based on the theory of reduction of ODEs to a canonical form by analytic gauge transformations. Let us consider an operator

$$L = \frac{d}{dz} - A(z)$$

with an analytic at $z = \infty$ matrix valued function $A(z)$ satisfying

$$A(z) = U + O\left(\frac{1}{z}\right)$$
$$A^T(-z) = A(z).$$

The gauge transformations of L have the form

$$L \mapsto g^{-1}(z) L g(z) \tag{3.126a}$$

where the matrix valued function $g(z)$ is analytic near $z = \infty$ with

$$g(z) = 1 + O(z^{-1}) \tag{3.126b}$$

satisfying

$$g(z) g^T(-z) = 1. \tag{3.126c}$$

According to the idea of Birkhoff [19] any operator L can be reduced by transformations (3.126) to an operator of the form (3.120). The orbits of the gauge transformations (3.126) form a finite-dimensional family. The local coordinates in this family are determined by the Stokes matrix of the operator Λ.

To give a constructive definition of the Stokes matrix one should use the asymptotic analysis of solutions of the equation (3.122) near $z = \infty$.

Let us fix a vector of the parameters $(u^1, \dots, u^n)$ with $u^i \neq u^j$ for $i \neq j$. We define first *Stokes rays* in the complex z-plane. These are the rays R_{ij} defined for $i \neq j$ of the form

$$R_{ij} := \left\{ z \,|\, Re[z(u^i - u^j)] = 0,\ Re[e^{i\epsilon} z(u^i - u^j)] > 0 \text{ for a small } \epsilon > 0. \right\}. \tag{3.127}$$

The ray R_{ji} is opposite to R_{ij}.

Let l be an arbitrary oriented line in the complex z-plane passing through the origin containing no Stokes rays. It divides $\mathbf{C}$ into two half-planes $\mathbf{C}_{\text{right}}$ and $\mathbf{C}_{\text{left}}$. There exist [13, 152] two matrix-valued solutions $\Psi^{\text{right}}(z)$ and $\Psi^{\text{left}}(z)$ of (3.122) analytic in the half-planes $\mathbf{C}_{\text{right/left}}$ resp. with the asymptotic

$$\Psi^{\text{right/left}}(z) = \left(1 + O\left(\frac{1}{z}\right)\right) e^{zU} \text{ for } z \to \infty,\ z \in \mathbf{C}_{\text{right/left}}. \tag{3.128}$$

These functions can be analytically continued (preserving the asymptotics) into some sectorial neighbourhoods of the half-planes. On the intersections of the neighbourhoods of $\mathbf{C}_{\text{right/left}}$ the solutions $\Psi^{\text{right}}(z)$ and $\Psi^{\text{left}}(z)$ must be related by a linear transformation. To be more specific let the line l have the form

$$l = \{z = \rho e^{i\phi_0} | \rho \in \mathbf{R},\ \phi_0 \text{ is fixed}\} \tag{3.129}$$

with the natural orientation of the real ρ-line. The half-planes $\mathbf{C}_{\rm right/left}$ will be labelled in such a way that the vectors

$$\pm i e^{i\phi_0} \tag{3.130}$$

belong to $\mathbf{C}_{\rm left/right}$ resp. For the matrices $\Psi^{\rm right}(z)$ and $\Psi^{\rm left}(z)$ analytic in the sectors

$$-\phi_0 - \epsilon < \arg z < \phi_0 + \epsilon$$

and

$$\phi_0 - \epsilon < \arg z < -\phi_0 + \epsilon$$

resp. for a sufficiently small positive ϵ we obtain

$$\begin{aligned} \Psi^{\rm left}(\rho e^{i\phi_0}) &= \Psi^{\rm right}(\rho e^{i\phi_0}) S_+ \\ \Psi^{\rm left}(-\rho e^{i\phi_0}) &= \Psi^{\rm right}(-\rho e^{i\phi_0}) S_- \end{aligned} \tag{3.131}$$

$\rho > 0$ for some constant nondegenerate matrices S_+ and S_-. The boundary-value problem (3.131) together with the asymptotic (3.128) is a particular case of *Riemann – Hilbert* b.v.p.

Proposition 3.10.

1. *The matrices $S_\pm$ must have the form*

$$S_+ = S, \quad S_- = S^T \tag{3.132a}$$

$$S \equiv (s_{ij}), \quad s_{ii} = 1, \quad s_{ij} = 0 \text{ if } i \neq j \text{ and } R_{ij} \subset \mathbf{C}_{\rm right}. \tag{3.132b}$$

2. *The monodromy at the origin is similar to the matrix*

$$M := S^T S^{-1} \tag{3.133a}$$

$$M_0 = C^{-1} S^T S^{-1} C \tag{3.133b}$$

for a nondegenerate matrix C. Particularly, the eigenvalues of the matrix $S^T S^{-1}$ are

$$(e^{2\pi i \mu_1}, \ldots, e^{2\pi i \mu_n}) \tag{3.134}$$

where $\mu_1, \ldots, \mu_n$ is the spectrum of the matrix V. The solution $\Psi^0(z)$ (3.125) has the form

$$\Psi^0(z) = \Psi^{\rm right}(z)\, C. \tag{3.135}$$

Proof. We will show first that the restriction for the matrices S_+ and S_- follows from the skew-symmetry of the matrix V. Indeed, the skew-symmetry is equivalent to constancy of the natural inner product in the space of solutions of (3.122)

$$\Psi^T(-z)\Psi(z) = const. \tag{3.136}$$

Let us proof that for the piecewise-analytic function

$$\Psi(z) := \Psi^{\text{right/left}}(z) \tag{3.137}$$

the identity

$$\Psi(z)\Psi^T(-z) = 1 \tag{3.138}$$

follows from the restriction (3.132a). Indeed, for $\rho > 0$, $z = \rho e^{i(\phi_0+0)}$, the l.h.s. of (3.138) reads

$$\Psi^{\text{right}}(z)\Psi^{\text{left}^T}(-z) = \Psi^{\text{right}}(z)S\Psi^{\text{right}^T}(-z). \tag{3.139}$$

For $z = \rho^{i(\phi_0-0)}$ we obtain the same expression for the l.h.s. So the piecewise-analytic function $\Psi(z)\Psi^T(-z)$ has no jump on the semiaxis $z = \rho^{i\phi_0}$, $\rho > 0$. Neither it has a jump on the opposite semiaxis (it can be verified similarly). So the matrix-valued function is analytic in the whole complex z-plane. As $z \to \infty$ we have from the asymptotic (3.128) that this function tends to the unity matrix. The Liouville theorem implies (3.138). So the condition (3.132a) is sufficient for the skew-symmetry of the matrix V. Inverting the considerations we obtain also the necessity of the condition.

To prove the restrictions (3.132b) for the Stokes matrix we put

$$\Psi^{\text{right/left}}(z) =: \Phi^{\text{right/left}}(z) \exp zU. \tag{3.140}$$

For the boundary values of the matrix-valued functions $\Phi^{\text{right/left}}(z)$ on the line l we have the following relations

$$\Phi^{\text{left}}(\rho e^{i\phi_0}) = \Phi^{\text{right}}(\rho e^{i\phi_0})\,\tilde{S} \tag{3.141a}$$

$$\Phi^{\text{left}}(-\rho e^{i\phi_0}) = \Phi^{\text{right}}(-\rho e^{i\phi_0})\,\tilde{S}^T \tag{3.141b}$$

$$\Phi^{\text{right/left}}(z) = 1 + O\left(\frac{1}{z}\right), \quad z \to \infty. \tag{3.141c}$$

Here

$$\tilde{S} := e^{zU} S e^{-zU}. \tag{3.141d}$$

From the asymptotic (3.128) we conclude that the matrix $\tilde{S}$ must tend to 1 when $z = \rho e^{i\phi_0}$, $\rho \to +\infty$. This gives (3.132b).

To prove the second statement of the proposition it is enough to compare the monodromy around $z = 0$ of the solution Ψ^{right} analytically continued through the half-line l_+ and of Ψ^0. Proposition is proved.

Definition 3.4. The matrix S in (3.132) is called *Stokes matrix* of the operator (3.120). The matrix M_0 is called *monodromy about the origin* of the operator. The matrix C in (3.135) is called *(central) connection matrix* of the operator. The set

$$S,\ M_0,\ C,\ \mu_1, \dots, \mu_n$$

with the S matrix of the form (3.132) satisfying the constraints

$$C^{-1} S^T S^{-1} C = M_0 \tag{3.142a}$$

$$\text{eigen}\, M_0 = (\mu_1, \ldots, \mu_n) \tag{3.142b}$$

is called *monodromy data* of the operator (3.120).

Observe that precisely $n(n-1)/2$ off-diagonal entries of the Stokes matrix can be nonzero due to (3.132).

The Stokes matrix changes when the line l passes through a separating ray R_{ij}. We will describe these changes in Appendix F.

The connection matrix C is defined up to transformations

$$C \mapsto CQ, \quad \det Q \neq 0, \; QM_0 = M_0 Q. \tag{3.143}$$

These transformations change the solution Ψ_0 preserving unchanged the matrix V. Note that if the matrix $M = S^T S^{-1}$ has simple spectrum then the solution Ψ^0 is completely determined (up to a normalization of the columns) by the Stokes matrix S. Non-semisimplicity of the matrix M can happen when some of the differences between the eigenvalues of V is equal to an integer (see below).

The eigenvalues of the skew-symmetric complex matrix V in general are complex numbers. We will obtain below sufficient conditions for them to be real.

To reconstruct the operator (3.120) from the monodromy data one is to solve the following *Riemann - Hilbert* boundary value problem: to construct functions $\Phi_{\text{right}}(z)$ analytic for $z \in \mathbf{C}_{\text{right}}$ for $|z| \geq 1$, $\Phi_{\text{left}}(z)$ analytic for $z \in \mathbf{C}_{\text{left}}$ for $|z| \geq 1$ having

$$\Phi_{\text{right/left}}(z) = 1 + O\left(\frac{1}{z}\right) \text{ for } |z| \to \infty \tag{3.144}$$

and $\Phi_0(z)$ analytic in the unit circle $|z| \leq 1$ such that the boundary values of these functions are related by the following equations

$$\Phi_0(z) = \Phi_{\text{right}}(z) e^{zU} C z^{-L_0} \quad \text{for } z \in \mathbf{C}_{\text{right}},\ |z| = 1 \tag{3.145a}$$

$$\Phi_0(z) = \Phi_{\text{left}}(z) e^{zU} S^{-1} C z^{-L_0} \quad \text{for } z \in \mathbf{C}_{\text{left}},\ |z| = 1 \tag{3.145b}$$

$$\Phi_{\text{left}}(z) = \Phi_{\text{right}}(z) e^{zU} S e^{-zU} \quad \text{for } z \in l_+,\ |z| > 1 \tag{3.145c}$$

$$\Phi_{\text{left}}(z) = \Phi_{\text{right}}(z) e^{zU} S^T e^{-zU} \quad \text{for } z \in l_-,\ |z| > 1 \tag{3.145d}$$

where the matrix L_0 must satisfy the conditions

$$e^{2\pi i L_0} = M_0 \tag{3.145e}$$

$$\text{eigen}\, L_0 = (\mu_1, \ldots, \mu_n). \tag{3.145f}$$

Exercise 3.7. Prove that the asymptotic expansion of the solution of the Riemann - Hilbert problem (3.145) for $z \to \infty$ has the form

$$\Phi^{\text{right/left}}(z) = \left(1 + \frac{\Gamma}{z} + O\left(\frac{1}{z^2}\right)\right) \tag{3.146a}$$

where

$$V = [\Gamma, U] = \Phi_0(0) L_0 \Phi_0^{-1}(0) \tag{3.146b}$$

Proposition 3.11. *The Riemann - Hilbert problem (3.145) together with (3.146) determines a meromorphic function*

$$V = V\left(\hat{u}; S, M_0, C, \mu_1, \ldots, \mu_n\right). \tag{3.147a}$$

Here $\hat{u}$ is a point of the universal covering of the space

$$\mathbf{C}^n \setminus \mathrm{diag} = \{(u_1, \ldots, u_n) | u_i \neq u_j \text{ for } i \neq j\} \tag{3.147b}$$

The regular points of the function (3.147) correspond to the monodromy data and the vectors u for which the Riemann - Hilbert problem (3.145) has a unique solution. For any such a regular point the matrices $\Phi_{\mathrm{right/left}}$ and Φ_0 are invertible everywhere.

This is a consequence of the general theory of Riemann - Hilbert problem (see [109, 113]).

Let us assume that the Riemann - Hilbert problem (3.145) for given monodromy data has a unique solution for a given u. Since the solvability of the problem is an open property, we obtain for the given (S, M_0, C, μ) locally a well-defined skew-symmetric matrix-valued analytic function $V(u)$. We show now that this is a solution of the system (3.74). More precisely,

Proposition 3.12. *If the dependence on u of the matrix $V(u)$ of the coefficients of the system (3.122) is specified by the system (3.74) then the monodromy data do not depend on u. Conversely, if the u-dependence of $V(u)$ preserves the matrices S, M_0, C and the numbers μ_α unchanged then $V(u)$ satisfies the system (3.74).*

Proof. We prove first the second part of the proposition. For the piecewise-analytic function

$$\Psi(u,z) = \begin{cases} \Phi^{\mathrm{right/left}}(u,z) e^{zU} & \text{for } z \in \mathbf{C}_{\mathrm{right/left}},\ |z| > 1 \\ \Phi_0(u,z) z^{L_0} & \text{for } |z| < 1 \end{cases}$$

determined by the Riemann - Hilbert problem (3.145) the combination

$$\partial_i \Psi(u,z) \cdot \Psi^{-1}(u,z) \tag{3.148}$$

for any $i = 1, \ldots, n$ has no jumps on the line l neither on the unit circle. So it is analytic in the whole complex z-plane. From (3.144) we obtain that for $z \to \infty$

$$\partial_i \Psi(u,z) \cdot \Psi^{-1}(u,z) = z E_i - [E_i, \Gamma] + O(1/z)$$

and from the constancy of M_0, C it follows that (3.148) is analytic also at the origin. Applying the Liouville theorem we conclude that the solution of the Riemann - Hilbert problem satisfies the linear system

$$\partial_i \Psi(u,z) = (z E_i - [E_i, \Gamma(u)]) \Psi(u,z), \quad i = 1, \ldots, n. \tag{3.149a}$$

Furthermore, due to z-independence of the monodromy data the function $\Psi(cu, c^{-1}z)$ is a solution of the same Riemann - Hilbert problem. From the uniqueness we obtain that the solution satisfies also the condition

$$\left(z\frac{d}{dz} - \sum u^i \partial_i\right)\Psi(u,z) = 0. \tag{3.149b}$$

Compatibility of the equations (3.149a) reads

$$0 = (\partial_i\partial_j - \partial_j\partial_i)\Psi(u,z) \equiv ([E_j, \partial_i\Gamma] - [E_i, \partial_j\Gamma] + [[E_i,\Gamma],[E_j,\Gamma]])\,\Psi(u,z).$$

Since the matrix

$$([E_j, \partial_i\Gamma] - [E_i, \partial_j\Gamma] + [[E_i,\Gamma],[E_j,\Gamma]])$$

does not depend on z, we conclude that

$$([E_j, \partial_i\Gamma] - [E_i, \partial_j\Gamma] + [[E_i,\Gamma],[E_j,\Gamma]]) = 0.$$

This coincides with the equations (3.70a,b). From (3.149b) we obtain the scaling condition

$$\Gamma(cu) = c^{-1}\Gamma(u).$$

This gives the last equation (3.70c).

Conversely, if the matrix $\Gamma(u)$ satisfies the system (3.70) (or, equivalently, $V = [\Gamma, U]$ satisfies the system (3.74)) then the equations (3.149a) are compatible with the equation (3.149b). Hence for a solution of the Riemann - Hilbert problem the matrices

$$(\partial_i - zE_i + [E_i,\Gamma])\Psi$$

and

$$\left(z\frac{d}{dz} - \sum u^i \partial_i\right)\Psi$$

satisfy the same differential equation (3.122). Hence

$$(\partial_i - zE_i + [E_i,\Gamma])\Psi(u,z) = \Psi(u,z)\,T_i \tag{3.150a}$$

and

$$\left(z\frac{d}{dz} - \sum u^i \partial_i\right)\Psi(u,z) = \Psi(u,z)\,T \tag{3.150b}$$

for some matrices $T_i = T_i(u)$, $T = T(u)$. Comparing the expansions of the both sides of (3.150) at $z \to \infty$ we obtain that $T_i = T = 0$. So the solution of the Riemann - Hilbert problem satisfies the equations (3.149). From this immediately follows that the monodromy data do not depend on u. Proposition is proved.

According to the proposition the monodromy data of the operators (3.120) locally parametrize the solutions of the system (3.74). For generic Stokes matrices S all the monodromy data are locally uniquelly determined by the Stokes matrix due to the relations (3.142). Note that, due to the conditions (3.132) there are precisely $n(n-1)/2$ independent complex parameters in the Stokes matrix of the operator (3.120). These can be considered as the local coordinates on the space of solutions of the system (3.74).

Local meromorphicity of the function (3.147) in u claimed by Proposition 3.11 is usually referred to as to the *Painlevé property* of the system (3.74). We summarize it as the following

Corollary 3.4. *Any solution $V(u)$ of the system (3.74) is a single-valued meromorphic function on the universal covering of domain $u^i \neq u^j$ for all $i \neq j$ i.e., on* $CP^{n-1} \setminus$ diagonals.

Remark 3.9. One can obtain the equations (3.74) also as the equations of isomonodromy deformations of an operator with regular singularities

$$\frac{d\phi}{d\lambda} + \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}\phi = 0. \tag{3.151a}$$

The points $\lambda = u_1, \ldots, \lambda = u_n$ are the regular singularities of the coefficients. If $A_1 + \ldots + A_n \neq 0$ then $\lambda = \infty$ is also a regular singularity. The monodromy preserving deformations of (3.151a) were described by Schlesinger [138]. They can be represented in the form of compatibility conditions of (3.151a) with linear system

$$\partial_i \phi = \left(\frac{A_i}{\lambda - u_i} + B_i \right) \phi, \quad i = 1, \ldots, n \tag{3.151b}$$

for some matrices B_i. To represent (3.74) as a reduction of the Schlesinger equations one put

$$A_i = E_i V, \quad B_i = -ad_{E_i} ad_U^{-1} V. \tag{3.151c}$$

Observe that the hamiltonian structure (3.107) of the equations (3.74) is obtained by the reduction (3.151c) of the hamiltonian structure of general Schlesinger equations found in [84].

Doing the substitution

$$\phi = \Psi(u)\chi \tag{3.152}$$

where $\Psi(u) = (\psi_{i\alpha}(u))$ is the matrix of eigenvectors of $V(u)$ normalized in such a way that

$$\partial_i \Psi = B_i \Psi$$

(this coincides with (3.71)) we obtain an equivalent form of the equations (3.151)

$$\frac{d\chi}{d\lambda} = -\eta \Psi^T (\lambda - U)^{-1} \Psi \hat{\mu} \chi \tag{3.153a}$$

$$\partial_i \chi = \frac{\eta \Psi^T E_i \Psi}{\lambda - u_i} \hat{\mu} \chi. \tag{3.153b}$$

To obtain (3.151) from (3.120) and (3.122) we apply the following trick (essentially due to Poincaré and Birkhoff, see the textbook [81, Section 19.4]). Do (just formally) the inverse Laplace transform

$$\psi(z) = z \oint e^{\lambda z} \phi(\lambda)\, d\lambda. \tag{3.154}$$

Substituting to (3.122), (3.144a) and integrating by parts we obtain (3.151) and (3.152).

We will show now that non-semisimplicity of the matrix $M = S^T S^{-1}$ is just the reason of appearing of linear nonhomogeneous terms in the Euler vector field $E(t)$. We consider here only the simplest case of Frobenius manifolds with the pairwise distinct scaling dimensions satisfying the inequalities

$$0 \le q_\alpha < q_n = d \le 1. \tag{3.155}$$

Such Frobenius manifolds were called reduced in Appendix B. We showed that any massive Frobenius manifold can be reduced to the form (3.155) by the transformations of Appendix B. Later we will show that these transformations essentially do not change the Stokes matrix.

Proposition 3.13. *1). For the case $d < 1$ the matrix M is diagonalizable.*

2). For $d = 1$ the matrix M is diagonalizable iff *$E(t)$ is a linear homogeneous vector field.*

Proof. If $d < 1$ then all the numbers $e^{2\pi i\mu_1}, \ldots, e^{2\pi i\mu_n}$ are pairwise distinct (I recall that the scaling dimensions are assumed to be pairwise distinct). This gives diagonalizability of the matrix M.

If $d = 1$ then $\mu_1 = -\frac{1}{2}$, $\mu_n = \frac{1}{2}$. So $e^{2\pi i\mu_1} = e^{2\pi i\mu_n} = -1$, and the characteristic roots of the matrix M are not simple. To prove diagonalizability of the matrix M we are to construct a fundamental system of solutions of the equation $\Lambda\psi = 0$ of the form

$$\psi(u, z) = (\psi(u) + O(z)) z^\mu.$$

We will show that the linear nonhomogeneous terms in the Euler vector field give just the obstruction to construct such a fundamental system.

Lemma 3.11. *If the Euler vector field of a Frobenius manifold with $d = 1$ is*

$$E = \sum_{\alpha=1}^{n-1} (1 - q_\alpha) t^\alpha \partial_\alpha + r\partial_n$$

then a fundamental system of solutions $\psi_{i\alpha}(u, z)$ of the equation $\Lambda\psi = 0$ exists such that

$$\psi_{i\alpha}(u, z) = (\psi_{i\alpha}(u) + O(z))\, z^{\mu_\alpha}, \quad \alpha \neq 1 \tag{3.156a}$$

$$\psi_{i1}(u, z) = \frac{1}{\sqrt{z}} \left[\psi_{i1}(u) + rz \log z\, \psi_{in}(u) + O(z)\right] \tag{3.156b}$$

where $\psi_{i\alpha}(u)$ are defined in (3.81).

Proof. Existence of the solutions of the form (3.156a) is a standard fact (see, e.g., [81]). Let us look for a solution $\psi_{i1}(u,z)$ in the form

$$\psi_{i1}(u,z) = \psi^{(0)} + \psi^{(1)} + \dots$$

where

$$\psi^{(0)} = \frac{1}{\sqrt{z}}\psi_{i1}(u)$$

and the successive approximations are determined by the recursion relations

$$z\partial_z \psi^{(k+1)} = zU\psi^{(k)} + V\psi^{(k+1)}, \quad k = 1, 2, \dots.$$

Again, existence and convergence of the expansion is a standard fact of the theory of ODEs with a regular singularity in the presence of a resonance. Using the identities

$$u_i \psi_{i1}(u) = \sum_{\alpha=1}^{n} g^{n\alpha}(t)\psi_{i\alpha}(u) \tag{3.157}$$

and

$$g^{n\alpha} = \begin{cases} (1-q_\alpha)t^\alpha & \text{if } \alpha \neq n \\ r & \text{for } \alpha = n \end{cases}$$

we obtain the first correction in the form

$$\psi^{(1)} = \sqrt{z}\left(r \log z\, \psi_{in}(u) + \sum_{\alpha=1}^{n-1} t^\alpha \psi_{i\alpha}(u)\right).$$

The subsequent corrections are at least of order $\sqrt{z}$. Lemma is proved.

We conclude that the basis of solutions of (3.122) being also eigenvectors of the monodromy around $z = 0$ can be constructed *iff* $r = 0$. Proposition is proved.

Corollary 3.5. *Under the assumptions of Lemma 3.11 the monodromy matrix M_0 in the origin $z = 0$ w.r.t. the basis (3.156) has the following form*

$$(M_0)_{\alpha\alpha} = e^{2\pi i \mu_\alpha}, \tag{3.158a}$$

$$(M_0)_{n1} = -2\pi i r, \tag{3.158b}$$

other entries vanish.

We conclude that the linear nonhomogeneous terms in the Euler vector field possibly existing when some of the charges $q_\alpha = 1$ are not determined by the Stokes matrix but by the monodromy at the origin M_0.

Similar arguments allow us to compute the monodromy at the origin also in the case of more complicated resonances.

Example 3.3. *) Consider the Frobenius manifold M^n constructed from the quantum cohomologies of a compact Kähler manifold X of the complex dimension d (see Lecture 2 above). I recall that $n = \dim H^*(X)$ and the flat coordinates t^α are in 1-to-1 correspondence with the cocycles

$$\phi_\alpha \in H^{2q_\alpha}(X).$$

Due to the isomonodromicity property it is enough to compute the monodromy at the origin in the classicall limit

$$\begin{array}{l} t^\alpha \to 0 \text{ when } q_\alpha \neq 1 \\ t^\alpha \to -\infty \text{ for } q_\alpha = 1 \end{array}.$$

In the limit the operator of multiplication by the Euler vector field coincides with the operator

$$R : H^*(X) \to H^*(X)$$

of cohomological multiplication by the first Chern class of X. It shifts gradings in $H^*(X)$ by 1. From this we easily derive the identity

$$z^\mu R = R z^{\mu+1}$$

where, as above $\mu = \mathrm{diag}(\mu_1, \ldots, \mu_n)$, $\mu_\alpha = q_\alpha - d/2$. Using this identity we obtain the classical limit of the fundamental matrix of the system (3.122) in the form

$$\Psi_0 = z^R z^{\mu+1}. \tag{3.160}$$

So the monodromy of (3.122) around $z = 0$ is given by the matrix

$$M_0 = (-1)^d r^{2\pi i R}. \tag{3.161}$$

We are close now to formulate the precise statement about parametrization of solutions of WDVV by Stokes matrices of the operators (3.120).

Lemma 3.13. *For two equivalent Frobenius manifolds satisfying the semisimplicity condition the corresponding solutions $V(u) = (V_{ij}(u))$ of the system (3.74) are related by a permutation of coordinates*

$$(u^1, \ldots, u^n)^T \mapsto P(u^1, \ldots, u^n)^T, \tag{3.162a}$$

$$V(u) \mapsto \epsilon P^{-1} V(u) P \epsilon, \tag{3.162b}$$

P is the matrix of the permutation, ϵ is an arbitrary diagonal matrix with ± 1 diagonal entries.

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*) This example was inspired by the observation of Di Francesco and Itzykson (after Lemma 2 in Section 2.5 of [39]) of a relation between the monodromy at the origin for the CP^2 sigma-model and the classical cohomologies of CP^2.

Observe that the permutations act on the differential operators Λ as

$$\Lambda \mapsto \epsilon P^{-1} \Lambda P \epsilon. \tag{3.163}$$

The Stokes matrix S of the operator Λ changes as

$$S \mapsto P^{-1} \epsilon \, S \, \epsilon \, P. \tag{3.164}$$

Note that the Legendre-type transformations (B.2) change the operator Λ only as in (3.163). So the corresponding transformations of the Stokes matrix have the form (3.164).

Summarizing the considerations of this section, we obtain

Theorem 3.2. *There exists a local one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Massive Frobenius manifolds} \\ \text{modulo transformations (B.2)} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Stokes matrices of differential} \\ \text{operators } \Lambda \text{ modulo transformations (3.164)} \end{array} \right\}.$$

Definition 3.5. The Stokes matrix S of the operator (3.120) considered modulo the transformations (3.164) will be called *Stokes matrix of the Frobenius manifold.*

Remark 3.10. In the paper [29] Cecotti and Vafa found a physical interpretation of the matrix entries S_{ij} for a Landau - Ginsburg TFT as the algebraic numbers of solitons propagating between classical vacua. In this interpretation S always is an integer-valued matrix. Due to (3.134) they arrive thus at the problem of classification of integral matrices S such that all the eigenvalues of $S^T S^{-1}$ are unimodular. This is the main starting point in the programme of classification of $N = 2$ superconformal theories proposed in [29].

It is interesting that *the same* Stokes matrix appears, according to [29], in the Riemann - Hilbert problem of [51] specifying the Zamolodchikov (or $t\,t^*$) hermitean metric on these Frobenius manifolds.

At the end of this section we explain the sense of the transformations (B.13) of WDVV from the point of view of the operators (3.120).

Proposition 3.14. *The rotation coefficients $\gamma_{ij}(u)$ and $\hat{\gamma}_{ij}(u)$ of two Frobenius manifolds related by the inversion (B.11) are related by the formula*

$$\hat{\gamma}_{ij} = \gamma_{ij} - A_{ij} \tag{3.165a}$$

where

$$A_{ij} := \frac{\sqrt{\partial_i t_1 \partial_j t_1}}{t_1}. \tag{3.165b}$$

The solutions $\psi(u, z)$ and $\hat{\psi}(u, z)$ of the corresponding systems (3.122) are related by the gauge transformation

$$\psi = \left(1 + \frac{A}{z}\right) \hat{\psi} \tag{3.166}$$

for $A = (A_{ij})$.

We leave the proof of this statement as an exercise for the reader.

The gauge transformations of the form

$$\psi(z) \mapsto g(z)\psi(z)$$

with rational invertible matrix valued function $g(z)$ preserving the form of the operator Λ are called *Schlesinger transformations* of the operator [84]. They preserve unchanged the monodromy property of the operator. However, they change some of the eigenvalues of the matrix V by an integer (see (B.16)).

It can be proved that all *elementary* Schlesinger transformations of the system (3.122) where $g(z) = (1 + A\,z^{-1})$ are superpositions of the transformation (B.13) and of the Legendre-type transformations (B.2). These generate all the group of Schlesinger transformations of (3.120). This group is a group of symmetries of WDVV according to Appendix B.

To conclude this long lecture we will discuss briefly the reality conditions of the solutions of WDVV. We say that the Frobenius manifold is real if it admits an antiholomorphic automorphism $\tau : M \to M$. This means that in some coordinates on M the structure functions $c_{\alpha\beta}^{\gamma}(t)$ all are real. The scaling dimensions q_α also are to be real.

The antiinvolution τ could either preserve or permute the canonical coordinates $u^1(t)$, $\ldots$, $u^n(t)$. We consider here only the case when the canonical coordinates are τ-invariant near some real point $t \in M$, $\tau^* u_i = \bar{u}_i$, $i = 1, \ldots, n$.

Exercise 3.8. Prove that the canonical coordinates are real near a point $t \in M$ where the intersection form is definite positive. Prove that in this case for even n half of the canonical coordinates are positive and half of them are negative, while for odd n one obtains $(n+1)/2$ negative and $(n-1)/2$ positive canonical coordinates.

For real canonical coordinates the diagonal metric $\eta_{ii}(u)$ is real as well. We put

$$J_i := \operatorname{sign} \eta_{ii}(u), \quad i = 1, \ldots, n \tag{3.167}$$

near the point u under consideration. The matrix $\Gamma(u)$ of the rotation coefficients and, hence, the matrix $V(u)$ obeys the symmetry

$$\Gamma^\dagger = J\Gamma J, \quad V^\dagger = -JVJ \tag{3.168a}$$

where

$$J = \operatorname{diag}(J_1, \ldots, J_n). \tag{3.168b}$$

Here dagger stands for the hermitean conjugation.

Proposition 3.15. *If the coefficients of the operator Λ for real u satisfy the symmetry (3.168) then the Stokes matrix w.r.t. the line $l = \{\operatorname{Im} z = 0\}$ satisfies the equation*

$$\bar{S}JSJ = 1. \tag{3.169}$$

Conversely, if the Stokes matrix satisfies the equation (3.169) and the Riemann - Hilbert problem (3.145) has a unique solution for a given real u then the corresponding solution of the system (3.74) satisfies (3.168).

Here the bar denotes the complex conjugation of all the entries of S.

Proof. Let l be the real line on the z-plane. As in the proof of Proposition 3.10 we obtain that the equation (3.169) is equivalent to the equation

$$\Psi_{\text{right/left}}(u,z) = \left(\Psi_{\text{left/right}}(u,\bar{z})\right)^{\dagger}. \tag{3.170}$$

Proposition is proved.

To derive from (3.168) the reality of the Frobenius manifold we are to provide also reality of the Euler vector field. For this we need the eigenvalues of the matrix $M = S^T S^{-1}$ to be unimodular.

Lemma 3.14. *The eigenvalues λ of a matrix $M = S^T S^{-1}$ with the matrix S satisfying (3.169) are invariant w.r.t. the transformations*

$$\lambda \mapsto \lambda^{-1}, \quad \lambda \mapsto \bar{\lambda}. \tag{3.171}$$

Proof is obvious.

We conclude that for a generic matrix S satisfying (3.169) the collection of the eigenvalues must consist of:

1). Quadruples $\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ for a nonreal λ with $|\lambda| \neq 1$.

2). Pairs $\lambda, \bar{\lambda}$ for a nonreal λ with $|\lambda| = 1$.

3). Pairs λ, λ^{-1} for a real λ distinct from ± 1.

4). The point $\lambda = 1$ for the matrices of odd dimension.

All these types of configurations of eigenvalues are stable under small perturbations of S. Absence of the eigenvalues of the types 1 and 3 specifies an open domain in the space of all complex S-matrices.

Example 3.4. For $n = 3$ and $J = \operatorname{diag}(-1,-1,1)$ the matrices satisfying (3.169) are parametrized by 3 real numbers a, b, c as

$$S = \begin{pmatrix} 1 & ia & b + \frac{i}{2}ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.172}$$

The eigenvalues of $S^T S^{-1}$ are unimodular *iff*

$$0 \le b^2 + c^2 + \frac{1}{4}a^2c^2 - a^2 \le 4. \tag{3.173}$$

Appendix D.
Geometry of flat pencils of metrics.

Proposition D.1. *For a flat pencil of metrics a vector field $f = f^i \partial_i$ exists such that the difference tensor (3.34) and the metric g_1^{ij} have the form*

$$\Delta^{ijk} = \nabla_2^i \nabla_2^j f^k \tag{D.1a}$$

$$g_1^{ij} = \nabla_2^i f^j + \nabla_2^j f^i + c^{ij} \tag{D.1b}$$

for a symmetric tensor c^{ij} such that

$$\nabla_2 c^{ij} = 0. \tag{D.1c}$$

The vector field satisfies the equations

$$\Delta_s^{ij} \Delta_l^{sk} = \Delta_s^{ik} \Delta_l^{sj} \tag{D.2}$$

where

$$\Delta_k^{ij} := g_{2ks} \Delta^{sij} = \nabla_{2k} \nabla_2^i f^j,$$

and

$$(g_1^{is} g_2^{jt} - g_2^{is} g_1^{jt}) \nabla_{2s} \nabla_{2t} f^k = 0. \tag{D.3}$$

Conversely, for a flat metric g_2^{ij} and for a solution f of the system (D.2), (D.3) the metrics g_2^{ij} and (D.1b) form a flat pencil.

Proof. Let us assume that x^1, ..., x^n is the flat coordinate system for the metric g_2^{ij}. In these coordinates we have

$$\Gamma_{2k}^{ij} = 0, \ \Delta_k^{ij} := g_{2ks} \Delta^{sij} = \Gamma_{1k}^{ij}. \tag{D.4}$$

The equation $R_l^{ijk} = 0$ in these coordinates reads

$$\left(g_1^{is} + \lambda g_2^{is}\right) \left(\partial_l \Delta_s^{jk} - \partial_s \Delta_l^{jk}\right) + \Delta_s^{ij} \Delta_l^{sk} - \Delta_s^{ik} \Delta_l^{sj} = 0. \tag{D.5}$$

Vanishing of the linear in λ term provides existence of a tensor f^{ij} such that

$$\Delta_k^{ij} = \partial_k f^{ij}.$$

The rest part of (D.5) gives (D.2). Let us use now the condition of symmetry (3.27) of the connection $\Gamma_{1\,k}^{ij} + \lambda \Gamma_{2\,k}^{ij}$. In the coordinate system this reads

$$\left(g_1^{is} + \lambda g_2^{is}\right) \partial_s f^{jk} = \left(g_1^{js} + \lambda g_2^{js}\right) \partial_s f^{ik}. \tag{D.6}$$

Vanishing of the terms in (D.6) linear in λ provides existence of a vector field f such that

$$f^{ij} = g_2^{is} \partial_s f^j.$$

This implies (D.1a). The rest part of the equation (D.6) gives (D.3). The last equation (3.26) gives (D.1b) - (D.1c). The first part of the proposition is proved. The converse statement follows from the same equations.

Remark D.1. The theory of S.P.Novikov and the author establishes a one-to-one correspondence between flat contravariant metrics on a manifold M and Poisson brackets of hydrodynamic type on the loop space

$$L(M) := \{\text{smooth maps } S^1 \to M\}$$

with certain nondegeneracy conditions [57, 58]. For a flat metric $g^{ij}(x)$ and the corresponding contravariant connection ∇^i the Poisson bracket of two functionals of the form

$$I = I[x] = \frac{1}{2\pi}\int_0^{2\pi} P(s, x(s))\, ds, \quad J = J[x] = \frac{1}{2\pi}\int_0^{2\pi} Q(s, x(s))\, ds,$$

$x = (x^i(s))$, $x(s+2\pi) = x(s)$ is defined by the formula

$$\{I, J\} := \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I}{\delta x^i(s)}\nabla^i \frac{\delta J}{\delta x^j(s)}\, dx^j(s) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I}{\delta x^i(s)} g^{ij}(x) d_s \frac{\delta J}{\delta x^j(s)}. \qquad (D.7)$$

Here the variational derivative $\delta I/\delta x^i(s) \in T_*M|_{x=x(s)}$ is defined by the equality

$$I[x+\delta x] - I[x] = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I}{\delta x^i(s)}\delta x^i(s)\, ds + o(|\delta x|); \qquad (D.8)$$

$\delta J/\delta x^j(s)$ is defined by the same formula, $d_s := ds\frac{\partial}{\partial s}$. The Poisson bracket can be uniquely extended to all "good" functionals on the loop space by Leibnitz rule [57, 58]. Flat pencils of metrics correspond to compatible pairs of Poisson brackets of hydrodynamic type. By the definition, Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are called compatible if an arbitrary linear combination with constant coefficients

$$a\{\ ,\ \}_1 + b\{\ ,\ \}_2$$

again is a Poisson bracket. Compatible pairs of Poisson brackets are important in the theory of integrable systems [108].

The main source of flat pencils is provided by the following statement.

Lemma D.1. *If for a flat metric in some coordinate system x^1, ..., x^n both the components $g^{ij}(x)$ of the metric and $\Gamma^{ij}_k(x)$ of the corresponding Levi-Civita connection depend linearly on the coordinate x^1 then the metrics*

$$g_1^{ij} := g^{ij} \text{ and } g_2^{ij} := \partial_1 g^{ij} \qquad (D.9)$$

form a flat pencil assuming that $\det(g_2^{ij}) \neq 0$. The corresponding Levi-Civita connections have the form

$$\Gamma^{ij}_{1k} := \Gamma^{ij}_k,\ \Gamma^{ij}_{2k} := \partial_1 \Gamma^{ij}_k. \qquad (D.10)$$

Proof. The equations (3.26), (3.27) and the equation of vanishing of the curvature have constant coefficients. Hence the transformation

$$g^{ij}(x^1, ..., x^n) \mapsto g^{ij}(x^1 + \lambda, ..., x^n), \; \Gamma^{ij}_k(x^1, ..., x^n) \mapsto \Gamma^{ij}_k(x^1 + \lambda, ..., x^n)$$

for an arbitrary λ maps the solutions of these equations to the solutions. By the assumption we have

$$g^{ij}(x^1 + \lambda, ..., x^n) = g^{ij}_1(x) + \lambda g^{ij}_2(x), \; \Gamma^{ij}_k(x^1 + \lambda, ..., x^n) = \Gamma^{ij}_{1k}(x) + \lambda \Gamma^{ij}_{2k}(x).$$

The lemma is proved.

All the above considerations can be applied also to complex (analytic) manifolds where the metrics are nondegenerate quadratic forms analyticaly depending on the point of M.

Appendix E.
WDVV and Painlevé-VI.

To reduce WDVV for $n = 3$ to a particular form of the Painlevé-VI equation we will use the commutation representation (3.153). For simplicity of the derivation we will assume the matrix V to be diagonalizable (although the result holds true without this assumption).

Since the matrix $\hat{\mu} = \text{diag}(\mu, 0, -\mu)$ where $\mu = -d/2$ has a zero eigenvalue, the component χ_2 of the vector-function $\chi = (\chi_1, \chi_2, \chi_3)^T$ drops from the r.h.s. of the system (3.153). For the vector $\tilde{\chi} := (\chi_1, \chi_3)^T$ we obtain a closed system

$$\frac{d\tilde{\chi}}{d\lambda} = -\mu \left[\frac{A_1}{\lambda - u_1} + \frac{A_2}{\lambda - u_2} + \frac{A_3}{\lambda - u_3} \right] \tilde{\chi} \equiv A(\lambda)\tilde{\chi} \qquad (E.1a)$$

$$\partial_i \tilde{\chi} = \mu \frac{A_i}{\lambda - u_i} \tilde{\chi}, \quad i = 1,\, 2,\, 3 \qquad (E.1b)$$

where the matrices A_i have the form

$$A_i = \begin{pmatrix} \psi_{i1}\psi_{i3} & -\psi_{i3}^2 \\ \psi_{i1}^2 & -\psi_{i1}\psi_{i3} \end{pmatrix}. \qquad (E.2)$$

From the definition of the intersection form (3.13) we obtain for the matrix $A(\lambda)$ the formula

$$A(\lambda) = \mu \begin{pmatrix} g_{13}(t,\lambda) & -g_{33}(t,\lambda) \\ g_{11}(t,\lambda) & -g_{13}(t,\lambda) \end{pmatrix} \qquad (E.3)$$

where the matrix $g_{\alpha\beta}(t,\lambda)$ is the inverse to the matrix $\left(g^{\alpha\beta}(t) - \lambda\, \eta^{\alpha\beta}\right)$ and $g^{\alpha\beta}(t)$ has the form (3.15). Note that the component χ_2 can be found by quadratures

$$\frac{d\chi_2}{d\lambda} = \mu \left[g_{12}\chi_1 - g_{23}\chi_3 \right]$$
$$\partial_i \chi_2 = \frac{\mu}{\lambda - u_i} \left[\psi_{i1}\psi_{i2}\chi_1 - \psi_{i2}\psi_{i3}\chi_3 \right].$$

The matrices A_i satisfy the equation

$$\sum_i A_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (E.4)$$

following from the normalization $\eta_{\alpha\beta} = \delta_{\alpha+\beta,4}$ of the t-coordinates. η-orthogonal changes of the t-coordinates determine simultaneous conjugations of the matrices A_i

$$A_i \mapsto K^{-1} A_i K, \quad K = \text{diag}(k_1, k_2). \qquad (E.5)$$

Let us introduce (essentially following [84]) coordinates p, q on the space of matrices A_i satisfying (E.4) modulo transformations (E.5) in the following way: q is the root λ of the equation

$$\left[A(\lambda)\right]_{12} = 0, \qquad (E.6)$$

and

$$p = [A(q)]_{11}\,. \tag{E.7}$$

Explicitly

$$\begin{aligned} q &= \left(g^{11}g^{22} - {g^{12}}^2\right)/g^{11} \\ p &= \mu \frac{g^{11}g^{22}}{{g^{12}}^3 + g^{11}g^{12}g^{13} - g^{11}g^{12}g^{22} - {g^{11}}^2 g^{23}} \end{aligned} \tag{E.8}$$

The entries of the matrices A_i can be expressed via the coordinates p, q and an auxilliary parameter k as follows

$$\psi_{i1}\psi_{i3} = -\frac{q-u_i}{2\mu^2 P'(u_i)}\left[P(q)p^2 + 2\mu\frac{P(q)}{q-u_i}p + \mu^2(q + 2u_i - \sum u_j)\right] \tag{E.9a}$$

$$\psi_{i3}^2 = -k\frac{q-u_i}{P'(u_i)} \tag{E.9b}$$

$$\psi_{i1}^2 = -k^{-1}\frac{q-u_i}{4\mu^4 P'(u_i)}\left[P(q)p^2 + 2\mu\frac{P(q)}{q-u_i}p + \mu^2(q + 2u_i - \sum u_j)\right]^2 \tag{E.9c}$$

where the polynomial $P(\lambda)$ has the form

$$P(\lambda) := (\lambda - u_1)(\lambda - u_2)(\lambda - u_3). \tag{E.10}$$

Substituting these formulae to the equations

$$\partial_i A(\lambda) = \frac{A_i}{(\lambda - u_i)^2} + \frac{[A(\lambda), A_i]}{\lambda - u_i} \tag{E.11}$$

of compatibility of the system (E.1) we obtain a closed system of equations for the functions p and q

$$\begin{aligned} \partial_i q &= \frac{P(q)}{P'(u_i)}\left[2p + \frac{1}{q-u_i}\right] \\ \partial_i p &= -\frac{P'(q)p^2 + (2q + u_i - \sum u_j)\,p + \mu(1-\mu)}{P'(u_i)} \end{aligned} \tag{E.12}$$

and a quadrature for the function $\log k$

$$\partial_i \log k = (2\mu - 1)\frac{q-u_i}{P'(u_i)}. \tag{E.13}$$

Eliminating p from the system we obtain a second order differential equation for the function $q = q(u_1, u_2, u_3)$

$$\partial_i^2 q = \frac{1}{2}\frac{P'(q)}{P(q)}\left(\partial_i q\right)^2 - \left[\frac{1}{2}\frac{P''(u_i)}{P'(u_i)} + \frac{1}{q-u_i}\right]\partial_i q$$

$$+\frac{1}{2}\frac{P(q)}{\left(P'(u_i)\right)^2}\left[(2\mu-1)^2+\frac{P'(u_i)}{(q-u_i)^2}\right], \quad i=1,\,2,\,3. \tag{E.14}$$

The system (E.14) is invariant w.r.t. transformations of the form

$$u_i \mapsto a u_i + b$$
$$q \mapsto a u_i + b.$$

Introducing the invariant variables

$$x=\frac{u_3-u_1}{u_2-u_1}$$
$$y=\frac{q}{u_2-u_1}-\frac{u_1}{u_2-u_1}$$

we obtain for the function $y=y(x)$ the following particular Painlevé-VI equation

$$y''=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right](y')^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]y'$$
$$+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2+\frac{x(x-1)}{(y-x)^2}\right]. \tag{E.15}$$

Conversely, for a solution $y(x)$ of the equation (E.15) we construct functions $q = q(u_1,u_2,u_3)$ and $p=p(u_1,u_2,u_3)$ putting

$$\begin{aligned} q &= (u_2-u_1)y\left(\frac{u_3-u_1}{u_2-u_1}\right)+u_1 \\ p &= \frac{1}{2}\frac{P'(u_3)}{P(q)}y'\left(\frac{u_3-u_1}{u_2-u_1}\right)-\frac{1}{2}\frac{1}{q-u_3}. \end{aligned} \tag{E.16}$$

Then we compute the quadrature (E.13) determining the function k (this provides us with one more arbitrary integration constant). After this we are able to compute the matrix $(\psi_{i\alpha}(u))$ from the equations (E.9) and

$$(\psi_{12},\psi_{22},\psi_{32})=\pm i\left(\psi_{21}\psi_{33}-\psi_{23}\psi_{31},\psi_{13}\psi_{31}-\psi_{11}\psi_{33},\psi_{11}\psi_{23}-\psi_{13}\psi_{21}\right). \tag{E.17}$$

The last step is in reconstructing the flat coordinates $t=t(u)$ and the tensor $c_{\alpha\beta\gamma}$ using the formulae (3.84).

We will show now that the hamiltonian structure of the Painlevé-VI is inherited from the $so(3)$-hamiltonian structure of the time-dependent Euler equations (3.113). We observe first that the squares of the matrix elements $V_{ij}=\mu(\psi_{i1}\psi_{j3}-\psi_{i3}\psi_{j1})$ can be expressed via p and q using identity

$$V_{ij}^2=-\mu^2\mathrm{tr}A_iA_j. \tag{E.18}$$

This gives

$$V_{ij}^2=-\frac{(p(q-u_k)+\mu)^2P(q)}{(q-u_k)P'(u_k)} \tag{E.19}$$

where i, j, k are three distinct indices from 1 to 3. The matrix elements V_{ij} can be uniquely reconstructed from (E.19) and from the following requirements: 1) the vector (V_{23}, V_{31}, V_{12}) must be proportional to (E.17) and 2) the identity

$$V_{12}V_{23}V_{31} = p^3 \frac{P(q)P\left(q + \frac{\mu}{p}\right)}{(u_1 - u_2)(u_2 - u_3)(u_3 - u_1)} \tag{E.20}$$

holds true (this specifies the common sign of the entries V_{ij}). It is easy to verify now that the transformation

$$V_{23},\, V_{31},\, V_{12} \mapsto q,\, p,\, \mu \tag{E.21}$$

transforms the standard $so(3)$-Lie-Poisson brackets for the entries V_{ij} to the canonical brackets for q and p commuting with the Casimir $\mu = \sqrt{-(V_{23}^2 + V_{31}^2 + V_{12}^2)}$. Observe that the change of coordinates (E.8) depends explicitly on the times u_1, u_2, u_3. Due to this dependence the hamiltonians of the systems (E.12) *are not* obtained by the reduction of the hamiltonians (3.106) on the spheres (3.114).

To obtain the hamiltonians of the equations (E.12) from the quadratic hamiltonians of the Euler equations (3.113) we use the following elementary statement.

Lemma E.1. *Let*

$$\frac{dy^i}{dt} = \left\{y^i, H(y,t)\right\} \tag{E.22}$$

be a hamiltonian system on a Poisson manifold with Poisson brackets $\{\ ,\ \}$ *and*

$$y = \varphi(x,t) \tag{E.23}$$

a local diffeomorphism depending explicitly on time t *Let the vector field* $\partial_t \varphi$ *be a hamiltonian one with the hamiltonian* $-\delta H$. *Then (E.22) is a hamiltonian system also in the* x*-coordinates*

$$\frac{dx^i}{dt} = \left\{x^i, \hat{H}(x,t)\right\} \tag{E.24a}$$

with the hamiltonian

$$\hat{H}(x,t) = H(\varphi(x,t),t) + \delta H(\varphi(x,t),t), \tag{E.24b}$$

Proof is obvious.

Applying this procedure to the transformation (E.8) we obtain, after some calculations

$$\delta H_i = \frac{p\frac{P(q)}{q-u_i} + \mu(q - u_i)}{P'(u_i)} \tag{E.25}$$

and the restriction of the hamiltonians (3.106) on the spheres (3.114) we obtain easyly using (E.19)

$$H_i = \frac{p^2 P(q) + \mu^2 (q - u_i)}{P'(u_i)}. \tag{E.26}$$

Using the reduced hamiltonians $H_i+\delta H_i$ we obtain the hamiltonian of the equation (E.15)

$$\hat{H} = \frac{y(y-1)(y-x)p^2 + y(y-1)p + \mu(1-\mu)(y-x)}{x(x-1)}. \qquad (E.27)$$

Applying this formalism to the polynomial solutions of Appendix A we obtain three algebraic solutions of Painlevé-VI as follows.

Intersection form of the type A_3 (the solution (A.7))

$$\begin{pmatrix} \frac{3}{4}\,{t_2}^2+\frac{1}{2}{t_3}^3 & \frac{5}{4}\,t_2\,t_3 & t_1 \\ \frac{5}{4}\,t_2\,t_3 & t_1+\frac{1}{2}{t_3}^2 & \frac{3}{4}\,t_2 \\ t_1 & \frac{3}{4}\,t_2 & \frac{1}{2}t_3 \end{pmatrix}. \qquad (E.28)$$

Characteristic polynomial

$$\det\left(g^{\alpha\beta}(t)-u\eta^{\alpha\beta}\right) = u^3+u^2\left(-3\,t_1-\frac{1}{2}{t_3}^2\right)$$

$$+u\left(3\,{t_1}^2-\frac{9}{4}\,{t_2}^2\,t_3+t_1\,{t_3}^2-\frac{1}{4}{t_3}^4\right)$$

$$-{t_1}^3-\frac{27}{64}\,{t_2}^4+\frac{9}{4}\,t_1\,{t_2}^2\,t_3-\frac{1}{2}{t_1}^2\,{t_3}^2-\frac{7}{8}\,{t_2}^2\,{t_3}^3+\frac{1}{4}t_1\,{t_3}^4+\frac{1}{8}{t_3}^6.$$

We obtain the following solution of (E.15) with $\mu=-1/4$ represented in the parametric form

$$y = \frac{1}{\omega_2-\omega_1}\left[\frac{-19t+4}{6t+4}-\omega_1\right] \qquad x = \frac{\omega_3-\omega_1}{\omega_2-\omega_1} \qquad (E.29a)$$

where ω_1, ω_2, ω_3 are the roots of the cubic equation

$$\omega^3-\omega^2-(9t+1)\omega-\frac{27}{8}t^2-7t+1=0. \qquad (E.29b)$$

Intersection form of the type B_3 (the solution (A.8))

$$\begin{pmatrix} \frac{5}{3}\,{t_2}^2\,t_3+\frac{1}{3}{t_3}^5 & \frac{2}{3}\,{t_2}^2+\frac{4}{3}\,t_2\,{t_3}^2 & t_1 \\ \frac{2}{3}\,{t_2}^2+\frac{4}{3}\,t_2\,{t_3}^2 & t_1+t_2\,t_3+\frac{1}{3}{t_3}^3 & \frac{2}{3}\,t_2 \\ t_1 & \frac{2}{3}\,t_2 & \frac{1}{3}t_3 \end{pmatrix}. \qquad (E.30)$$

Characteristic polynomial

$$\det\left(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}\right) = u^3 + u^2\left(-3\,t_1 - t_2\,t_3 - \frac{1}{3}{t_3}^3\right)$$

$$+u\left(3\,{t_1}^2 - \frac{8}{9}\,{t_2}^3 + 2\,t_1\,t_2\,t_3 - \frac{7}{3}\,{t_2}^2\,{t_3}^2 + \frac{2}{3}\,t_1\,{t_3}^3 - \frac{1}{9}{t_3}^6\right)$$

$$-{t_1}^3 + \frac{8}{9}\,t_1\,{t_2}^3 - {t_1}^2\,t_2\,t_3 - \frac{8}{9}\,{t_2}^4\,t_3 + \frac{7}{3}\,t_1\,{t_2}^2\,{t_3}^2 - \frac{1}{3}{t_1}^2\,{t_3}^3$$

$$-\frac{1}{27}{t_2}^3\,{t_3}^3 - \frac{5}{9}\,{t_2}^2\,{t_3}^5 + \frac{1}{9}t_1\,{t_3}^6 + \frac{1}{9}t_2\,{t_3}^7 + \frac{1}{27}{t_3}^9.$$

This gives a solution of (E.15) with $\mu = -1/3$ of the form

$$\begin{aligned} y &= \frac{(2-s)(s+1)(s^2-3)^2}{(2+s)(5s^4-10s^2+9)} \\ x &= \frac{(2-s)^2(1+s)}{(2+s)^2(1-s)}. \end{aligned} \tag{E.31}$$

Intersection form of the type H_3 (the solution (A.9))

$$\begin{pmatrix} \frac{3}{5}\,{t_2}^3 + \frac{9}{5}\,{t_2}^2\,{t_3}^3 + \frac{1}{20}{t_3}^9 & \frac{7}{5}\,{t_2}^2\,t_3 + \frac{7}{10}\,t_2\,{t_3}^4 & t_1 \\ \frac{7}{5}\,{t_2}^2\,t_3 + \frac{7}{10}\,t_2\,{t_3}^4 & t_1 + t_2\,{t_3}^2 + \frac{1}{10}{t_3}^5 & \frac{3}{5}\,t_2 \\ t_1 & \frac{3}{5}\,t_2 & \frac{1}{5}t_3 \end{pmatrix}. \tag{E.32}$$

Characteristic polynomial

$$\det\left(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}\right) = u^3 + u^2\left(-3\,t_1 - t_2\,{t_3}^2 - \frac{1}{10}{t_3}^5\right)$$

$$+u\left(3\,{t_1}^2 - \frac{9}{5}\,{t_2}^3\,t_3 + 2\,t_1\,t_2\,{t_3}^2 - \frac{6}{5}\,{t_2}^2\,{t_3}^4 + \frac{1}{5}t_1\,{t_3}^5 - \frac{1}{100}{t_3}^{10}\right)$$

$$-{t_1}^3 - \frac{27}{125}\,{t_2}^5 + \frac{9}{5}\,t_1\,{t_2}^3\,t_3 - {t_1}^2\,t_2\,{t_3}^2 - \frac{23}{25}\,{t_2}^4\,{t_3}^3 + \frac{6}{5}\,t_1\,{t_2}^2\,{t_3}^4$$

$$-\frac{1}{10}{t_1}^2\,{t_3}^5 - \frac{1}{50}{t_2}^3\,{t_3}^6 - \frac{2}{25}\,{t_2}^2\,{t_3}^9 + \frac{1}{100}t_1\,{t_3}^{10} + \frac{1}{100}t_2\,{t_3}^{12} + \frac{1}{1000}{t_3}^{15}.$$

We obtain the following solution of (E.15) with $\mu = -2/5$

$$\begin{aligned} y &= \frac{1}{\omega_2 - \omega_1}\left[\frac{-272t^4 - 20t^3 - 62t^2 + 10t + 1}{12t^3 + 36t^2 + 1} - \omega_1\right] \\ x &= \frac{\omega_3 - \omega_1}{\omega_2 - \omega_1} \end{aligned} \tag{E.33a}$$

$$\omega^3-(10t+1)\omega^2-(180t^3+120t^2+1)\omega-216t^5-920t^4-20t^3-80t^2+10t+1=0. \quad (E.33b)$$

Intersection form of the type $\tilde{W}(A_2)$ (the solution (A.13)) is given in the formula (5.85). Although this is not an algebraic solution of WDVV the corresponding solution of the Painlevé-VI (for $\mu=-\frac{1}{2}$) is algebraic. It has the form

$$\begin{aligned} y &= \frac{1}{\omega_1-\omega_2}\left[\frac{9+4t^3}{8t}+\omega_1\right] \\ x &= \frac{\omega_3-\omega_1}{\omega_2-\omega_1} \end{aligned} \qquad (E.34a)$$

where ω_i are the roots of the equation

$$\omega^3+\frac{1}{2}t^2\omega^2-\frac{9}{2}t\omega-\frac{1}{8}(27+16t^3)=0. \qquad (E.34b)$$

Intersection form of the type $\tilde{W}(B_2)$ (the solution (A.14)) reads as follows

$$\left(g^{\alpha\beta}(t)\right)=\begin{pmatrix} \frac{1}{4}e^{2t_3}+\frac{1}{2}t_2^2e^{t_3} & \frac{3}{4}t_2e^{t_3} & t_1 \\ \frac{3}{4}t_2e^{t_3} & t_1-\frac{1}{4}t_2^2+\frac{1}{2}e^{t_3} & \frac{1}{2}t_2 \\ t_1 & \frac{1}{2}t_2 & 1 \end{pmatrix}. \qquad (E.35)$$

It gives the following algebraic solution of the Painlevé-VI (again for $\mu=-\frac{1}{2}$)

$$\begin{aligned} y &= \frac{(1+s)^2(s+2)}{4(1+2s^2)} \\ x &= \frac{(s+2)^2}{8s}. \end{aligned} \qquad (E.36)$$

Intersection form of the type $\tilde{W}(G_2)$ (the solution (A.15)) is given by the matrix

$$\begin{pmatrix} 2\left(9\,e^{4\,t_3}+6\,e^{2\,t_3}\,{t_2}^2+\frac{2}{3}\,e^{t_3}\,{t_2}^3\right) & \frac{3}{2}\left(6\,e^{2\,t_3}\,t_2+2\,e^{t_3}\,{t_2}^2\right) & t_1 \\ \frac{3}{2}\left(6\,e^{2\,t_3}\,t_2+2\,e^{t_3}\,{t_2}^2\right) & 3\,e^{2\,t_3}+t_1+4\,e^{t_3}\,t_2-\frac{1}{6}{t_2}^2 & \frac{1}{2}t_2 \\ t_1 & \frac{1}{2}t_2 & \frac{1}{2} \end{pmatrix}$$

The solution of Painlevé-VI equation with $\mu=-1/2$ has the form

$$\begin{aligned} y &= \frac{3(3-t)(t+1)(t^2-3)^2}{(3+t)^2(t^6+3t^4-9t^2+9)} \\ x &= \frac{(3-t)^3(1+t)}{(1-t)(3+t)^3}. \end{aligned} \qquad (E.37)$$

An approach to classification of particular solutions of Painlevé-VI equation that can be expressed via classical transcendental functions was proposed by Okamoto [121]. This is based on the theory of canonical transformations of Painlevé-VI as of a time-dependent Hamiltonian system. For our particular Painlevé-VI for a non-integer d the solutions of Okamoto have the form

$$q \equiv u_i$$

for some i, p can be found from the equations (E.12). Particularly, for $q \equiv u_3$ the solution can be expressed via hypergeometric functions

$$p = \frac{1}{u_2 - u_1} \frac{d}{dx} \log f(\mu, 1-\mu, 1; x) \tag{E.38a}$$

where $f(\mu, 1-\mu, 1; x)$ stands for the general solution of the hypergeometric equation

$$x(1-x)f'' + (1-2x)f' - \mu(1-\mu)f = 0 \tag{E.38b}$$

Also for $\mu = 0$ or $\mu = 1$ there is a particular solution of [121] of the form $p \equiv 0$. This gives a rational solution of (3.74) of the form

$$V_{ij} = \frac{a_i a_j (u_i - u_j)}{\sum a_i^2 u_i} \tag{E.39a}$$

when the constants a_i must satisfy the constraint

$$\sum_{i=1}^{n} a_i^2 = 0 \tag{E.39b}$$

(this is a solution of (3.74) for an arbitrary n).

It is easy to check that none of our algebraic solutions of the Painlevé-VI is of the above form (it is sufficient to check that $\det\left(g^{\alpha\beta} - q\eta^{\alpha\beta}\right) \neq 0$). The Legendre-type transformations of Appendix B correspond to the transpositions of the matrix $A(\lambda)$. This generates solutions of the Painlevé-VI equation of the form (3.15) with the change $\mu \mapsto -\mu$. Particularly, for the function q this gives a transformation of the form

$$q \mapsto \tilde{q} = \left(g^{22}g^{33} - {g^{23}}^2\right)/g^{33}. \tag{E.40}$$

The inversion (B.13) maps a solution of (E.15) to a solution of the same equation with $\mu \mapsto -(1+\mu)$.

These are the only canonical transformations of the Okamoto type of our Painlevé-VI equation.

We conclude that the constructed algebraic solutions of the Painlevé-VI equations cannot be reduced to previously known solutions of these equations. *)

*) I learned recently from N.Hitchin [77] that he also found a particular solution of the Painlevé-VI equation in terms of algebraic functions using the Gauss - Manin connection of the type A_3. In a more recent preprint [78] Hitchin constructs an infinite family of algebraic solutions of certain equations of the Painlevé-VI type. It would be interesting to compare these equations with those of the form (E.15).

Exercise E.1. 1). Show that the canonical coordinates u_1, u_2, u_3 for a 3-dimensional Frobenius manifold with the free energy of the form (C.2) with $\gamma = \gamma(t_3)$ being an arbitrary solution of the Chazy equation (C.5) have the form

$$u_i = t_1 + \frac{1}{2} t_2^2 \omega_i(t_3), \quad i = 1, 2, 3 \tag{E.41}$$

where $\omega_i(\tau)$ are the roots of the cubic equation (C.7).

2). Show that the corresponding solutions of the Euler equations (3.113) has the form

$$\Omega_i = \frac{\omega_i}{2\sqrt{(\omega_j - \omega_i)(\omega_i - \omega_k)}} \tag{E.42}$$

(here i, j, k are distinct indices) where the dependence $t_3 = t_3(s)$ is determined by the equation

$$\frac{\omega_3(t_3) - \omega_1(t_3)}{\omega_2(t_3) - \omega_1(t_3)} = s. \tag{E.43}$$

Appendix F.

Analytic continuation of solutions of WDVV and braid group

Recall that the characteristic feature of ODEs of the Painlevé type is absence of movable critical singularities for a generic solution of the ODE. For example, generic solution of the system (3.74) is a meromorphic (matrix-valued) function of $(u^1, \ldots, u^n)$ outside the diagonals $u^i = u^j$ for some $i \neq j$. The diagonals form the locus of fixed critical singularities of the solutions. Particularly, for $n = 3$ the solutions of (3.74) have the critical singularities on the diagonals $u_1 = u_2$, $u_1 = u_3$, $u_2 = u_3$. The transformation of Appendix E translates these to the critical points $z = 0,\ 1,\ \infty$ of the PVI equation (E.1).

The behaviour of solutions of equations of the Painlevé type "in the large" is very complicated. Multivaluedness of the analytic continuation of the solutions around the critical locus can be called *nonlinear monodromy* of equations of the Painlevé type.

We propose here an approach to the problem of description of the nonlinear monodromy of equations of the Painlevé type in the setting of the isomonodromy integration method. This method is based on the representation of equations of the Painlevé type as monodromy preserving deformations of an auxilliary linear differential operator with rational coefficients. Solutions of the equation of the Painlevé type are parametrized by the monodromy data of the auxilliary linear differential operator. To avoid confusions we will call these *linear monodromy* of the auxilliary linear differential operator.

Our aim is to describe the nonlinear monodromy of solutions of equations of the Painlevé type in terms of linear monodromy of the associated linear differential operator with rational coefficients.

As an example we consider here isomonodromy deformations of the operator

$$\Lambda = \frac{d}{dz} - U - \frac{1}{z}V, \tag{F.1}$$

i.e., the system (3.74) where, as above, $U = \mathrm{diag}\,(u^1, \ldots, u^n)$ is a constant diagonal matrix with $u^i \neq u^j$ for $i \neq j$ and V is a skew-symmetric matrix. In this Appendix we will explain how to relate the *nonlinear monodromy* of the isomonodromy deformations of the operator (F.1) around the fixed critical locus $u^i = u^j$ for $i \neq j$ with the linear monodromy of the operator.

We define first *Stokes factors* of the operator (see [13]).

For any ordered pair $i\,j$ with $i \neq j$ we define the *Stokes ray* R_{ij}

$$R_{ij} = \left\{ z = -ir\left(\bar u^i - \bar u^j\right) | r \geq 0 \right\}. \tag{F.2}$$

Note that the ray R_{ji} is the opposite to R_{ij}. The line $R_{ij} \cup R_{ji}$ divides $\mathbf{C}$ into two half-planes P_{ij} and P_{ji} where the half-plan P_{ij} is on the left of the ray R_{ij}. We have

$$|e^{zu^i}| > |e^{zu^j}| \text{ for } z \in P_{ij}. \tag{F.3}$$

Separating rays are those who coincide with some of the Stokes rays.

Let l be an oriented line going through the origin not containing Stokes rays. It divides $\mathbf{C}$ into two half-planes P_{left} and P_{right}. We order the separating rays $R_1, \ldots, R_{2m}$ starting with the first one in $P_{\rm right}$. In the formulae below the labels of the separating rays will be considered modulo $2m$.

Let Ψ_j be the matrix solution of the equation

$$\Lambda\Psi = 0 \tag{F.4}$$

uniquely determined by the asymptotic

$$\Psi = \left(1 + O\left(\frac{1}{z}\right)\right) e^{zU} \tag{F.5}$$

in the sector from $R_j e^{-i\epsilon/2}$ to $R_{m+j}e^{-i\epsilon}$. Here ϵ is a sufficiently small positive number. This can be extended analytically into the open sector from R_{j-1} to R_{m+j}. On the intersection of two such subsequent sectors we have

$$\Psi_{j+1} = \Psi_j K_{R_j} \tag{F.6}$$

for some nondegenerate matrix K_{R_j}.

For a given choice of the oriented line l we obtain thus a matrix K_R for any separating ray R. These matrices will be called *Stokes factors* of the operator Λ.

Lemma F.1. *The matrices $K = K_R$ satisfy the conditions*

$$K_{ii} = 1,\ i = 1, \dots, n,\ K_{ij} \neq 0 \text{ for } i \neq j \text{ only when } R_{ji} \subset R \tag{F.7a}$$

$$K_{-R} = K_R^{-T}. \tag{F.7b}$$

The solutions $\Psi_{\rm right}$ and $\Psi_{\rm left}$ of Lecture 3 have the form

$$\Psi_{\rm right} = \Psi_1, \quad \Psi_{\rm left} = \Psi_{m+1}. \tag{F.8}$$

The Stokes matrix S is expressed via the Stokes factors in the form

$$S = K_{R_1} K_{R_2} \dots K_{R_m}. \tag{F.9}$$

Conversely, for a given configuration of the line l and of the Stokes rays the Stokes factors of the form (F.7) are uniquelly determined from the equation (F.9).

Proofs of all of the statement of the lemma but (F.7b) can be found in [13]. The relation (F.7b) follows from the skew-symmetry of V as in Proposition 3.10.

Example F.1. For generic $u_1, \dots, u_n$ all the Stokes rays are pairwise distinct. Then the Stokes factors have only one nonzero off-diagonal element, namely

$$\left(K_{R_{ij}}\right)_{ji} \neq 0. \tag{F.10}$$

Let the matrix $V = V(u)$ depend now on $u = (u^1, \dots, u^n)$ in such a way that small deformations of u are isomonodromic. After a large deformation the separating rays could pass through the line l. The corresponding change of the Stokes matrix is described by the following

Corollary F.1. *If a separating ray R passes through the positive half-line l_+ moving clockwise then the solutions $\Psi_{\rm right}$, $\Psi_{\rm left}$ and the Stokes matrix S are transformed as follows*

$$\Psi_{\rm right} = \Psi'_{\rm right} K_R^T,\ \Psi'_{\rm left} = \Psi_{\rm left} K_R,\ S' = K_R^T S K_R. \tag{F.11}$$

Remark F.1. A similar statement holds as well without skew-symmetry of the matrix V. Instead of the matrices K_R and K_R^T in the formulae there will be two independent matrices K_R and K_{-R}^{-1}.

Particularly, let us assume that the real parts of u^i are pairwise distinct. We order them in such a way that

$$\mathrm{Re}\, u^1 < \ldots < \mathrm{Re}\, u^n. \tag{F.12}$$

The real line with the natural orientation will be chosen as the line l. The corresponding Stokes matrix will be upper triangular for such a choice. Any closed path in the space of pairwise distinct ordered parameters u determines a transformation of the Stokes matrix S that can be read of (F.11) (just permutation of the Stokes factors). We obtain an action of the pure braid group on the space of the Stokes matrices. We can extend it onto all the braid group adding permutations of $u_1, \ldots, u_n$. Note that the eigenvalues of the matrix $S^T S^{-1}$ (the monodromy of (F.1) in the origin) are preserved by the action (F.11).

Proposition F.1. *If the solution $V(u)$ of the equations of the isomonodromy deformations of the operator Λ with the given Stokes matrix S is an algebraic function with branching along the diagonals $u^i = u^j$ then S belongs to a finite orbit of the action of the pure braid group.*

Proof. We know from Proposition 3.11 that the matrix function $V(u)$ is meromorphic on the universal covering of $CP^{n-1} \setminus \cup\{u^i = u^j\}$. Closed paths in the deformation space will interchange the branches of this function. Due to the assumptions we will have only finite number of branches. Proposition is proved.

In the theory of Frobenius manifolds the parameters u^i (i.e. the canonical coordinates) are determined only up to a permutation. So we obtain the action of the braid group B_n on the space of Stokes matrices. Explicitly, the standard generator σ_i of B_n ($1 \le i \le n-1$) interchanging u^i and u^{i+1} moving u^i clockwise around u^{i+1} acts as follows

$$\Psi'_{\rm right} = P\Psi_{\rm right} K^{-1},\ \Psi'_{\rm left} = P\Psi_{\rm left} K, \tag{F.13a}$$

$$S' = KSK \tag{F.13b}$$

where the matrix $K = K_i(S)$ has the form

$$K_{ss} = 1, s = 1, \ldots, n,\ s \ne i,\ i+1,$$

$$K_{ii} = -s_{i\,i+1},\quad K_{i\,i+1} = K_{i+1\,i} = 1,\ K_{i+1\,i+1} = 0 \tag{F.14}$$

other matrix entries of K vanish, $P = P_i$ is the matrix of permutation $i \leftrightarrow i+1$.

It is clear that finite orbits of the full braid group B_n must consist of finite orbits of the subgroup of pure braids. So it is sufficient to find finite orbits of the action (F.13).

Exercise F.1. Verify that the braid

$$(\sigma_1 \dots \sigma_{n-1})^n \tag{F.15}$$

acts trivially on the space of Stokes matrices.

The braid (F.15) is the generator of the center of the braid group B_n for $n \geq 3$ [20]. We obtain thus an action of the quotient

$$A_n^* = B_n/\text{center}$$

on the space of the Stokes matrices. Note that A_n^* coincides with the mapping class group of the plane with n marked points [20]. For $n = 3$ the group A_n^* is isomorphic to the modular group $PSL(2, \mathbf{Z})$.

In order to construct examples of finite orbits of the action (F.13b), (F.14) we represent it in a more geometric way (cf. [29]).

Let V be a n-dimensional space supplied with a symmetric bilinear form (,). For any vector $f \in V$ satisfying $(f, f) = 2$ the *reflection* $R_f : V \to V$ is defined by the formula

$$R_f(x) = x - (x, f)f.$$

It preserves the hyperplane orthogonal to f and it inverts f

$$R_f(f) = -f.$$

The hyperplane is called the *mirror* of the reflection. The linear map R_f preserves also the bilinear form (,). So it can be projected onto the quotient V/V_0 where $V_0 \subset V$ is the annihilator of the bilinear form.

We say that a basis $e_1, \dots, e_n$ in $(V,\ (\ ,\ ))$ is an *admissible* one if

$$(e_i, e_i) = 2,\ i = 1, \dots, n$$

and the reflections $R_1 := R_{e_1}, \dots, R_n := R_{e_n}$ generate a finite group of linear transformations of the space V/V_0. Equivalently, this means that, applying the reflections $R_1, \dots, R_n$ and their products to the vectors $e_1, \dots, e_n$ and projecting them onto V/V_0 we obtain a finite system of vectors.

An alternative definition of admissibility can be given in terms of (generalized) root systems. We say that a system of nonzero vectors f_α in a space V with a symmetric bilinear form (,) is a *generalized root system* if *(i)* the vectors span the space V, and their projections is a finite system of vectors spanning V/V_0, *(ii)* the square lengths (f_α, f_α) all equal 2 and *(iii)* the system is invariant w.r.t. the reflections R_{f_α}.

Exercise F.2. Prove that the basis $e_1, \ldots, e_n$ in $(V, (\ ,\ ))$ is an admissible one *iff* it is a part of a generalized root system in $(V, (\ ,\ ))$. [Hint: use the identity

$$R_{ij} = R_i R_j R_i$$

where R_{ij} is the reflection in the hyperplane orthogonal to $R_i(e_j)$.]

We introduce now an action of the braid group B_n on the set of admissible bases in $(V,\ (\ ,\ ))$. The standard generator σ_i $(i = 1, \ldots, n-1)$ of the braid group (as above) acts as follows

$$\begin{aligned} \sigma_i(e_k) &= e_k,\ \ k \neq i, i+1 \\ \sigma_i(e_i) &= R_i(e_{i+1}) \equiv e_{i+1} - (e_i, e_{i+1})e_i \\ \sigma_i(e_{i+1}) &= e_i. \end{aligned} \tag{F.16}$$

Observe that the action of the braid (F.15) is not trivial.

Exercise F.3. Prove that the action of the braid $(\sigma_{n-1} \ldots \sigma_1)^n$ on an admissible basis $e_1,\ \ldots,\ e_n$ coincides with the transformation $-R_1 R_2 \ldots R_n$.

It is easy to relate the action (F.16) to the action (F.13). To an admissible basis we associate an upper triangular matrix $S = S(e_1, \ldots, e_n) = (s_{ij})$ taking a half of the Gram matrix

$$s_{ii} = 1, \quad s_{ij} = (e_i, e_j) \text{ for } i < j, \quad s_{ij} = 0 \text{ for } i > j. \tag{F.17}$$

Lemma F.2. *1). The transformation law (F.13a) of the columns of the matrix* Ψ_{right} *is dual to (F.16). 2). The map*

$$(e_1, \ldots, e_n) \mapsto S(e_1, \ldots, e_n)$$

intertwines the action (F.16) of the braid group on admissible bases with the action (F.13b) on the Stokes matrices

$$S\left(\sigma_i(e_1, \ldots, \sigma_n)\right) = \sigma_i\left(S(e_1, \ldots, e_n)\right).$$

Proof. The matrix K of the transformation (F.16) coincides with (F.14). This proves the lemma.

The computation of Appendix H of the intersection form of a Frobenius manifold in terms of the Stokes matrix will elucidate the correspondence (F.17).

Now we can prove

Proposition F.2. *Let* $e_1, \ldots, e_n$ *be an admissible basis in* $(V,\ (\ ,\ ))$. *Then the orbit of the matrix* $S = S(e_1, \ldots, e_n)$ *under the action (F.13b) of the braid group* B_n *is finite.*

Proof. Let f_α, $\alpha = 1, \ldots, N$ for some $N < \infty$ be the generalized root system consisted of the vectors $e_1,\ \ldots,\ e_n$ and their images under the reflections $R_1,\ \ldots,\ R_n$ and their iterations. Applying any element of the braid group to the basis $(e_1, \ldots, e_n) \mapsto (e'_1, \ldots, e'_n)$ we will always obtain a vector of the root system. So there is only finite number of possibilities for the Gram matrix (e_i', e_j') depending only on the projections of e_i' onto V/V_0. Hence we obtain a finite orbit of $S = S(e_1, \ldots, e_n)$. Proposition is proved.

We recall that a finite Coxeter group is a finite group W of linear transformations of n-dimensional Euclidean space V generated by reflections (see details in Lecture 4 below). For any finite Coxeter group we will construct a finite orbit of the action (F.13b) of the braid group B_n (avoid confusions with the Coxeter group of the type B_n!).

Let $R_1, \ldots, R_n$ be a system of generated reflections of W. Let $e_1, \ldots, e_n$ be the vectors orthogonal to the mirrors of the reflections normalized by the condition $(e_i, e_i) = 2$ (here $(\ ,\ )$ is the W-invariant Euclidean inner product on V). The Stokes matrix (F.17) in this case coincides with the upper half of the Coxeter matrix of the group with respect to the given system of generated reflections (probably, such a matrix S was considered first by Coxeter in [34]. From Proposition F.2 we obtain

Corollary F.2. *The B_n-orbit of the upper half of the Coxeter matrix of an arbitrary finite Coxeter group w.r.t. arbitrary system of generating reflections is finite.*

Example F.2. For $n = 3$ we put $s_{12} = x$, $s_{13} = y$, $s_{23} = z$. The transformations of the braid group act as follows:

$$\sigma_1 : \ (x, y, z) \mapsto (-x, z - xy, y), \tag{F.18a}$$

$$\sigma_2 : \ (x, y, z) \mapsto (y - xz, x, -z). \tag{F.18b}$$

These preserve the polynomial

$$x^2 + y^2 + z^2 - xyz. \tag{F.19}$$

Indeed, the characteristic equation of the matrix $S^T S^{-1}$ has the form

$$(\lambda - 1)[\lambda^2 + (x^2 + y^2 + z^2 - xyz - 2)\lambda + 1] = 0. \tag{F.20}$$

The action of the group B_3 (in fact, this can be reduced to the action of $PSL(2, \mathbf{Z})$) admits also an invariant Poisson bracket

$$\begin{aligned} \{x, y\} &= xy - 2z \\ \{y, z\} &= yz - 2x\,. \\ \{z, x\} &= zx - 2y \end{aligned} \tag{F.21}$$

The polynomial (F.19) is the Casimir of the Poisson bracket. Thus an invariant symplectic structure is induced on the level surfaces

$$x^2 + y^2 + z^2 - xyz = \text{const}.$$

A B_n-invariant Poisson bracket exists also on the space of Stokes matrices of the order n. But it has more complicated structure.

For integer x, y, z this action on the invariant surface $x^2 + y^2 + z^2 = xyz$ was discussed first by Markoff in 1876 in the theory of Diophantine approximations [27]. The general action (F.13b), (F.14) (still on integer valued matrices) appeared also in the theory of exceptional vector bundles over projective spaces [128]. Essentially it was also found from physical considerations in [29] (again for integer matrices S) describing "braiding of Landau - Ginsburg superpotential". The invariant Poisson structure (F.21) looks to be new.

There are 3 finite Coxeter groups in the three-dimensional space: the groups of symmetries of tetrahedron, cube and icosahedron. According to Corollary F.2 they give the finite B_3-orbits of the points

$$(0,-1,-1),\ (0,-1,-\sqrt{2}),\ (0,-1,-\frac{\sqrt{5}+1}{2}).$$

The orbits consist of 16, 36, 40 points resp. Two more finite orbits of the braid group can be obtained using another system of generating reflections in the icosahedron group. The first orbit of 40 points consists of the images of the point

$$\left(\frac{1-\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right).$$

The corresponding mirrors are the planes passing through the origin and through the three edges of some face of icosahedron. The orbit of the point

$$\left(0,\frac{1-\sqrt{5}}{2},-\frac{1+\sqrt{5}}{2}\right)$$

consists of 72 points. Two of the mirrors in these case pass through the origin and through two of the edges of some face of icosahedron; the third mirror passes through the origin and a median of the face (not between the first two mirrors). We see that in the icosahedron group not all the systems of generating reflections are equivalent w.r.t. the action of the braid group.

We will see below that the first three finite orbits are the Stokes matrices of the polynomial solutions (A.7), (A.8), (A.9) resp. I do not know the solutions of WDVV with the last two Stokes matrices.

We construct now 3-dimensional generalized root systems with a degenerate bilinear form. Let us take the Stokes matrix

$$S=\begin{pmatrix}1 & -2\cos\phi & -2\sin\phi\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The symmetrization $S+S^T$ for an arbitrary ϕ gives a bilinear form of rank 2 on the three-dimensional space with a marked basis e_1, e_2, e_3. The basis will be admissible *iff* ϕ is commensurable with π

$$\phi=\pi\frac{k}{n}.$$

Particularly, for $k=1$, $n=3$ we obtain the Stokes matrix of the solution (4.71) of WDVV, and for $k=1$, $n=4$ the Stokes matrix of the solution (4.75).

The last example of a 3-dimensional generalized root system is given by the Stokes matrix

$$S = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The symmetrized bilinear form has rank 1. This is the Stokes matrix of the twisted Frobenius manifold (C.87).

It is an interesting problem to classify periodic orbits of the action (F.13b), (F.14) of the braid group, and to figure out what of them correspond to algebraic solutions of the Painlevé-type equations (3.74).

Remark F.1. Cecotti and Vafa in [29] conjectured that

$$S = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

is the Stokes matrix of the CP^2 topological sigma model (see above Lecture 2). Points of the orbit of the Stokes matrix are in one-to-one correspondence with triples of Markoff numbers (see in [25]). So the orbit is not finite, and the function (2.77) is not algebraic.

Appendix G.
Monodromy group of a Frobenius manifold.

In Lecture 3 I have defined the intersection form of arbitrary Frobenius manifold M. This is another flat (contravariant) metric $(\ , \ )^*$ on M defined by the formula (3.13). In this Appendix we will study the Euclidean structure on M determined by the new metric. Let us assume that the Frobenius manifold M is *analytic*. This means that the structure functions $(c_{\alpha\beta}^{\gamma}(t))$ are analytic in t. As it follows from the resullts of Lecture 3 the assumption is not restrictive in the semisimple case.

The contravariant metric $(\ , \ )^*$ degenerates on the sublocus where the determinant

$$\Delta(t) := \det\left(g^{\alpha\beta}(t)\right). \tag{G.1}$$

vanishes. Let $\Sigma \subset M$ be specified by the equation

$$\Sigma := \left\{t | \Delta(t) := \det(g^{\alpha\beta}(t)) = 0\right\}. \tag{G.2}$$

This is a proper analytic subset in M. We will call it *the discriminant locus* of the Frobenius manifold. The analytic function $\Delta(t)$ will be called *the discriminant* of the manifold.

On $M \setminus \Sigma$ we have a locally Euclidean metric determined by the inverse of the intersection form. This specifies an isometry

$$\Phi : \Omega \to \widehat{M \setminus \Sigma} \tag{G.3}$$

of a domain Ω in the standard n-dimensional (complex) Euclidean space E^n to the universal covering of $M \setminus \Sigma$. Action of the fundamental group $\pi_1(M \setminus \Sigma)$ on the universal covering can be lifted to an action by isometries on E^n. We obtain a representation

$$\mu : \pi_1(M \setminus \Sigma) \to Isometries(E^n). \tag{G.4}$$

Definition G.1. The group

$$W(M) := \mu\left(\pi_1(M \setminus \Sigma)\right) \subset Isometries(E^n) \tag{G.5}$$

is called *the monodromy group of the Frobenius manifold*.

By the construction

$$M \setminus \Sigma = \Omega / W(M).$$

To construct explicitly the isometry (G.3) we are to fix a point $t \in M \setminus \Sigma$ and to find the flat coordinates of the intersection form in a neibourghood of the point. The flat coordinates $x = x(t^1, \dots, t^n)$ are to be found from the system of differential equations

$$\hat\nabla^\alpha \hat\nabla_\beta x := g^{\alpha\epsilon}(t)\partial_\epsilon\partial_\beta x + \Gamma_\beta^{\alpha\epsilon}(t)\partial_\epsilon x = 0, \quad \alpha, \beta = 1, \dots, n. \tag{G.6}$$

(Here $\hat\nabla$ is the Levi-Civita connection for $(\ , \ )^*$; the components of the metric and of the connection are given by the formulae (3.15), (3.36).) This is an overdetermined holonomic system. Indeed, vanishing of the curvature of the intersection form (Proposition 3.2) provides compatibility of the system. More precisely,

Proposition G.1. *Near a point $t_0 \in M$ where*

$$\det\left(g^{\alpha\beta}(t_0)\right) \neq 0$$

the space of solutions of the system (G.6) modulo constants has dimension n. Any linearly independent (modulo constants) solutions $x^1(t), \ldots, x^n(t)$ of (G.6) can serve as local coordinates near t_0. The metric $g^{\alpha\beta}(t)$ in these coordinates is constant

$$g^{ab} := \frac{\partial x^a}{\partial t^\alpha}\frac{\partial x^b}{\partial t^\beta} g^{\alpha\beta}(t) = const.$$

This is a reformulation of the standard statement about the flat coordinates of a zero-curvature metric.

Exercise G.1. For $d \neq 1$ prove that:

1. $x_a(t)$ are quasihomogeneous functions of t^1, ..., t^n of the degree

$$\deg x_a(t) = \frac{1-d}{2}; \tag{G.7}$$

2. that

$$t_1 \equiv \eta_{1\alpha} t^\alpha = \frac{1-d}{4} g_{ab} x^a x^b \tag{G.8}$$

where $(g_{ab}) = \left(g^{ab}\right)^{-1}$.

Example G.1. For the two-dimensional Frobenius manifold with the polynomial free energy

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^{k+1}, \quad k \geq 2 \tag{G.9}$$

the system (G.6) can be easily solved in elementary functions. The flat coordinates x and y can be introduced in such a way that

$$\begin{aligned} t_1 &= 4\sqrt{k(k^2-1)}\,\mathrm{Re}\,(x+iy)^k \\ t_2 &= x^2 + y^2. \end{aligned} \tag{G.10}$$

Thus the monodromy group of the Frobenius manifold (G.9) is the group $I_2(k)$ of symmetries of the regular k-gon.

For the polynomial solutions (A.7), (A.8), (A.9) the calculation of the monodromy group is more involved. In the next Lecture we will see that the monodromy groups of these three polynomial solutions of WDVV coincide with the groups A_3, B_3, H_3 of symmetries of the regular tetrahedron, cube and icosahedron in the three-dimensional space.

More generally, for the Frobenius manifolds of Lecture 4 where $M = \mathbf{C}^n/W$ for a finite Coxeter group W, the solutions of the system (G.6) are the Euclidean coordinates in $\mathbf{C}^n$ as the functions on the space of orbits. If we identify the space of orbits with the universal unfolding of the corresponding simple singularity [5, 6, 25, 139] then the map

$$M \ni t \mapsto (x^1(t), \ldots, x^n(t)) \in \mathbf{C}^n \tag{G.11}$$

coincides with the period mapping. The components $x^a(t)$ are sections of the bundle of vanishing cycles being locally horizontal w.r.t. the Gauss – Manin connection. Note that globally (G.11) is a multivalued mapping. The multivaluedness is just described by the action of the Coxeter group W coinciding with the monodromy group of the Frobenius manifolds.

Basing on this example we introduce

Definition G.2. The system (G.6) is called *Gauss – Manin equation* of the Frobenius manifold.

Note that the coefficients of Gauss – Manin equation on an analytic Frobenius manifold also are analytic in t. This follows from (3.15), (3.36). However, the solutions may not be analytic everywhere. Indeed, if we rewrite Gauss – Manin equations in the form solved for the second order derivatives

$$\partial_\alpha \partial_\beta x - \Gamma^\gamma_{\alpha\beta}(t)\partial_\gamma x = 0 \tag{G.12}$$

$$\Gamma^\gamma_{\alpha\beta}(t) := -g_{\alpha\epsilon}\Gamma^{\epsilon\beta}_\gamma(t), \quad (g_{\alpha\epsilon}) := (g^{\alpha\epsilon})^{-1}$$

then the coefficients will have poles on the discriminant Σ.

So the solutions of Gauss – Manin equation (G.6) are analytic in t on $M \setminus \Sigma$. Continuation of some basic solution $x^1(t), \ldots, x^n(t)$ along a closed path γ on $M \setminus \Sigma$ can give new basis of solutions $\tilde{x}^1(t), \ldots, \tilde{x}^n(t)$. Due to Proposition G.1 it must have the form

$$\tilde{x}^a(t) = A^a_b(\gamma)x^b(t) + B^a(\gamma) \tag{G.13}$$

for some constants $A^a_b(\gamma), B^a(\gamma)$. The matrix $A^a_b(\gamma)$ must be orthogonal w.r.t. the intersection form

$$A^a_b(\gamma)g^{bc}A^d_c(\gamma) = g^{ad}. \tag{G.14}$$

The formula (G.13) determines the representation (G.4) of the fundamental group $\pi_1(M \setminus \Sigma, t) \ni \gamma$ to the group of isometries of the n-dimensional complex Euclidean space E^n. This is just the monodromy representation (G.5).

Proposition G.2. *For $d \neq 1$ the monodromy group is a subgroup in $O(n)$ (linear orthogonal transformations).*

Proof. Due to Exercise G.1 for $d \neq 1$ one can choose the coordinates $x^a(t)$ to be invariant w.r.t. the scaling transformations

$$x^a(c^{\deg t^1}t^1, \ldots, c^{\deg t^n}t^n) = c^{\frac{1-d}{2}}x^a(t^1, \ldots, t^n). \tag{G.15}$$

The monodromy preserves such an invariance. Proposition is proved.

Example G.2. If some of the scaling dimensions $q_\alpha = 1$ then the Frobenius structure may admit a discrete group of translations along these variables. The Gauss - Manin equations then will be a system with periodic coefficients. The corresponding monodromy transformation (i.e. the shift of solutions of (G.6) along the periods) will contribute to the monodromy group of the Frobenius manifold.

To see what happens for $d = 1$ let us find the monodromy group for the two-dimensional Frobenius manifold with

$$F = \frac{1}{2} {t^1}^2 t^2 + e^{t^2}. \tag{G.16}$$

I recall that this describes the quantum cohomology of CP^1. The manifold M is the cylinder $\left(t^1, t^2 (\mathrm{mod}\, 2\pi i)\right)$. The Euler vector field is

$$E = t^1 \partial_1 + 2\partial_2. \tag{G.17}$$

The intersection form has the matrix

$$\left(g^{\alpha\beta}\right) = \begin{pmatrix} 2e^{t^2} & t^1 \\ t^1 & 2 \end{pmatrix}. \tag{G.18}$$

The Gauss – Manin system reads

$$\begin{aligned} 2e^{t^2} {\partial_1}^2 x + t^1 \partial_1 \partial_2 x = 0 \\ 2e^{t^2} \partial_1 \partial_2 x + t^1 {\partial_2}^2 x + e^{t^2} \partial_1 x = 0 \\ t^1 {\partial_1}^2 x + 2\partial_1 \partial_2 x + \partial_1 x = 0 \\ t^1 \partial_1 \partial_2 x + 2{\partial_2}^2 x = 0. \end{aligned} \tag{G.19}$$

The basic solutions are

$$\begin{aligned} x^1 &= -i t^2 \\ x^2 &= 2 \arcsin \frac{1}{2} t^1 e^{-\frac{t^2}{2}}. \end{aligned} \tag{G.20}$$

The intersection form in these coordinates is

$$\frac{1}{2} \left(-{dx^1}^2 + {dx^2}^2 \right). \tag{G.21}$$

The Euler vector field reads

$$E = \frac{\partial}{\partial x^1}. \tag{G.22}$$

The discriminant locus is specified by the equation

$$t^1 = \pm 2 e^{\frac{t^2}{2}} \tag{G.23a}$$

or, equivalently

$$x^2 = \pm \pi. \tag{G.23b}$$

The monodromy group is generated by the transformations of the following two types.

The transformations of the first type are obtained by continuation of the solutions (G.20) along the loops around the discriminant locus. This gives the transformations

$$\left(x^1, x^2\right) \mapsto \left(x^1, (-1)^n x^2 + 2\pi n\right), \quad n \in \mathbf{Z}. \tag{G.24a}$$

The transformations of the second type

$$\left(x^1, x^2\right) \mapsto \left(x^1 + 2\pi m, (-1)^m x^2\right), \quad m \in \mathbf{Z} \tag{G.24b}$$

are generated by the closed loops $t^2 \mapsto t^2 + 2\pi i n$ on M. Altogether this gives an extension

$$\left(x^1, x^2\right) \mapsto \left(x^1 + 2\pi m, (-1)^{m+n} x^2 + 2\pi n\right) \tag{G.24c}$$

of the simplest affine Weyl group of the type $A_1^{(1)}$. So M is the quotient of $\mathbf{C}^2$ over the extended affine Weyl group (G.24) (although t^2 is not a globally single-valued function on the quotient). The coordinate

$$t^1 = 2e^{ix^1/2} \sin \frac{x^2}{2} \tag{G.25}$$

is the basic invariant of the group (G.24) homogeneous w.r.t. the Euler vector field (G.22). That means that any other invariant is a polynomial (or a power series) in t^1 with the coefficients being arbitrary $2\pi i$-periodic functions in t^2. Another flat coordinate

$$t^2 = ix^1$$

is invariant w.r.t. the affine Weyl group (G.24a) and it gets a shift w.r.t. the transformations (G.24b).

Example G.3. The monodromy group of a Frobenius manifold can be in principle computed even if we do not know the structure of it using the isomonodromicity property of the solutions of the Gauss - Manin system. For the example of the CP^2 model (see above Lecture 2) it is enough to compute the monodromy group on the sublocus $t^1 = t^3 = 0$. We obtain that the linear part of the monodromy of the CP^2-model is generated by the monodromy group of the operator with rational coefficients

$$\begin{pmatrix} 3q & 0 & \lambda \\ 0 & \lambda & 3 \\ \lambda & 3 & 0 \end{pmatrix} \frac{\partial \xi}{\partial \lambda} + \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 1/2 & 0 \\ 3/2 & 0 & 0 \end{pmatrix} \xi = 0 \tag{G.26a}$$

for

$$q = e^t, \quad t = t_2$$

and of the operator with $2\pi i$-periodic coefficients

$$\begin{pmatrix} 3q & 0 & \lambda \\ 0 & \lambda & 3 \\ \lambda & 3 & 0 \end{pmatrix} \frac{\partial \xi}{\partial t} + \begin{pmatrix} \frac{3}{2}q & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \xi = 0. \tag{G.26b}$$

The last one can be also reduced to an operator with rational coefficients (with irregular singularity at $q = \infty$) by the substitution $t \to q$. It is still an open problem to solve these equations and to compute the monodromy.

The monodromy group can be defined also for twisted Frobenius manifolds. Particularly, the inversion (B.11) acts as a conformal transformation of the intersection form. We will see in Appendix J an example of such a situation.

Another important sublocus in a Frobenius manifold satisfying the semisimplicity condition is *the nilpotent locus* $\Sigma_{\rm nil}$ consisting of all the points t of M where the algebra on T_tM is not semisimple. According to Proposition 3.3 the nilpotent discriminant is contained in the discriminant locus of the polynomial

$$\det\left(g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}\right) = \Delta(t^1 - \lambda, t^2, \dots, t^n). \tag{G.27}$$

The discriminant $\Delta_{\rm nil}(t)$ of (G.27) as of the polynomial on λ will be called *nilpotent discriminant*.

Example G.4. For the Frobenius manifold of Example 1.7 the discriminant locus

$$\Sigma = \{\text{set of polynomials } \lambda(p) \text{ having a critical value } \lambda = 0\}. \tag{G.28}$$

The discriminant $\Delta(t)$ coincides with the discriminant of the polynomial $\lambda(p)$ (I recall that the coefficients of the polynomial $\lambda(p)$ are certain functions on t). The nilpotent locus of M is the *caustic* (see [6])

$$\Sigma_{\rm nil} = \{\text{set of polynomials } \lambda(p) \text{ with multiple critical points } p_0,$$

$$\left.\lambda^{(k)}(p_0) = 0 \text{ for } k = 1, 2, \dots, k_0 \geq 2\right\}. \tag{G.29}$$

The nilpotent discriminant is a divisor of the discriminant of the polynomial $\mathrm{discr}_p(\lambda(p) - \lambda)$ as of the polynomial in λ.

On the complement $M \setminus \Sigma$ a metric

$$ds^2 := g_{\alpha\beta}(t)dt^\alpha dt^\beta \tag{G.30}$$

is well-defined. Here the matrix $g_{\alpha\beta}(t)$ is the inverse to the matrix $g^{\alpha\beta}(t)$ (3.15). The metric has a pole on the discriminant locus. We will show that the singularity of the metric can be eliminated after lifting to a covering of M.

Let $\hat{M}$ be the two-sheet covering of the Frobenius manifold M ramifying along the discriminant locus

$$\hat{M} := \left\{(w, t),\ w \in \mathbf{C},\ t \in M | w^2 = \Delta(t)\right\}. \tag{G.31}$$

We have a natural projection

$$\pi : \hat{M} \to M. \tag{G.32}$$

Lemma G.1. *The pullback $\pi^*\, ds^2$ of the metric (G.30) onto $\hat{M}$ is analytic on $\hat{M} \setminus (\Sigma \cap \Sigma_{\rm nil})$.*

Proof. Outside of Σ on M we can use the canonical coordinates. In these the metric has the form

$$ds^2 = \sum_{i=1}^{n} \frac{\eta_{ii}(u)}{u^i}(du^i)^2. \tag{G.33}$$

The canonical coordinates serve near a point $t_0 \in M \setminus (\Sigma \cap \Sigma_{\rm nil})$ as well, but some of them vanish. The vanishing is determined by a splitting $(i_1, \ldots, i_p) \cup (j_1, \ldots, j_q) = (1, 2, \ldots, n)$, $p + q = n$ such that

$$u^{i_1}(t_0) = 0, \ldots, u^{i_p}(t_0) = 0, \quad u^{j_i}(t_0) \neq 0, \ldots, u^{j_q} \neq 0. \tag{G.34}$$

The local coordinates near the corresponding point $\pi^{-1}(t_0) \in \hat{M}$ are

$$\sqrt{u^{i_1}}, \ldots, \sqrt{u^{i_p}}, u^{j_1}, \ldots, u^{i_q}. \tag{G.35}$$

Rewriting (G.33) near the point t_0 as

$$ds^2 = 4\sum_{s=1}^{p} \eta_{i_s i_s}(u)(d\sqrt{u^{i_s}})^2 + \sum_{s=1}^{q} \eta_{j_s j_s}(u)(du^{j_s})^2 \tag{G.36}$$

we obtain analyticity of ds^2 on $\hat{M}$. Lemma is proved.

Due to Lemma G.1 the flat coordinates $x^a(t)$ as the functions on $\hat{M}$ can be extended to any component of $\Sigma \setminus \Sigma_{\rm nil}$. The image

$$\left(x^1(t), \ldots, x^n(t)\right)_{t \in \Sigma \setminus \Sigma_{\rm nil}} \tag{G.37}$$

is the discriminant locus written in the flat coordinates $x^1, \ldots, x^n$.

Lemma G.2. *Any component of $\Sigma \setminus \Sigma_{\rm nil}$ in the coordinates $x^1(t), \ldots, x^n(t)$ is a hyperplane.*

Proof. We will show that the second fundamental form of the hypersurface $\Delta(t) = 0$ in $\hat{M}$ w.r.t. the metric (G.30) vanishes. Near $\Sigma \setminus \Sigma_{\rm nil}$ we can use the canonical coordinates $u^1, \ldots, u^n$. Let $\Sigma \setminus \Sigma_{\rm nil}$ be specified locally by the equation $u^n = 0$. Let us first calculate the second fundamental form of the hypersurface

$$u^n = u_0^n \neq 0. \tag{G.38}$$

The unit normal vector to the hypersurface is

$$N = \sqrt{\frac{u_0^n}{\eta_{nn}}}\partial_n. \tag{G.39}$$

The vectors ∂_i, $i \neq n$ span the tangent plane to the hypersurface. The second fundamental form is

$$b_{ij} := \left(\hat{\nabla}_i \partial_j, N\right) = \sqrt{\frac{\eta_{nn}}{u_0^n}}\Gamma_{ij}^n, \quad 1 \leq i, j \leq n-1$$

$$= -\frac{\delta_{ij}}{2}\sqrt{\frac{u_0^n}{\eta_{nn}}}\partial_n\left(\frac{\eta_{ii}}{u^i}\right) = -\delta_{ij}\sqrt{\frac{u_0^n}{\eta_{nn}}}\frac{\psi_{i1}\psi_{n1}\gamma_{in}}{u^i} \qquad (G.40)$$

in the notations of Lecture 3 (but Γ_{ij}^k here are the Christoffel coefficients for the intersection form). This vanishes when $u_0^n \to 0$. Lemma is proved.

I recall that a linear orthogonal transformation $A : \mathbf{C}^n \to \mathbf{C}^n$ of a complex Euclidean space is called *reflection* if $A^2 = 1$ and A preserves the points of a hyperplane in $\mathbf{C}^n$.

Theorem G.1. *For $d \neq 1$ the monodromy along a small loop around the discriminant on an analytic Frobenius manifold satisfying semisimplicity condition is a reflection.*

Proof. Since the flat coordinates are analytic and single valued on the two-sheet covering $\hat{M}$ the monodromy transformations A along loops around Σ are involutions, $A^2 = 1$.They preserves the hyperplanes (G.28). Note that the hyperplanes necessary pass through the origin for $d \neq 1$. Theorem is proved.

In Lecture 4 we will show that any finite reflection group arises as the monodromy group of a Frobenius manifold. This gives a very simple constructionn of polynomial Frobenius manifolds with $d < 1$. Similarly, the construction of Example G.2 can be generalized to arbitrary affine Weyl groups (properly extended). This gives a Frobenius structure with $d = 1$ and linear nonhomogeneous Euler vector field E on their orbit spaces (i.e. $\mathcal{L}_E t^n \neq 0$). We will consider the details of this construction in a separate publication. Finally, in Appendix J we will construct a twisted Frobenius manifold whose monodromy is a simplest extended complex crystallographic group. This will give Froobenius manifolds again with $d = 1$ but with linear homogeneous Euler vector field (i.e. $\mathcal{L}_E t^n = 0$).

Appendix H.

Generalized hypergeometric equation associated with a Frobenius manifold and its monodromy.

The main instrument to calculate the monodromy of a Frobenius manifold coming from the loops winding around the discriminant is a differential equation with rational coefficients wich we are going to define now. This will be the equation for the flat coordinates of the linear pencil of the metrics

$$(\ , \ )^*_\lambda := (\ , \ )^* - \lambda < \ , \ >^* = \left(g^{\alpha\beta} - \lambda\eta^{\alpha\beta}\right) \tag{H.1}$$

as the functions of the parameter λ. We will obtain also an integral transform relating the flat coordinates of the deformed connection (3.3) and the flat coordinnates of the deformed metric (H.1).

Let $x_1 = x_1(t)$, ..., $x_n = x_n(t)$ be flat coordinates for the metric $g^{\alpha\beta}(t)$. The flat coordinates for the flat pencil H.1 can be construct easily.

Lemma H.1. *The functions*

$$\tilde{x}_a(t,\lambda) := x_a(t^1 - \lambda, t^2, \dots, t^n), \quad a = 1, \dots, n \tag{H.2}$$

are flat coordinates for the deformed metric (H.1).

Proof. The linear combination (H.1) can be written in the form

$$g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} = g^{\alpha\beta}(t^1 - \lambda, t^2, \dots, t^n). \tag{H.3}$$

Lemma is proved.

Corollary H.1. *The gradient $\xi_\epsilon := \partial_\epsilon x(\lambda, t)$ of the flat coordinates of the pencil $g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}$ satisfies the following system of linear differential equations in λ*

$$\left(\lambda\eta^{\alpha\epsilon} - g^{\alpha\epsilon}(t)\right)\frac{d}{d\lambda}\xi_\epsilon = \eta^{\alpha\epsilon}\left(-\frac{1}{2} + \mu_\epsilon\right)\xi_\epsilon. \tag{H.4}$$

Proof. We have from (3.38)

$$\Gamma_1^{\alpha\epsilon} = \left(\frac{d+1}{2} - q_\epsilon\right)c_1^{\alpha\epsilon} = \left(\frac{1}{2} - \mu_\epsilon\right)\eta^{\alpha\epsilon}. \tag{H.5}$$

So the equation (G.6) for $\beta = 1$ reads

$$\left(g^{\alpha\epsilon}(t) - \lambda\eta^{\alpha\epsilon}\right)\partial_1\partial_\epsilon x + \eta^{\alpha\epsilon}\left(\frac{1}{2} - \mu_\epsilon\right)\partial_\epsilon x = 0. \tag{H.6}$$

Due to Lemma H.1 we have

$$\partial_1 = -\frac{d}{d\lambda}. \tag{H.7}$$

Corollary is proved.

The equation (H.4) is a system of linear ordinary differential equations with rational coefficients depending on the parameters $t^1, \ldots, t^n$. The coefficients have poles on the shifted discriminant locus

$$\Sigma_\lambda := \left\{ t \,|\Delta(t^1 - \lambda, t^2, \ldots, t^n) = 0 \right\}. \tag{H.8}$$

Lemma H.2. *Monodromy of the system (H.4) of differential equations with rational coefficients around Σ_λ coincides with the monodromy of the Frobenius manifold around the discriminant Σ. The monodromy does not depend on the parameters t.*

Proof. The first statement is obvious. The second one follows from the compatibility of the equations (G.6) with the equations in λ (H.4).

Definition H.1. The differential equation (H.4) with rational coefficients will be called *generalized hypergeometric equation associated with the Frobenius manifold.*

In this definition we are motivated also by [122] where it was shown that the differential equations for the functions ${}_nF_{n-1}$ are particular cases of the system (H.15) (this is an equivalent form of (H.4), see below) however, in general without the skew-symmetry of the matrix V. In the semisimple case the equation (H.4) has only regular singularities on (H.8) and in the infinite point $\lambda = \infty$.

We construct now an integral transform relating the flat coordinates $\tilde{t}(t, z)$ of the deformed connection (3.3) and the flat coordinates $x(t, \lambda)$ of the pencil of metrics (H.1). We recall that the coordinates $x(t, \lambda)$ are the solutions of the differential equations

$$(g^{\alpha\epsilon} - \lambda\eta^{\alpha\epsilon})\, \partial_\epsilon \partial_\beta x + \Gamma^{\alpha\epsilon}_\beta \partial_\epsilon x = 0. \tag{H.9}$$

Proposition H.1. *Let $\tilde{t}(t, z)$ be a flat coordinate of the deformed connection (3.3) normalized by the condition*

$$z\partial_z \tilde{t} = \mathcal{L}_E \tilde{t}. \tag{H.10}$$

Then the function

$$x(t, \lambda) := \oint z^{\frac{d-3}{2}} e^{-\lambda z} \tilde{t}(t, z)\, dz \tag{H.11}$$

is a flat coordinate for the pencil (H.1).

Here the integral is considered along any closed loop in the extended complex plane $z \in \mathbf{C} \cup \infty$. We will specify later how to choose the contour of the integration to obtain a well-defined integral.

Proof is based on the following

Lemma H.3. *The following identity holds true*

$$\left(dt^\alpha, \hat{\nabla}_\gamma d\tilde{t}\right)^* dt^\gamma = z^{-1} d \left[< dt^\alpha, \left(z\partial_z + \frac{d-3}{2} \right) d\tilde{t} >^* \right]. \tag{H.12}$$

Here $\tilde{t} = \tilde{t}(t, z)$ is a flat coordinate of the deformed affine connection (3.3), $\hat{\nabla}$ is the Levi-Civita connection for the intersection form, $d = dt^\gamma \partial_\gamma$.

Proof. The l.h.s. of (H.12) reads

$$\left(g^{\alpha\sigma}\partial_\sigma\partial_\gamma\tilde{t} + \Gamma_\gamma^{\alpha\sigma}\partial_\sigma\tilde{t}\right) dt^\gamma =$$

$$= \left(z\sum_\rho E^\rho c_\rho^{\alpha\sigma} c_{\sigma\gamma}^\nu \partial_\nu \tilde{t} + \sum_\sigma \left(\frac{d+1}{2} - q_\sigma\right) c_\gamma^{\alpha\sigma}\partial_\sigma\tilde{t}\right) dt^\gamma.$$

On the other side, using the equation (H.10) we obtain

$$\partial_\gamma\left(z\partial_z + \frac{d-3}{2}\right)\partial_\epsilon\tilde{t} = z^2 c_{\epsilon\gamma}^\sigma U_\sigma^\nu \partial_\nu\tilde{t} + z c_{\epsilon\gamma}^\sigma\partial_\sigma\tilde{t} + z\sum_\sigma c_{\epsilon\gamma}^\sigma(1-q_\sigma)\partial_\sigma\tilde{t} + z\frac{d-3}{2}c_{\epsilon\gamma}^\sigma\partial_\sigma\tilde{t} =$$

$$= z^2 E^\rho c_{\rho\sigma}^\nu c_{\epsilon\gamma}^\sigma \partial_\nu\tilde{t} + z\sum_\sigma c_{\epsilon\gamma}^\sigma\left(\frac{d+1}{2} - q_\sigma\right)\partial_\sigma\tilde{t}.$$

Multiplying by $\eta^{\alpha\epsilon}$ and using the associativity condition

$$c_{\epsilon\gamma}^\sigma c_{\rho\sigma}^\nu = c_{\rho\epsilon}^\sigma c_{\sigma\gamma}^\nu$$

we obtain, after multiplication by dt^γ and division over z, the expression (H.12). Lemma is proved.

Proof of Proposition. For the function $x = x(t,\lambda)$ of the form (H.11) we obtain, using Lemma and integrating by parts

$$\left(dt^\alpha, \hat{\nabla}_\gamma dx\right)^* dt^\gamma = dt^\gamma \oint z^{\frac{d-3}{2}} e^{-\lambda z}\left(dt^\alpha, \hat{\nabla}_\gamma d\tilde{t}\right)^* dz$$

$$= d_t \oint z^{\frac{d-3}{2}} e^{-\lambda z} < dt^\alpha, \partial_z d\tilde{t} + \frac{d-3}{2z} d\tilde{t} >^* dz$$

$$= d_t\left\{\oint\left(\lambda z^{\frac{d-3}{2}} e^{-\lambda z} - \frac{d-3}{2} z^{\frac{d-5}{2}} e^{-\lambda z}\right) < dt^\alpha, d\tilde{t} >^* dz\right.$$

$$\left. + \frac{d-3}{2}\oint z^{\frac{d-5}{2}} e^{-\lambda z} < dt^\alpha, d\tilde{t} >^* dz\right\}$$

$$= \lambda d_t \oint z^{\frac{d-3}{2}} e^{-\lambda z} < dt^\alpha, d\tilde{t} >^* dz = \lambda\eta^{\alpha\epsilon}\partial_\epsilon\partial_\gamma x\, dt^\gamma.$$

So x satisfies the differential equation (H.9). Proposition is proved.

We study now the monodromy of our generalized hypergeometric equation (G.6) in a neiborghood of a semisimple point $t \in M$. First we rewrite the differential equations (3.5) and (H.9) and the integral transform (H.11) in the canonical coordinates u^i.

Proposition H.2. *Let $x = x(t,\lambda)$ be a flat coordinate of the metric (H.1). Put*

$$\phi_i(u,\lambda) := \partial_i x(t,\lambda)/\sqrt{\eta_{ii}(u)}, \quad t = t(u). \qquad (H.13)$$

The vector-function $\phi = (\phi_1, \ldots, \phi_n)^T$, $\phi_i = \phi_i(u, \lambda)$ *satisfies the system*

$$\partial_j \phi_i = \gamma_{ij} \phi_j, \;\; i \neq j \tag{H.14a}$$

$$\sum_{k=1}^{n} (u^k - \lambda) \partial_k \phi_i = -\frac{1}{2} \phi_i \tag{H.14b}$$

and also the following differential equation in λ

$$(\lambda \cdot 1 - U) \frac{d}{d\lambda} \phi = -\left(\frac{1}{2} \cdot 1 + V(u) \right) \phi \tag{H.15}$$

where $U = \operatorname{diag}(u^1, \ldots, u^n)$ *and* $V(u) = \big((u^j - u^i)\gamma_{ij}(u)\big)$ *(cf. (3.156) above).*

If $\psi = (\psi_1, \ldots, \psi_n)^T$ *is a solution to the linear system (3.118), (3.122) then*

$$\phi(u, \lambda) = \oint e^{-\lambda z} \psi(u, z) \frac{dz}{\sqrt{z}}$$

satisfies the system (H.9).

The proof is omitted.

In the semisimple case also the system (H.15) will be called generalized hypergeometric equation associated to the Frobenius manifold.

For the example $n = 2$ the substitution

$$\phi_1(u_1, u_2, \lambda) = \frac{1}{\sqrt{\lambda - u_1}} \psi_1(t), \quad \phi_2(u_1, u_2, \lambda) = \frac{1}{\sqrt{\lambda - u_2}} \psi_2(t)$$

$$\lambda = u_1 + t(u_2 - u_1)$$

reduces the system (H.15) to a very elementary particular case of the Gauss equation

$$t(t-1) \frac{d^2 \psi_1}{dt^2} + \left(t - \frac{1}{2} \right) \frac{d\psi_1}{dt} - \mu^2 \psi_1 = 0,$$

$\mu = -d/2$. The solutions are expressed via elementary functions

$$\psi_1 = a \cos(\mu \arccos t) + b \sin(\mu \arccos t)$$

for arbitrary constants a, b.

We consider now the reduced case (see above Appendix B) where $0 \leq q_\alpha \leq d < 1$. In this case instead of the loop integrals (H.11) (or (H.16)) it's better to use more convenient Laplace integrals. We will use these Laplace integrals to express the monodromy of our generalized hypergeometric equation in terms of the Stokes matrix of the Frobenius manifold. This will establish a relation between the Stokes matrix and the monodromy group of Frobenius manifold with $d < 1$.

Let $\Psi(u,z) = (\psi_{ia}(u,z))$, $i, a = 1, \ldots, n$ be a solution of the equation (3.122) analytic in a half-plane. Let us assume that

$$0 \leq q_\alpha \leq d < 1. \tag{H.17}$$

We construct the functions $x_a(u,\lambda)$ taking the Laplace transform of these solutions:

$$\partial_i x_a(u,\lambda) = \sqrt{\eta_{ii}(u)}\hat{\psi}_{ia}(u,\lambda) \tag{H.18a}$$

where

$$\hat{\psi}_{ia}(u,\lambda) := \frac{1}{\sqrt{-2\pi}}\int_0^\infty e^{-\lambda z}\psi_{ia}(u,z)\frac{dz}{\sqrt{z}}. \tag{H.18b}$$

We can normalize them uniquelly by the homogeneity requirement

$$\left(\lambda\frac{d}{d\lambda} - \mathcal{L}_E\right)x_a(u,\lambda) = \frac{1-d}{2}x_a(u,\lambda). \tag{H.19}$$

Theorem H.1. *Functions $x_a(u,\lambda)$ are flat coordinates of the deformed metric (H.1).*

Proof coincides with the proof of Proposition H.2 (due to the inequalities (H.17) the boundary terms at $z = 0$ vanish).

Corollary H.2. *The intersection form*

$$g^{ab} := (dx_a(u,\lambda), dx_b(u,\lambda))^*_\lambda = \sum_{i=1}^n (u^i - \lambda)\hat{\psi}_{ia}(u,\lambda)\hat{\psi}_{ib}(u,\lambda) \tag{H.20}$$

does not depend on λ neither on u.

The coordinates $x_a(u,\lambda)$ are multivalued analytic functions of λ. They have also singularities at the points $\lambda = u^i$. The monodromy of these functions coincide with the monodromy of the differential operator (H.15) with regular singular points at $\lambda = u^i$, $i = 1, \ldots, n$ and $\lambda = \infty$. We will calculate now this monodromy in terms of the Stokes matrix of the original operator.

To calculate the monodromy I will use the following elementary way of analytic continuation of Laplace transforms of a function analytic in a halfplane.

Lemma H.4. *Let the function $\psi(z)$ be analytic in the right halfplane and*

$$\begin{aligned} |\psi(z)| &\to 1 \text{ for } z \to \infty \\ z|\psi(z)| &\to 0 \text{ for } |z| \to 0 \end{aligned} \tag{H.21}$$

uniformly in the sector $-\frac{\pi}{2} + \epsilon \leq \arg z \leq \frac{\pi}{2} - \epsilon$ for arbitrary small $\epsilon > 0$. Then the Laplace transform

$$\hat{\psi}(\lambda) := \int_0^\infty e^{-\lambda z}\psi(z)\,dz \tag{H.22}$$

can be analytically continued in the complex λ-plane with a cut along the negative real half-line.

Proof. $\hat{\psi}(\lambda)$ is an analytic function in the right half-plane $\mathrm{Re}\lambda > 0$. Let us show that for these λ the equality

$$\hat{\psi}(\lambda) = \int_0^\infty e^{-\lambda z e^{i\alpha}} \psi(z e^{i\alpha})\, d(z e^{i\alpha}) \tag{H.23}$$

holds true for any α such that

$$-\frac{\pi}{2} - \arg\lambda < \alpha < \frac{\pi}{2} - \arg\lambda.$$

Indeed, let us consider the contour integral

$$\oint_C e^{-\lambda z} \psi(z)\, dz.$$

Integrals along the arcs tend to zero when $r \to 0$, $R \to \infty$ (see Fig.15). In the limit we obtain (H.23).

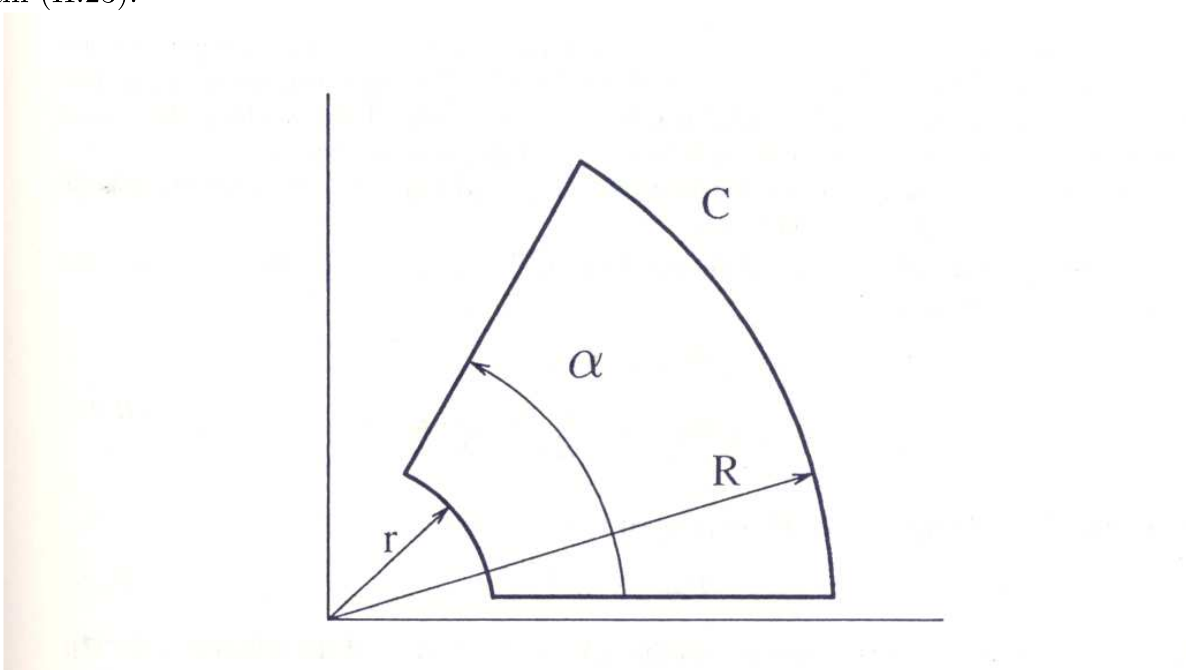

Fig.15

Now observe that the r.h.s. of (H.23) is analytic in the halfplane

$$-\frac{\pi}{2} - \alpha < \arg\lambda < \frac{\pi}{2} - \alpha.$$

Varying α from $-\frac{\pi}{2} + 0$ to $\frac{\pi}{2} - 0$ we obtain the needed analytic continuation.

Let us fix some oriented line $l = l_+ \cup l_-$ not containing the separating rays of the operator (3.120). Let $\Psi^{\mathrm{right}\,/\mathrm{left}}(u, z) = \left(\psi_{ia}^{\mathrm{right}\,/\mathrm{left}}(u, z)\right)$ be the canonical solutions (3.128) of (3.122) in the corresponding half-planes. Their Laplace transforms will be defined by the integrals

$$\hat{\psi}_{ia}^{\text{right}}(u,\lambda) = \frac{1}{\sqrt{-2\pi}} \int_{il_-} e^{-\lambda z} \psi_{ia}^{\text{right}}(u,z) \frac{dz}{\sqrt{z}} \tag{H.24}$$

(analytic function outside the cut $u_a + i\bar{l}_+$) where we chose the branch of $\sqrt{z}$ with the cut along l_-, and

$$\hat{\psi}_{ia}^{\text{left}}(u,\lambda) = \frac{1}{\sqrt{-2\pi}} \int_{il_+} e^{-\lambda z} \psi_{ia}^{\text{left}}(u,z) \frac{dz}{\sqrt{z}} \tag{H.25}$$

(analytic in λ outside the cut $u_a + i\bar{l}_+$). By $x_a^{\text{right}}(u,\lambda)$ and $x_a^{\text{left}}(u,\lambda)$ we denote the corresponding coordinates (H.18). By

$$A := S + S^T \tag{H.26}$$

we denote the symmetrized Stokes matrix. This matrix does not degenerate due to inequalities (H.17).

Note that the rays $u_1 + i\bar{l}_+, \ldots, u_n + i\bar{l}_+$ are pairwise distinct (this is equivalent to nonintersecting of the Stokes rays (F.2) with the line l). We chose generators g_a in the fundamental group $\pi_1(\mathbf{C} \setminus (u^1 \cup \ldots \cup u^n))$ taking the loops going from ∞ along these rays to u^a then around $\lambda = u^a$ and then back to infinity along the same ray.

The monodromy group of the differential equation (H.15) w.r.t. the chosen basis of the fundamental group is described by

Theorem H.2. *Monodromy of the functions $x_1^{\text{right}}(u,\lambda), \ldots, x_n^{\text{right}}(u,\lambda)$ around the point $\lambda = u_b$ is the reflection*

$$\begin{aligned} &x_a^{\text{right}}(u,\lambda) \mapsto x_a^{\text{right}}(u,\lambda) \text{ for } a \neq b \\ &x_b^{\text{right}}(u,\lambda) \mapsto x_b^{\text{right}}(u,\lambda) - \sum_{a=1}^{n} A_{ba} x_a^{\text{right}}(u,\lambda). \end{aligned} \tag{H.27}$$

Remark. Monodromy at infinity is specified by the matrix

$$-T = -S^T S^{-1}. \tag{H.28}$$

This is in agreement with the Coxeter identity [34] for the product of the reflections (H.27):

$$R_1 \ldots R_n = -S^T S^{-1} \tag{H.29}$$

where R_b is the matrix of the reflection (H.27).

Proof (cf. [14]). When λ comes clockwise/counter-clockwise to the cut $u_a + i\bar{l}_-$ the ray of integration in the Laplace integral (H.24) for $b = a$ (only!) rotates counter-clockwise/clockwise to l_+/l_-. To continue the integrals through the cut we express them via $x_b^{\text{left}}(u,\lambda)$ using the formula (3.131). Since the functions $x_b^{\text{left}}(u,\lambda)$ and the functions

$x_a^{\rm right}(u,\lambda)$ for $a \neq b$ have no jump on the cut $u_a + i\bar{l}_-$ we obtain the monodromy transformation

$$x_b^{\rm right} + \sum_{R_{ab}\subset P_{\rm right}} s_{ab} x_a^{\rm right} \mapsto -\left(x_b^{\rm right} + \sum_{R_{ab}\subset P_{\rm left}} s_{ba} x_a^{\rm right} \right)$$

the sign "$-$" is due to the change of the branch of $\sqrt{z}$ when the ray of the integration is moving through l_-. This coincides with (H.27). Theorem is proved.

We can construct another system of flat coordinates using the Laplace transform of the columns of the matrix $\Psi_0(u,z) = \Psi_{\rm right}(u,z)C$. I recall that the matrix C consists of the eigenvectors of $S^T S^{-1}$

$$S^T S^{-1} C = C e^{2\pi i \mu}. \tag{H.30}$$

Here we introduce a diagonal matrix

$$\mu = {\rm diag}\,(\mu_1, \dots, \mu_n). \tag{H.31}$$

The numbers $\mu_\alpha = q_\alpha - \frac{d}{2}$ are ordered in such a way that

$$\mu_\alpha + \mu_{n-\alpha+1} = 0. \tag{H.32}$$

This gives a useful identity

$$\mu\eta + \eta\mu = 0. \tag{H.33}$$

Note that the case $0 \leq \, q_\alpha \leq \, d < 1$ corresponds to

$$-\frac{1}{2} < \mu_\alpha < \frac{1}{2}. \tag{H.34}$$

We normalize the eigenvectors (H.30) in such a way that the entries of the matrix $\Psi_0(u,z) = (\psi_{0\,i\alpha}(u,z))$ have the following expansions near the origin

$$\psi_{0\,i\alpha}(u,z) = z^{\mu_\alpha}\left(\psi_{i\alpha}(u) + O(z)\right) \text{ when } z\to 0,\ z\in P_{\rm right}. \tag{H.35}$$

We continue analytically the matrix $\Psi_0(u,z)$ in the left half-plane $z \in P_{\rm left}$ with a cut along the ray il_+.

Lemma H.5. *Under the assumption (H.34) and the normalization (H.35) the matrix C satisfies the relations*

$$C\eta e^{\pi i \mu} C^T = S, \quad C\eta e^{-\pi i\mu} C^T = S^T. \tag{H.36}$$

Proof. We use the identity

$$\Psi_{\rm right}(u,z)\Psi_{\rm left}^T(u,-z) = 1 \tag{H.37}$$

(see above). Let z belong to the sector from il_- to l_+. Then $-z = ze^{-\pi i}$ belongs to the sector from il_+ to l_-. For such z we have

$$\Psi_0(u, ze^{-\pi i}) = \Psi_{\text{left}}(u, -z)S^{-T}C. \tag{H.38}$$

Substituting (H.38) and (H.33) to (H.37), we obtain

$$\Psi_0(u, z)C^{-1}SC^{-T}\Psi_0^T(u, ze^{-\pi i}) = 1,$$

or

$$C^{-1}SC^{-T} = \Psi_0^{-1}(u, z)\Psi_0^{-T}(u, ze^{-\pi i}). \tag{H.39}$$

Let z tend to zero keeping it within the sector from il_- to l_+. Using the identity

$$\Psi^{-1}(u)\Psi^{-T}(u) \equiv \eta,$$

where

$$\Psi(u) := (\psi_{i\alpha}(u))$$

in the leading term in z we obtain from (H.39) and (H.35)

$$C^{-1}SC^{-T} = z^{-\mu}\eta z^{-\mu}e^{\pi i\mu} = \eta e^{\pi i\mu}$$

and we can truncate the terms $O(z)$ off the expansion due to (H.34). This proves the first of the equations (H.36). Transposing this equation and applying again the identity (H.33) we obtain the second equation (H.36). Lemma is proved.

We introduce the coordinates $y_\alpha(u, \lambda)$ such that

$$\partial_i y_\alpha(u, \lambda) = \sqrt{\eta_{ii}(u)}\int_{il_-} e^{-\lambda z}\psi_{0\,i\alpha}(u, z)\frac{dz}{\sqrt{z}} \tag{H.40}$$

normalizing them as in (H.19). They are still flat coordinates of the metric $(\ ,\ )^* - \lambda <\ ,\ >^*$. We calculate now the matrix of the corresponding covariant metric in these coordinates.

Lemma H.6. *In the coordinates $y_\alpha(u, \lambda)$ the metric $g_{\alpha\beta}(u, \lambda) := \left(g^{\alpha\beta}(u) - \lambda\eta^{\alpha\beta}\right)^{-1}$ is a constant matrix of the form*

$$-\frac{1}{\pi}\eta\cos\pi\mu. \tag{H.41}$$

Proof. The contravariant metric $g^{\alpha\beta} - \lambda\eta^{\alpha\beta}$ in the coordinates (H.40) has the matrix

$$\hat{g}^{\alpha\beta} = \sum_{i=1}^n (u^i - \lambda)\int_{il_-} e^{-\lambda z}\psi_{0\,i\alpha}(u, z)\frac{dz}{\sqrt{z}}\int_{il_-} e^{-\lambda z}\psi_{0\,i\beta}(u, z)\frac{dz}{\sqrt{z}}. \tag{H.42}$$

The matrix is λ-independent. So we can calculate it taking the asymptotic with $\lambda \to \infty$. From (H.35) we obtain asymptotically

$$\int_{il_-} e^{-\lambda z}\psi_{0\,i\alpha}(u, z)\frac{dz}{\sqrt{z}} \simeq \Gamma\left(\frac{1}{2} + \mu_\alpha\right)\lambda^{-\left(\frac{1}{2}+\mu_\alpha\right)}\psi_{i\alpha}(u). \tag{H.43}$$

So

$$\hat{g}^{\alpha\beta} = -\lambda\Gamma\left(\frac{1}{2}+\mu_\alpha\right)\Gamma\left(\frac{1}{2}+\mu_\beta\right)\lambda^{-(1+\mu_\alpha+\mu_\beta)}\eta_{\alpha\beta} =$$

$$-\delta_{\alpha+\beta,n+1}\Gamma\left(\frac{1}{2}+\mu_\alpha\right)\Gamma\left(\frac{1}{2}-\mu_\alpha\right) = -\frac{\pi\delta_{\alpha+\beta,n+1}}{\cos\pi\mu_\alpha}. \qquad (H.44)$$

Taking the inverse matrix, we obtain (H.41). Lemma is proved.

Corollary H.3. *In the coordinates x_a^{right} the intersection form has the matrix*

$$G = A = S + S^T. \qquad (H.45)$$

Proof. From (H.30) we obtain the transformation of the coordinates

$$(y_1, \ldots, y_n) = \sqrt{-2\pi}(x_1, \ldots, x_n)C. \qquad (H.46)$$

Here we denote $x_a = x_a^{\text{right}}$. So the matrix G has the form, due to Lemma H.6

$$G = 2C\eta\cos\pi\mu C^T = S + S^T.$$

Corollary is proved.

The corollary shows that x_1^{right}, ..., x_n^{right} are the coordinates w.r.t. the basis of the root vectors of the system of generating reflections (H.27). This establishes a relation of the monodromy of the differential operator (3.120) to the monodromy group of the Frobenius manifold.

I recall that the root vectors e_1, ..., e_n of the system of reflections R_1, ..., R_n are defined by the following two conditions

$$R_i e_i = -e_i,\ i = 1, \ldots, n$$

$$(e_i, e_i) = 2.$$

The reflection R_i in the basis of the root vectors acts as

$$R_i(e_j) = e_j - (e_i, e_j)e_i.$$

The matrix

$$A_{ij} := (e_i, e_j)$$

is called *Coxeter matrix* of the system of reflections. For the monodromy group of our generalized hypergeometric equation (H.15) the Coxeter matrix coincides with the symmetrized Stokes matrix (H.45). The assumption $d < 1$ is equivalent to nondegenerateness of the symmetrized Stokes matrix.

For the finite Coxeter groups (see Lecture 4 below) the system of generating reflections can be chosen in such a way that all A_{ij} are nonpositive. For the particular subclass of

Weyl groups of simple Lie algebras the Coxeter matrix coincides with the symmetrized Cartan matrix of the Lie algebra.

The coordinates $y_1, \ldots, y_n$ are dual to the basis of eigenvectors of the *Coxeter transformation* $R_1 \ldots R_n$ due to the formula (H.29). The basis of the eigenvectors $f_1, \ldots, f_n$ of the Coxeter transform due to (H.41) is normalized as

$$(f_\alpha, f_\beta) = -\frac{1}{\pi} \cos \pi \mu_\beta \delta_{\alpha+\beta, n+1}. \qquad (H.47)$$

At the end of this Appendix we consider an application of the integral formula (H.11) to computation of the flat coordinates of the intersection form on a trivial Frobenius manifold. In this case we have a linear contravariant metric (3.37) parametrized by a graded Frobenius algebra A. The gradings q_α of the basic vectors e_α of the algebra are determined up to a common nonzero factor

$$q_\alpha \mapsto \kappa q_\alpha, \quad d \mapsto \kappa d.$$

The normalized flat coordinates $\tilde{t}_\alpha(t, z)$ of the deformed connection can be found easily ((3.5) is an equation with constant coefficients)

$$\tilde{t}_\alpha(t, z) = z^{q_\alpha - d} < e_\alpha, e^{z\mathbf{t}} - 1 >, \quad \alpha = 1, \ldots, n \qquad (H.48)$$

for

$$\mathbf{t} = t^\alpha e_\alpha \in A.$$

So the integral formula (H.11) for the flat coordinates of the pencil (H.1) reads

$$x_\alpha(t, \lambda) =< e_\alpha, \int z^{-\frac{3}{2}+\mu_\alpha} \left(e^{z(\mathbf{t}-\lambda)} - e^{-\lambda z} \right) dz >=$$

$$= \Gamma\left(-\frac{1}{2} + \mu_\alpha\right) < e_\alpha, (\lambda - \mathbf{t})^{\frac{1}{2}-\mu_\alpha} > \qquad (H.49)$$

for $\mu_\alpha = q_\alpha - \frac{d}{2}$ if $\mu_\alpha \neq \frac{1}{2}$. For $\lambda = 0$ renormalizing (H.49) we obtain the flat coordinates $x_\alpha(t)$ of the intersection form (3.37)

$$x_\alpha(t) =< e_\alpha, \mathbf{t}^{\frac{1}{2}-\mu_\alpha} >, \quad \alpha = 1, \ldots, n. \qquad (H.50)$$

The r.h.s. is a polynomial in $t^2, \ldots, t^n$ but it ramifies as a function of t^1.

Particularly, for an arbitrary Frobenius algebra A with the trivial grading $q_\alpha = d = 0$ the intersection form (3.37) coincides with the metric on the dual space A^* introduced by Balinski and Novikov [12]. The formula (H.50) in this case gives the quadratic transformation of ref. [12] to the flat coordinates $x_1, \ldots, x_n$ of the metric

$$\mathbf{t} = \mathbf{x}^2, \quad \mathbf{x} = x^\alpha e_\alpha \in A. \qquad (H.51)$$

Remark H.1. Inverting the Laplace-type integrals (H.11) and integrating by parts we arrive to an integral representation of the deformed coordinates $\tilde{t}(t, z)$ via the flat coordinates of the intersection form (therefore, via solutions of our generalized hypergeometric equation). This gives "oscillatory integrals" for the solutions of (3.5), (3.6):

$$\tilde{t}(t, z) = -z^{\frac{1-d}{2}} \oint e^{z\lambda(x,t)} dx \qquad (H.52)$$

where $\lambda = \lambda(x, t)$ is a function inverse to $x = x(t, \lambda) = x(t^1 - \lambda, t^2, \ldots, t^n)$ for a flat coordinate $x(t)$ of the intersection form.

Appendix I.

Determination of a superpotential of a Frobenius manifold.

In this Appendix we will show that *any* irreducible massive Frobenius manifold with $d < 1$ can be described by the formulae (2.94) for some LG superpotential $\lambda(p;t)$. The superpotential will always be a function of *one* variable p (may be, a multivalued one) depending on the parameters $t = (t^1, \dots, t^n)$.

We first construct a function $\lambda(p;t)$ for any massive Frobenius manifold such that the critical values of it are precisely the canonical coordinates on the Frobenius manifold

$$u^i(t) = \lambda(q^i(t);t), \quad \frac{d\lambda}{dp}|_{p=q^i(t)} = 0, \quad i = 1, \dots, n. \tag{I.1}$$

For the construction we will use the flat coordinates of the flat pencil (H.1) of metrics on M. I recall that these can be represented as

$$x_a(\lambda; t^1, \dots, t^n) = x_a(t^1 - \lambda, t^2, \dots, t^n) \tag{I.2}$$

where $x_a(t)$ are flat coordinates of the intersection form.

Due to Lemma G.1 the coordinates $x_a(t^1 - \lambda, t^2, \dots, t^n)$ are analytic in t and λ outside of the locus

$$\Delta(t^1 - \lambda, t^2, \dots, t^n) = 0. \tag{I.3}$$

On the semisimple part of the locus we have

$$\lambda = u^i(t) \tag{I.4}$$

for some i. Near such a point $x_a(t^1 - \lambda, t^2, \dots, t^n)$ is analytic in $\sqrt{\lambda - u^i(t)}$.

Let us fix some $t_0 \in M \setminus (\Sigma \cup \Sigma_{\rm nil})$ and some a between 1 and n such that

$$\partial_1 x_a(t_0^1 - u^i(t_0^1, t_0^2, \dots, t_0^n), t_0^2, \dots, t_0^n) \neq 0 \tag{I.5}$$

for any $i = 1, \dots, n$ (such a exists since x_1, ..., x_n are local coordinates) and put

$$p = p(\lambda, t) := x_a(t^1 - \lambda, t^2, \dots, t^n). \tag{I.6}$$

By $\lambda = \lambda(p,t)$ we denote the inverse function.

Proposition I.1. *For t close to t_0 the critical points of the function $\lambda(p,t)$ are*

$$q^i = p(u^i(t), t), \quad i = 1, \dots, n. \tag{I.7}$$

The corresponding critical values equal $u^i(t)$.

Proof. Near $\lambda = u^i(t_0)$ we have

$$x_a = q^i + x_a^1 \sqrt{\lambda - u^i} + O(\lambda - u^i)$$

and $x_a^1 \neq 0$ by the assumption. For the inverse function we have locally

$$\lambda = u^i(t) + (x_a^1)^{-1}(p - q^i)^2 + o\left((p - q^i)^2\right). \qquad (I.8)$$

This proves that λ has the prescribed critical values.

Let us assume now that $d < 1$. In this case we will construct a particular flat coordinate $p = p(t)$ of the intersection form such that the function inverse to $p = p(t, \lambda) = p(t^1 - \lambda, t^2, \ldots, t^n)$ is the LG superpotential of the Frobenius manifold.

I will use the flat coordinates $x_a^{\text{right}}(u, \lambda)$ constructed in the previous Appendix.

Lemma I.1. *For $\lambda \to u_i$*

$$x_a^{\text{right}}(u, \lambda) = q_{ai}(u) + \delta_{ai}\sqrt{2\eta_{ii}(u)}\sqrt{u_i - \lambda} + O(u_i - \lambda) \qquad (I.9)$$

for some functions $q_{ai}(u)$.

Proof. Using the asymptotic

$$\psi_{ia}^{\text{right}}(u, z) = \left[\delta_{ai} + O\left(\frac{1}{z}\right)\right] e^{z u_i}$$

for $z \to \infty$ we obtain (I.9). Lemma is proved.

We consider now the following particular flat coordinate

$$p = p(t; \lambda) := \sum_{a=1}^{n} x_a^{\text{right}}(u, \lambda). \qquad (I.10)$$

By $\lambda = \lambda(p; t)$ we denote, as above, the inverse function.

Theorem I.1. *For the metrics $< \, , \, >$, $(\, , \,)$ and for the trilinear form (1.46) the following formulae hold true*

$$< \partial', \partial'' >_t = -\sum_{i=1}^{n} \operatorname*{res}_{p=q^i} \frac{\partial'(\lambda(p,t)dp)\, \partial''(\lambda(p,t)dp)}{d\lambda(p,t)} \qquad (I.11)$$

$$(\partial', \partial'')_t = -\sum_{i=1}^{n} \operatorname*{res}_{p=q^i} \frac{\partial'(\log\lambda(p,t)dp)\, \partial''(\log\lambda(p,t)dp)}{d\log\lambda(p,t)} \qquad (I.12)$$

$$c(\partial', \partial'', \partial''')_t = -\sum_{i=1}^{n} \operatorname*{res}_{p=q^i} \frac{\partial'(\lambda(p,t)dp)\, \partial''(\lambda(p,t)dp)\, \partial'''(\lambda(p,t)dp)}{dp\, d\lambda(p,t)}. \qquad (I.13)$$

In these formulae

$$d\lambda := \frac{\partial\lambda(p,t)}{\partial p}dp, \quad d\log\lambda := \frac{\partial\log\lambda(p,t)}{\partial p}dp.$$

Proof. From (I.9) we obtain

$$p(\lambda, t) = \sum_a x_a^{\text{right}}(u, \lambda) = q_i(u) + \sqrt{2\eta_{ii}(u)}\sqrt{u_i - \lambda} + O(u_i - \lambda) \qquad (I.14)$$

near $\lambda = u_i$ where we put

$$q_i(u) := \sum_a q_{ai}(u). \qquad (I.15)$$

So $q_i(u)$ is a critical point of λ with the critical value u_i. Near this point

$$\lambda = u_i - \frac{(p - q_i)^2}{2\eta_{ii}(u)} + O(p - q_i)^3. \qquad (I.16)$$

From this formula we immediately obtain that for $\partial' = \partial_i$, $\partial'' = \partial_j$ the r.h.s. of the formula (I.11) is equal to

$$\eta_{ii}(u)\delta_{ij} =< \partial_i, \partial_j >_u .$$

This proves the equality (I.11). The other equalities are proved in a similar way. Theorem is proved.

Example I.1. Using the flat coordinates from Example G.2 we obtain the LG superpotential of the CP^1-model (see Lecture 2)

$$\lambda(p; t_1, t_2) = t_1 - 2e^{\frac{t_2}{2}} \cos p. \qquad (I.17)$$

The cosine is considered as an analytic function on the cylinder $p \simeq p + 2\pi$, so it has only 2 critical points $p = 0$ and $p = \pi$.

Other examples will be considered in Lectures 4 and 5.

Lecture 4.

Frobenius structure on the space of orbits of a Coxeter group.

Let W be a *Coxeter group*, i.e. a finite group of linear transformations of real n-dimensional space V generated by reflections. In this Lecture we construct Frobenius manifolds whose monodromy is a given Coxeter group W. All of these Frobenius structures will be polynomial. The results of Appendix A suggest that the construction of this Lecture gives all the polynomial solution of WDVV with $d < 1$ satisfying the semisimplicity assumption, although this is still to be proved.

We always can assume the transformations of the group W to be orthogonal w.r.t. a Euclidean structure on V. The complete classification of irreducible Coxeter groups was obtained in [33]; see also [22]. The complete list consists of the groups (dimension of the space V equals the subscript in the name of the group) A_n, B_n, D_n, E_6, E_7, E_8, F_4, G_2 (the Weyl groups of the corresponding simple Lie algebras), the groups H_3 and H_4 of symmetries of the regular icosahedron and of the regular 600-cell in the 4-dimensional space resp. and the groups $I_2(k)$ of symmetries of the regular k-gone on the plane. The group W also acts on the symmetric algebra $S(V)$ (polynomials of the coordinates of V) and on the $S(V)$-module $\Omega(V)$ of differential forms on V with polynomial coefficients. The subring $R = S(V)^W$ of W-invariant polynomials is generated by n algebraicaly independent homogeneous polynomials y^1, ..., y^n [22]. The submodule $\Omega(V)^W$ of the W-invariant differential forms with polynomial coefficients is a free R-module with the basis $dy^{i_1} \wedge ... \wedge dy^{i_k}$ [22]. Degrees of the basic invariant polynomials are uniquely determined by the Coxeter group. They can be expressed via the *exponents* m_1, ..., m_n of the group, i.e. via the eigenvalues of a Coxeter element C in W [22]

$$d_i := \deg y^i = m_i + 1, \tag{4.1a}$$

$$\{\text{eigen } C\} = \{\exp \frac{2\pi i(d_1 - 1)}{h}, ..., \exp \frac{2\pi i(d_n - 1)}{h}\}. \tag{4.1b}$$

The maximal degree h is called *Coxeter number* of W. I will use the reversed ordering of the invariant polynomials

$$d_1 = h > d_2 \geq ... \geq d_{n-1} > d_2 = 2. \tag{4.2}$$

The degrees satisfy the *duality condition*

$$d_i + d_{n-i+1} = h + 2, \ i = 1, ..., n. \tag{4.3}$$

The list of the degrees for all the Coxeter groups is given in Table 1.

W	$d_1, \ldots, d_n$
A_n	$d_i = n + 2 - i$
B_n	$d_i = 2(n - i + 1)$
$D_n, \ n = 2k$	$d_i = 2(n - i), \ i \leq k,$
	$d_i = 2(n - i + 1), \ k + 1 \leq i$
$D_n, \ n = 2k + 1$	$d_i = 2(n - i), \ i \leq k,$
	$d_{k+1} = 2k + 1,$
	$d_i = 2(n - i + 1), \ k + 2 \leq i$
E_6	12, 9, 8, 6, 5, 2
E_7	18, 14, 12, 10, 8, 6, 2
E_8	30, 24, 20, 18, 14, 12, 8, 2
F_4	12, 8, 6, 2
G_2	6, 2
H_3	10, 6, 2
H_4	30, 20, 12, 2
$I_2(k)$	k, 2

Table 1.

I will extend the action of the group W to the complexified space $V \otimes \mathbf{C}$. The space of orbits

$$M = V \otimes \mathbf{C}/W$$

has a natural structure of an affine algebraic variety: the coordinate ring of M is the (complexified) algebra R of invariant polynomials of the group W. The coordinates y^1, ..., y^n on M are defined up to an invertible transformation

$$y^i \mapsto y^{i'}(y^1, ..., y^n), \tag{4.4}$$

where $y^{i'}(y^1, ..., y^n)$ is a graded homogeneous polynomial of the same degree d_i in the variables y^1, ..., y^n, $\deg y^k = d_k$. Note that the Jacobian $\det(\partial y^{i'}/\partial y^j)$ is a constant (it should not be zero). The transformations (4.4) leave invariant the vector field $\partial_1 := \partial/\partial y^1$ (up to a constant factor) due to the strict inequality $d_1 > d_2$. The coordinate y^n is determined uniquely within a factor. Also the vector field

$$E = \frac{1}{h}\left(d_1 y^1 \partial_1 + ... + d_n y^n \partial_n\right) = \frac{1}{h} x^a \frac{\partial}{\partial x^a} \tag{4.5}$$

(the generator of scaling transformations) is well-defined on M. Here we denote by x^a the coordinates in the linear space V.

Let (,) denotes the W-invariant Euclidean metric in the space V. I will use the orthonormal coordinates x^1, ..., x^n in V with respect to this metric. The invariant y^n can be chosen as

$$y^n = \frac{1}{2h}((x^1)^2 + ... + (x^n)^2). \tag{4.6}$$

We extend $(\, , \,)$ onto $V \otimes \mathbf{C}$ as a complex quadratic form.

The factorization map $V \otimes \mathbf{C} \to M$ is a local diffeomorphism on an open subset of $V \otimes \mathbf{C}$. The image of this subset in M consists of *regular orbits* (i.e. the number of points of the orbit equals $\# W$). The complement is the *discriminant locus* $\mathrm{Discr}\, W$. By definition it consists of all irregular orbits. Note that the linear coordinates in V can serve also as local coordinates in small domains in $M \setminus \mathrm{Discr}\, W$. It defines a metric $(\, , \,)$ (and $(\, , \,)^*$) on $M \setminus \mathrm{Discr}\, W$. The contravariant metric can be extended onto M according to the following statement (cf. [134, Sections 5 and 6]).

Lemma 4.1. *The Euclidean metric of V induces polynomial contravariant metric $(\, , \,)^*$ on the space of orbits*

$$g^{ij}(y) = (dy^i, dy^j)^* := \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^a} \tag{4.7}$$

and the corresponding contravariant Levi-Civita connection

$$\Gamma^{ij}_k(y) dy^k = \frac{\partial y^i}{\partial x^a} \frac{\partial^2 y^j}{\partial x^a \partial x^b} dx^b \tag{4.8}$$

also is a polynomial one.

Proof. The right-hand sides in (4.7)/(4.8) are W-invariant polynomials/differential forms with polynomial coefficients. Hence $g^{ij}(y)/\Gamma^{ij}_k(y)$ are polynomials in y^1, ..., y^n. Lemma is proved.

Remark 4.1. The matrix $g^{ij}(y)$ does not degenerate on $M \setminus \mathrm{Discr}\, W$ where the factorization $V \otimes \mathbf{C} \to M$ is a local diffeomorphism. So the polynomial (also called *discriminant* of W)

$$D(y) := \det(g^{ij}(y)) \tag{4.9}$$

vanishes precisely on the discriminant locus $\mathrm{Discr}\, W$ where the variables x^1, ..., x^n fail to be local coordinates. Due to this fact the matrix $g^{ij}(y)$ is called *discriminant matrix* of W. The contravariant metric (4.7) was introduced by V.I.Arnold [3] in the form of operation of convolution of invariants $f(x)$, $g(x)$ of a reflection group

$$f, \; g \mapsto (df, dg)^* = \sum \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial x^a}.$$

Note that the image of V in the real part of M is specified by the condition of positive semidefiniteness of the matrix $(g^{ij}(y))$ (cf. [125]). The Euclidean connection (4.8) on the space of orbits is called *Gauss - Manin connection.*

The main result of this lecture is

Theorem 4.1. *There exists a unique, up to an equivalence, Frobenius structure on the space of orbits of a finite Coxeter group with the intersection form (4.7), the Euler vector field (4.5) and the unity vector field $e := \partial / \partial y^1$.*

We start the proof of the theorem with the following statement.

Proposition 4.1. *The functions $g^{ij}(y)$ and $\Gamma^{ij}_k(y)$ depend linearly on y^1.*

Proof. From the definition one has that $g^{ij}(y)$ and $\Gamma^{ij}_k(y)$ are graded homogeneous polynomials of the degrees

$$\deg g^{ij}(y) = d_i + d_j - 2 \tag{4.10}$$

$$\deg \Gamma^{ij}_k(y) = d_i + d_j - d_k - 2. \tag{4.11}$$

Since $d_i + d_j \leq 2h = 2d_1$ these polynomials can be at most linear in y^1. Proposition is proved.

Corollary 4.1 (K.Saito) *The matrix*

$$\eta^{ij}(y) := \partial_1 g^{ij}(y) \tag{4.12}$$

has a triangular form

$$\eta^{ij}(y) = 0 \text{ for } i + j > n + 1, \tag{4.13}$$

and the antidiagonal elements

$$\eta^{i(n-i+1)} =: c_i \tag{4.14}$$

are nonzero constants. Particularly,

$$c := \det(\eta^{ij}) = (-1)^{\frac{n(n-1)}{2}} c_1 ... c_n \neq 0. \tag{4.15}$$

Proof. One has

$$\deg \eta^{ij}(y) = d_i + d_j - 2 - h.$$

Hence $\deg \eta^{i(n-i+1)} = 0$ (see (4.3)) and $\deg \eta^{ij} < 0$ for $i+j > n+1$. This proves triangularity of the matrix and constancy of the antidiagonal entries c_i. To prove nondegenerateness of $(\eta^{ij}(y))$ we consider, following Saito, the discriminant (4.9) as a polynomial in y^1

$$D(y) = c(y^1)^n + a_1(y^1)^{n-1} + ... + a_n$$

where the coefficients a_1, ..., a_n are quasihomogeneous polynomials in y^2, ..., y^n of the degrees h, $2h$, ..., nh resp. and the leading coefficient c is given in (4.15). Let γ be the eigenvector of a Coxeter transformation C with the eigenvalue $\exp(2\pi i/h)$. Then

$$y^k(\gamma) = y^k(C\gamma) = y^k(\exp(2\pi i/h)\gamma) = \exp(2\pi i d_k/h) y^k(\gamma).$$

For $k > 1$ we obtain

$$y^k(\gamma) = 0, \;\; k = 2, ..., n.$$

But $D(\gamma) \neq 0$ [22]. Hence the leading coefficient $c \neq 0$. Corollary is proved.

Corollary 4.2. *The space M of orbits of a finite Coxeter group carries a flat pencil of metrics $g^{ij}(y)$ (4.7) and $\eta^{ij}(y)$ (4.12) where the matrix $\eta^{ij}(y)$ is polynomially invertible globally on M.*

We will call (4.12) *Saito metric* on the space of orbits. This was introduced by Saito, Sekiguchi and Yano in [131] using the classification of Coxeter groups for all of them but E_7 and E_8. The general proof of flateness (not using the classification) was obtained in [130].

This metric will be denoted by $< \, , \, >^*$ (and by $< \, , \, >$ if considered on the tangent bundle TM). Let us denote by

$$\gamma_k^{ij}(y) := \partial_1 \Gamma_k^{ij}(y) \tag{4.16}$$

the components of the Levi-Civita connection for the metric $\eta^{ij}(y)$. These are quasihomogeneous polynomials of the degrees

$$\deg \gamma_k^{ij}(y) = d_i + d_j - d_k - h - 2. \tag{4.17}$$

Corollary 4.3 (K.Saito). *There exist homogeneous polynomials $t^1(x)$, ..., $t^n(x)$ of degrees d_1, ..., d_n resp. such that the matrix*

$$\eta^{\alpha\beta} := \partial_1 (dt^\alpha, dt^\beta)^* \tag{4.18}$$

is constant.

The coordinates t^1, ..., t^n on the orbit space will be called *Saito flat coordinates.* They can be chosen in such a way that the matrix (4.18) is antidiagonal

$$\eta^{\alpha\beta} = \delta^{\alpha+\beta,n+1}.$$

Then the Saito flat coordinates are defined uniquely up to an η-orthogonal transformation

$$t^\alpha \mapsto a^\alpha_\beta t^\beta,$$

$$\sum_{\lambda+\mu=n+1} a^\alpha_\lambda a^\beta_\mu = \delta^{\alpha+\beta,n+1}.$$

Proof. From flatness of the metric $\eta^{ij}(y)$ it follows that the flat coordinates $t^\alpha(y)$, $\alpha = 1$, ..., n exist at least localy. They are to be determined from the following system

$$\eta^{is} \partial_s \partial_j t + \gamma_j^{is} \partial_s t = 0 \tag{4.19}$$

(see (3.30)). The inverse matrix $(\eta_{ij}(y)) = (\eta^{ij}(y))^{-1}$ also is polynomial in y^1, ..., y^n. So rewriting the system (4.19) in the form

$$\partial_k \partial_l t + \eta_{il} \gamma_k^{is} \partial_s t = 0 \tag{4.20}$$

we again obtain a system with polynomial coefficients. It can be written as a first-order system for the entries $\xi_l = \partial_l t$,

$$\partial_k \xi_l + \eta_{il} \gamma_k^{is} \xi_s = 0, \;\; k, l = 1, ..., n \tag{4.21}$$

(the integrability condition $\partial_k \xi_l = \partial_l \xi_k$ follows from vanishing of the curvature). This is an overdetermined holonomic system. So the space of solutions has dimension n. We can choose a fundamental system of solutions $\xi_l^\alpha(y)$ such that $\xi_l^\alpha(0) = \delta_l^\alpha$. These functions are analytic in y for sufficiently small y. We put $\xi_l^\alpha(y) =: \partial_l t^\alpha(y)$, $t^\alpha(0) = 0$. The system of solutions is invariant w.r.t. the scaling transformations

$$y^i \mapsto c^{d_i} y^i, \ i = 1, ..., n.$$

So the functions $t^\alpha(y)$ are quasihomogeneous in y of the same degrees d_1, ..., d_n. Since all the degrees are positive the power series $t^\alpha(y)$ should be polynomials in y^1, ..., y^n. Because of the invertibility of the transformation $y^i \mapsto t^\alpha$ we conclude that $t^\alpha(y(x))$ are polynomials in x^1, ..., x^n. Corollary is proved.

We need to calculate particular components of the metric $g^{\alpha\beta}$ and of the corresponding Levi-Civita connection in the coordinates t^1, ..., t^n (in fact, in arbitrary homogeneous coordinates y^1, ..., y^n).

Lemma 4.2. *Let the coordinate t^n be normalized as in (4.6). Then the following formulae hold:*

$$g^{n\alpha} = \frac{d_\alpha}{h} t^\alpha \tag{4.22}$$

$$\Gamma_\beta^{n\alpha} = \frac{(d_\alpha - 1)}{h} \delta_\beta^\alpha. \tag{4.23}$$

(In the formulae there is no summation over the repeated indices!)

Proof. We have

$$g^{n\alpha} = \frac{\partial t^n}{\partial x^a} \frac{\partial t^\alpha}{\partial x^a} = \frac{1}{h} x^a \frac{\partial t^\alpha}{\partial x^a} = \frac{d_\alpha}{h} t^\alpha$$

due to the Euler identity for the homogeneous functions $t^\alpha(x)$. Furthermore,

$$\Gamma_\beta^{n\alpha} dt^\beta = \frac{\partial t^n}{\partial x^a} \frac{\partial^2 t^\alpha}{\partial x^a \partial x^b} dx^b = \frac{1}{h} x^a \frac{\partial^2 t^\alpha}{\partial x^a \partial x^b} dx^b = \frac{1}{h} x^a d\left(\frac{\partial t^\alpha}{\partial x^a}\right)$$

$$= \frac{1}{h} d\left(x^a \frac{\partial t^\alpha}{x^a}\right) - \frac{1}{h} \frac{\partial t^\alpha}{\partial x^a} dx^a = \frac{(d_\alpha - 1)}{h} dt^\alpha.$$

Lemma is proved.

We can formulate now

Main lemma. *Let t^1, ..., t^n be the Saito flat coordinates on the space of orbits of a finite Coxeter group and*

$$\eta^{\alpha\beta} = \partial_1 (dt^\alpha, dt^\beta)^* \tag{4.24}$$

be the corresponding constant Saito metric. Then there exists a quasihomogeneous polynomial $F(t)$ of the degree $2h + 2$ such that

$$(dt^\alpha, dt^\beta)^* = \frac{(d_\alpha + d_\beta - 2)}{h} \eta^{\alpha\lambda} \eta^{\beta\mu} \partial_\lambda \partial\mu F(t). \tag{4.25}$$

The polynomial $F(t)$ determines on the space of orbits a polynomial Frobenius structure with the structure constants

$$c_{\alpha\beta}^{\gamma}(t) = \eta^{\gamma\epsilon}\partial_\alpha\partial_\beta\partial_\epsilon F(t) \tag{4.26a}$$

the unity

$$e = \partial_1 \tag{4.26b}$$

the Euler vector field

$$E = \sum \frac{\deg t^\alpha}{h} t^\alpha \partial_\alpha$$

and the invariant inner product η.

Proof. Because of Corollary 4.3 in the flat coordinates the tensor $\Delta_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta}$ should satisfy the equations (D.1) - (D.3) where $g_1^{\alpha\beta} = g^{\alpha\beta}(t)$, $g_2^{\alpha\beta} = \eta^{\alpha\beta}$. First of all according to (D.1a) we can represent the tensor $\Gamma_\gamma^{\alpha\beta}(t)$ in the form

$$\Gamma_\gamma^{\alpha\beta}(t) = \eta^{\alpha\epsilon}\partial_\epsilon\partial_\gamma f^\beta(t) \tag{4.27}$$

for a vector field $f^\beta(t)$. The equation (3.27) (or, equivalently, (D.3)) for the metric $g^{\alpha\beta}(t)$ and the connection (4.27) reads

$$g^{\alpha\sigma}\Gamma_\sigma^{\beta\gamma} = g^{\beta\sigma}\Gamma_\sigma^{\alpha\gamma}.$$

For $\alpha = n$ because of Lemma 4.2 this gives

$$\sum_\sigma d_\sigma t^\sigma \eta^{\beta\epsilon}\partial_\sigma\partial_\epsilon f^\gamma = (d_\gamma - 1) g^{\beta\gamma}.$$

Applying to the l.h.s. the Euler identity (here deg $\partial_\epsilon f^\gamma = d_\gamma - d_\epsilon + h$) we obtain

$$(d_\gamma - 1) g^{\beta\gamma} = \sum_\epsilon \eta^{\beta\epsilon}(d_\gamma - d_\epsilon + h)\partial_\epsilon f^\gamma = (d_\gamma + d_\beta - 2)\eta^{\beta\epsilon}\partial_\epsilon f^\gamma. \tag{4.28a}$$

From this one obtains the symmetry

$$\frac{\eta^{\beta\epsilon}\partial_\epsilon f^\gamma}{d_\gamma - 1} = \frac{\eta^{\gamma\epsilon}\partial_\epsilon f^\beta}{d_\beta - 1}.$$

Let us denote

$$\frac{f^\gamma}{d_\gamma - 1} =: \frac{F^\gamma}{h}. \tag{4.28b}$$

We obtain

$$\eta^{\beta\epsilon}\partial_\epsilon F^\gamma = \eta^{\gamma\epsilon}\partial_\epsilon F^\beta.$$

Hence a function $F(t)$ exists such that

$$F^\alpha = \eta^{\alpha\epsilon}\partial_\epsilon F. \tag{4.28c}$$

It is clear that $F(t)$ is a quasihomogeneous polynomial of the degree $2h+2$. From the formula (4.28) one immediately obtains (4.25).

Let us prove now that the coefficients (4.26a) satisfy the associativity condition. It is more convenient to work with the dual structure constants

$$c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F.$$

Because of (4.27), (4.28) one has

$$\Gamma_\gamma^{\alpha\beta} = \frac{d_\beta - 1}{h} c_\gamma^{\alpha\beta}.$$

Substituting this in (D.2) we obtain associativity. Finaly, for $\alpha = n$ the formulae (4.22), (4.23) imply

$$c_\beta^{n\alpha} = \delta_\beta^\alpha.$$

Since $\eta^{1n} = 1$, the vector (4.26b) is the unity of the algebra. Lemma is proved.

Proof of Theorem.

Existence of a Frobenius structure on the space of orbits satisfying the conditions of Theorem 4.1 follows from Main lemma. We are now to prove uniqueness. Let us consider a polynomial Frobenius structure on M with the Euler vector field (4.5) and with the Saito invariant metric. In the Saito flat coordinates we have

$$dt^\alpha \cdot dt^\beta = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F(t) dt^\gamma.$$

The r.h.s. of (3.13) reads

$$i_E(dt^\alpha \cdot dt^\beta) = \frac{1}{h}\sum_\gamma d_\gamma t^\gamma \eta_{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F(t) = \frac{1}{h}(d_\alpha + d_\beta - 2)\eta_{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t).$$

This should be equal to $(dt^\alpha, dt^\beta)^*$. So the function $F(t)$ must satisfy (4.25). It is determined uniquely by this equation up to terms quadratic in t^α. Such an ambiguity does not affect the Frobenius structure. Theorem is proved.

We will show know that the Frobenius manifolds we have constructed satisfy the semisimplicity condition. This will follow from the following construction.

Let $R = \mathbf{C}[y_1, ..., y_n]$ be the coordinate ring of the orbit space M. The Frobenius algebra structure on the tangent planes $T_y M$ for any $y \in M$ provides the R-module *Der R* of invariant vector fields with a structure of Frobenius algebra over R. To describe this structure let us consider a homogeneous basis of invariant polynomials y_1, ..., y_n of the Coxeter group. Let $D(y_1, ..., y_n)$ be the discriminant of the group. We introduce a polynomial of degree n in an auxiliary variable u putting

$$P(u; y_1, ..., y_n) := D(y_1 - u, y_2, ..., y_n). \tag{4.29}$$

Let $D_0(y_1, ..., y_n)$ be the discriminant of this polynomial in u. It does not vanish identicaly on the space of orbits.

Theorem 4.2. *The map*

$$1 \mapsto e, \quad u \mapsto E \tag{4.30a}$$

can be extended uniquely to an isomorphism of R-algebras

$$R[u]/(P(u;y)) \to Der\, R. \tag{4.30b}$$

Corollary 4.4. *The algebra on T_yM has no nilpotents outside the zeroes of the polynomial $D_0(y_1, ..., y_n)$.*

We start the proof with an algebraic remark: let T be a n-dimensional space and $U : T \to T$ an endomorphism (linear operator). Let

$$P_U(u) := \det(U - u \cdot 1)$$

be the characteristic polynomial of U. We say that the endomorphism U is semisimple if all the n roots of the characteristic polynomial are simple. For a semisimple endomorphism there exists a cyclic vector $e \in T$ such that

$$T = \text{span}\,(e, Ue, ..., U^{n-1}e).$$

The map

$$\mathbf{C}[u]/(P_U(u)) \to T, \quad u^k \mapsto U^k e, \; k = 0, 1, ..., n-1 \tag{4.31}$$

is an isomorphism of linear spaces.

Let us fix a point $y \in M$. We define a linear operator

$$U = (U^i_j(y)) : T_yM \to T_yM \tag{4.32}$$

(being also an operator on the cotangent bundle of the space of orbits) taking the ratio of the quadratic forms g^{ij} and η^{ij}

$$< U\omega_1, \omega_2 >^* = (\omega_1, \omega_2)^* \tag{4.33}$$

or, equivalently,

$$U^i_j(y) := \eta_{js}(y) g^{si}(y). \tag{4.34}$$

Lemma 4.3. *The characteristic polynomial of the operator $U(y)$ is given up to a nonzero factor c^{-1} (4.15) by the formula (4.29).*

Proof. We have

$$P(u; y^1, \ldots, y^n) := \det(U - u \cdot 1) = \det(\eta_{js})\det(g^{si} - u\eta^{si}) =$$

$$c^{-1}\det(g^{si}(y^1 - u, y^2, \ldots, y^n) = c^{-1} D(y^1 - u, y^2, \ldots, y^n).$$

Lemma is proved.

Corollary 4.5. *The operator $U(y)$ is semisimple at a generic point $y \in M$.*

Proof. Let us prove that the discriminant $D_0(y^1, \dots, y^n)$ of the characteristic polynomial $P(u; y^1, \dots, y^n)$ does not vanish identicaly on M. Let us fix a Weyl chamber $V_0 \subset V$ of the group W. On the inner part of V_0 the factorization map

$$V_0 \to M_{Re}$$

is a diffeomorphism. On the image of V_0 the discriminant $D(y)$ is positive. It vanishes on the images of the n walls of the Weyl chamber:

$$D(y)_{i-\text{th wall}} = 0, \quad i = 1, \dots, n. \tag{4.35}$$

On the inner part of the i-th wall (where the surface (4.35) is regular) the equation (4.35) can be solved for y^1:

$$y^1 = y_i^1(y^2, \dots, y^n). \tag{4.36}$$

Indeed, on the inner part

$$(\partial_1 D(y))_{i-\text{th wall}} \neq 0.$$

This holds since the polynomial $D(y)$ has simple zeroes at the generic point of the discriminant of W (see, e.g., [4]) .

Note that the functions (4.36) are the roots of the equation $D(y) = 0$ as the equation in the unknown y^1. It follows from above that this equation has simple roots for generic y^2, ..., y^n. The roots of the characteristic equation

$$D(y^1 - u, y^2, \dots, y^n) = 0 \tag{4.37a}$$

are therefore

$$u_i = y^1 - y_i^1(y^2, \dots, y^n), \quad i = 1, \dots, n. \tag{4.37b}$$

Genericaly these are distinct. Lemma is proved.

Lemma 4.4. *The operator U on the tangent planes T_yM coincides with the operator of multiplication by the Euler vector field E.*

Proof. We check the statement of the lemma in the Saito flat coordinates:

$$\sum_\sigma \frac{d_\sigma}{h} t^\sigma c_{\sigma\beta}^\alpha = \frac{h - d_\beta + d_\alpha}{h} \eta^{\alpha\epsilon} \partial_\epsilon \partial_\beta F =$$

$$\sum_\lambda \frac{d_\lambda + d_\alpha - 2}{h} \eta_{\beta\lambda} \eta^{\alpha\epsilon} \eta^{\lambda\mu} \partial_\epsilon \partial_\mu F = \eta_{\beta\lambda} g^{\alpha\lambda} = U_\beta^\alpha.$$

Lemma is proved.

Proof of Theorem 4.2.

Because of Lemmas 4.3, 4.4 the vector fields

$$e, \ E, \ E^2, \dots, E^{n-1} \tag{4.38}$$

genericaly are linear independent on M. It is easy to see that these are polynomial vector fields on M. Hence e is a cyclic vector for the endomorphism U acting on $Der\, R$. So in generic point $y \in M$ the map (4.30a) is an isomorphism of Frobenius algebras

$$\mathbf{C}[u]/(P(u;x)) \to T_x M.$$

This proves Theorem 2.

Remark 4.2. The Euclidean metric (4.7) also defines an invariant inner product for the Frobenius algebras (on the cotangent planes $T_* M$). It can be shown also that the trilinear form

$$(\omega_1 \cdot \omega_2, \omega_3)^*$$

can be represented (localy, outside the discriminant locus $Discr\, W$) in the form

$$(\hat\nabla^i \hat\nabla^j \hat\nabla^k \hat F(x))\partial_i \otimes \partial_j \otimes \partial_k$$

for some function $\hat F(x)$. Here $\hat\nabla$ is the Gauss-Manin connection (i.e. the Levi-Civita connection for the metric (4.7)). The unity dt^n/h of the Frobenius algebra on T_*M is not covariantly constant w.r.t. the Gauss-Manin connection.

Remark 4.3. The vector fields

$$l^i := g^{is}(y)\partial_s, \quad i = 1, \ldots, n \tag{4.39}$$

form a basis of the R-module $\mathrm{Der}_R(-\log(D(y))$ of the vector fields on M tangent to the discriminant locus [4]. By the definition, a vector field $u \in \mathrm{Der}_R(-\log(D(y))$ *iff*

$$uD(y) = p(y)D(y)$$

for a polynomial $p(y) \in R$. The basis (4.39) of $\mathrm{Der}_R(-\log(D(y))$ depends on the choice of coordinates on M. In the Saito flat coordinates commutators of the basic vector fields can be calculated via the structure constants of the Frobenius algebra on T_*M. The following formula holds:

$$[l^\alpha, l^\beta] = \frac{d_\beta - d_\alpha}{h} c_\epsilon^{\alpha\beta} l^\epsilon. \tag{4.40}$$

This can be proved using (4.25).

Example 4.1. $W = I_2(k)$, $k \geq 0$. The action of the group on the complex z-plane is generated by the transformations

$$z \mapsto e^{\frac{2\pi i}{k}} z, \quad z \mapsto \bar z.$$

The invariant metric on $\mathbf{R}^2 = \mathbf{C}$ is

$$ds^2 = dz d\bar z,$$

the basic invariant polynomials are

$$t^1 = z^k + \bar z^k, \quad \deg t^1 = k,$$

$$t^2 = \frac{1}{2k} z\bar z, \quad \deg t^2 = 2.$$

We have

$$g^{11}(t) = (dt^1, dt^1) = 4\frac{\partial t^1}{\partial z}\frac{\partial t^1}{\partial \bar z} = 4k^2 (z\bar z)^{k-1} = (2k)^{k+1}(t^2)^{k-1}$$

$$g^{12}(t) = (dt^1, dt^2) = 2\left(\frac{\partial t^1}{\partial z}\frac{\partial t^2}{\partial \bar z} + \frac{\partial t^1}{\partial \bar z}\frac{\partial t^2}{\partial z}\right) = (z^k + \bar z^k) = t^1$$

$$g^{22}(t) = 4\frac{\partial t^2}{\partial z}\frac{\partial t^2}{\partial \bar z} = \frac{2}{k} t^2.$$

The Saito metric (4.12) is constant in these coordinates. The formula (4.25) gives

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + \frac{(2k)^{k+1}}{2(k^2-1)}(t^2)^{k+1}.$$

This coincides with (1.24a) (up to an equivalence) for $\mu = \frac{1}{2}(k-1)/(k+1)$. Particularly, for $k = 3$ this gives the Frobenius structure on $\mathbf{C}^2/A_2$, for $k = 4$ on $\mathbf{C}^2/B_2$, for $k = 6$ on $\mathbf{C}^2/G_2$.

Example 4.2. $W = A_n$. The group acts on the $(n+1)$-dimensional space $\mathbf{R}^{n+1} = \{(\xi_0, \xi_1, \ldots, \xi_n)\}$ by the permutations

$$(\xi_0, \xi_1, \ldots, \xi_n) \mapsto (\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_n}).$$

Restricting the action onto the hyperplane

$$\xi_0 + \xi_1 + \ldots, + \xi_n = 0 \tag{4.41}$$

we obtain the desired action of A_n on the n-dimensional space (4.41). The invariant metric on (4.41) is obtained from the standard Euclidean metric on $\mathbf{R}^{n+1}$ by the restriction.

The invariant polynomials on (4.41) are symmetric polynomials on $\xi_0, \xi_1, \ldots, \xi_n$. The elementary symmetric polynomials

$$a_k = (-1)^{n-k+1}(\xi_0\xi_1 \ldots \xi_k + \ldots), \quad k = 1, \ldots, n \tag{4.42}$$

can be taken as a homogeneous basis in the graded ring of the W-invariant polynomials on (4.41). So the complexified space of orbits $M = \mathbf{C}^n/A_n$ can be identified with the space of polynomials $\lambda(p)$ of an auxiliary variable p of the form (1.65).

Let us show that the Frobenius structure (4.25) on M coincides with the structure (1.66) (this will give us the simplest proof of that the formulae (1.66) give an example of Frobenius manifold). It will be convenient first to rewrite the formulae (1.66) in a slightly modified way (cf. (2.94))

Lemma 4.5.

1. *For the example 3 of Lecture 1 the inner product* $< \, , \, >_\lambda$ *and the 3-d rank tensor* $c(\,.\,,\,.\,,\,.\,) =< \,.\,\cdot\,.\,,\,.\, >_\lambda$ *have the form*

$$< \partial', \partial'' >_\lambda = -\sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\,\partial''(\lambda(p)dp)}{d\lambda(p)} \tag{4.43}$$

$$c(\partial', \partial'', \partial''') = -\sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\,\partial''(\lambda(p)dp)\,\partial'''(\lambda(p)dp)}{dp\,d\lambda(p)}. \tag{4.44}$$

2. *Let* $q^1, \ldots, q^n$ *be the critical points of the polynomial* $\lambda(p)$,

$$\lambda'(q^i) = 0, \quad i = 1, \ldots, n$$

and

$$u^i = \lambda(q^i), \quad i = 1, \ldots, n \tag{4.45}$$

be the corresponding critical values. The variables $u^1, \ldots, u^n$ *are local coordinates on* M *near the points* λ *where the polynomial* $\lambda(p)$ *has no multiple roots. These are canonical coordinates for the multiplication (1.66a). The metric (1.66b) in these coordinates has the diagonal form*

$$< \, , \, > |_\lambda = \sum_{i=1}^n \eta_{ii}(u)(du^i)^2, \quad \eta_{ii}(u) = \frac{1}{\lambda''(q^i)}. \tag{4.46}$$

3. *The metric on* M *induced by the invariant Euclidean metric in a point* λ *where the polynomial* $\lambda(p)$ *has simple roots has the form*

$$(\partial', \partial'')_\lambda = -\sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda(p)dp)\,\partial''(\log\lambda(p)dp)}{d\log\lambda(p)}. \tag{4.47}$$

Here ∂', ∂'', ∂''' are arbitrary tangent vectors on M in the point λ, the derivatives $\partial'(\lambda(p)dp)$ etc. are taken keeping $p = const$; $\lambda'(p)$ and $\lambda''(p)$ are the first and the second derivatives of the polynomial $\lambda(p)$ w.r.t. p. In other words, the formulae (4.43) - (4.44) mean that (1.65) is the LG superpotential for the Frobenius manifold (1.66) [42].

Proof. The first formula follows immediately from (1.66b) since the sum of residues of a meromorphic differential ω on the Riemann p-sphere vanishes:

$$\operatorname*{res}_{p=\infty} \omega + \sum \operatorname*{res}_{|\lambda|<\infty} \omega = 0. \tag{4.48}$$

Here we apply the residue theorem to the meromorphic differential

$$\omega = \frac{\partial'(\lambda(p)dp)\,\partial''(\lambda(p)dp)}{d\lambda(p)}.$$

From (4.48) it also follows that the formula (1.66a) can be rewritten as

$$c(\partial', \partial'', \partial''') = \operatorname*{res}_{p=\infty} \frac{\partial'(\lambda(p)dp)\, \partial''(\lambda(p)dp)\, \partial'''(\lambda(p)dp)}{dp\, d\lambda(p)}. \tag{4.49}$$

Let

$$f(p) = \partial'(\lambda(p)), \quad g(p) = \partial''(\lambda(p)), \; h(p) = \partial'''(\lambda(p)),$$
$$f(p)g(p) = q(p) + r(p)\lambda'(p)$$

for polynomials $q(p)$, $r(p)$, $\deg q(p) < n$. In the algebra $\mathbf{C}[p]/(\lambda'(p)$ we have then

$$f \cdot g = q.$$

On the other side, for the residue (4.49) we obtain

$$\operatorname*{res}_{p=\infty} \frac{\partial'(\lambda(p)dp)\, \partial''(\lambda(p)dp)\, \partial'''(\lambda(p)dp)}{dp\, d\lambda(p)} = \operatorname*{res}_{p=\infty} \frac{q(p)h(p)dp}{d\lambda(p)} + \operatorname*{res}_{p=\infty} r(p)h(p)dp.$$

The second residue in the r.h.s. of the formula equals zero while the first one coincides with the inner product $< q, h >_\lambda = < f \cdot g, h >_\lambda$.

Let us prove the second statement of Lemma. Let $\lambda(p)$ be a polynomial without multiple roots. Independence of the critical values u^1, ..., u^n as functions of the polynomial is a standard fact (it also follows from the explicit formula (4.55) for the Jacobi matrix). Let us choose ξ_1, ..., ξ_n as the coordinates on the hyperplane (4.41). These are not orthonormal: the matrix of the (contravariant) W-invariant metric in these coordinates has the form

$$g^{ab} = \delta^{ab} - \frac{1}{n+1}. \tag{4.50}$$

We have

$$\lambda(p) = (p + \xi_1 + \ldots + \xi_n) \prod_{a=1}^{n} (p - \xi_a), \quad \lambda'(p) = \prod_{i=1}^{n} (p - q^i), \tag{4.51}$$

$$\partial_i \lambda(p) = \frac{1}{p - q^i} \frac{\lambda'(p)}{\lambda''(q^i)}. \tag{4.52}$$

The last one is the Lagrange interpolation formula since

$$\partial_i \lambda(p)|_{p=q^j} = \delta_{ij}. \tag{4.53}$$

Substituting $p = \xi_a$ to the identity

$$(\partial_i \xi_1 + \ldots + \partial_i \xi_n) \prod_{b=1}^{n} (p - \xi_b) - \sum_{a=1}^{n} \frac{\lambda(p)}{p - \xi_a} \partial_i \xi_a = \partial_i \lambda(p) \tag{4.54}$$

we obtain the formula for the Jacobi matrix

$$\partial_i \xi_a = -\frac{1}{(\xi_a - q^i)\lambda''(q^i)}, \quad i, a = 1, \ldots, n. \tag{4.55}$$

Note that for a polynomial $\lambda(p)$ without multiple roots we have $\xi_a \neq q^i$, $\lambda''(q^i) \neq 0$.

For the metric (4.43) from (4.53) we obtain

$$< \partial_i, \partial_j >= -\delta_{ij} \frac{1}{\lambda''(q^i)}. \tag{4.56}$$

For the tensor (4.44) for the same reasons only $c(\partial_i, \partial_i, \partial_i)$ could be nonzero and

$$c(\partial_i, \partial_i, \partial_i) \equiv < \partial_i \cdot \partial_i, \partial_i >= -\frac{1}{\lambda''(q^i)}. \tag{4.57}$$

Hence

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i \tag{4.58}$$

in the algebra (1.66).

To prove the last statement of the lemma we observe that the metric (4.47) also is diagonal in the coordinates u^1, ..., u^n with

$$g_{ii}(u) := (\partial_i, \partial_i) = -\frac{1}{u^i \lambda''(q^i)}. \tag{4.59}$$

The inner product of the gradients $(d\xi_a, d\xi_b)$ w.r.t. the metric (4.59) is

$$\sum_{i=1}^n \frac{1}{g_{ii}(u)} \frac{\partial \xi_a}{\partial u^i} \frac{\partial \xi_b}{\partial u^i} = -\sum_{i=1}^n \frac{u^i}{(\xi_a - q^i)(\xi_b - q^i)\lambda''(q^i)}$$

$$= -\sum_{i=1}^n \operatorname*{res}_{d\lambda=0} \frac{\lambda(p)}{(p-\xi_a)(p-\xi_b)\lambda'(p)} = \left[\operatorname*{res}_{p=\infty} + \operatorname*{res}_{p=\xi_a} + \operatorname*{res}_{p=\xi_b} \right] \frac{\lambda(p)}{(p-\xi_a)(p-\xi_b)\lambda'(p)}$$

$$= \delta^{ab} - \frac{1}{n+1}.$$

So the metric (4.47) coincides with the W-invariant Euclidean metric (4.7). Lemma is proved.

Exercise 4.1. Prove that the function

$$V(u) := -\frac{1}{2(n+1)} \left[\xi_0^2 + \ldots + \xi_n^2 \right] |_{\xi_0 + \ldots + \xi_n = 0} \tag{4.60}$$

is the potential for the metric (4.59):

$$\partial_i V(u) = \eta_{ii}(u).$$

Let us check that the curvature of the metric (1.66b) vanishes. I will construct explicitly the flat coordinates for the metric (cf. [42, 131]). Let us consider the function $p = p(\lambda)$ inverse to the polynomial $\lambda = \lambda(p)$. It can be expanded in a Puiseaux series as $\lambda \to \infty$

$$p = p(k) = k + \frac{1}{n+1} \left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + \ldots + \frac{t^1}{k^n} \right) + O\left(\frac{1}{k^{n+1}} \right) \tag{4.61}$$

where $k := \lambda^{\frac{1}{n+1}}$, the coefficients

$$t^1 = t^1(a_1, \dots, a_n), \dots, t^n = t^n(a_1, \dots, a_n) \tag{4.62}$$

are determined by this expansion. The inverse functions can be found from the identity

$$(p(k))^{n+1} + a_n(p(k))^{n-1} + \dots + a_1 = k^{n+1}. \tag{4.63}$$

This gives a triangular change of coordinates of the form

$$a_i = -t^i + f_i(t^{i+1}, \dots, t^n), \quad i = 1, \dots, n. \tag{4.64}$$

So the coefficients t^1, ..., t^n can serve as global coordinates on the orbit space M (they give a distinguished basis of symmetric polynomials of $(n+1)$ variables).

Exercise 4.2. Show the following formula [42, 131] for the coordinates t^α

$$t^\alpha = -\frac{n+1}{n-\alpha+1} \operatorname*{res}_{p=\infty} \left(\lambda^{\frac{n-\alpha+1}{n+1}}(p)\, dp \right). \tag{4.65}$$

Let us prove that the variables t^α are the flat coordinates for the metric (1.66b),

$$< \partial_\alpha, \partial_\beta >= \delta_{\alpha+\beta,n+1}. \tag{4.66}$$

To do this (and also in other proofs) we will use the following "thermodynamical identity".

Lemma 4.6. *Let $\lambda = \lambda(p, t^1, \dots, t^n)$ and $p = p(\lambda, t^1, \dots, t^n)$ be two mutually inverse functions depending on the parameters $t^1, \dots, t^n$. Then*

$$\partial_\alpha(\lambda\, dp)_{p=const} = -\partial_\alpha(p\, d\lambda)_{\lambda=const}, \tag{4.67}$$

$\partial_\alpha = \partial/\partial t^\alpha$.

Proof. Differentiating the identity

$$\lambda(p(\lambda, t), t) \equiv \lambda$$

w.r.t. t^α we obtain

$$\frac{d\lambda}{dp}\partial_\alpha p(\lambda, t)_{\lambda=const} + \partial_\alpha \lambda(p(\lambda, t), t)_{p=const} = 0.$$

Lemma is proved.

Observe that $k = \lambda^{\frac{1}{n+1}}$ can be expanded as a Laurent series in $1/p$

$$k = p + O\left(\frac{1}{p}\right).$$

By $[\ \ ]_+$ I will denote the polynomial part of a Laurent series in $1/p$. For example, $[k]_+ = p$. Similarly, for a differential $f\,dk$ where f is a Laurent series in $1/p$ we put $[f\,dk]_+ := [f\,dk/dp]_+ dp$.

Lemma 4.7. *The following formula holds true*

$$\partial_\alpha (\lambda\, dp)_{p=const} = -[k^{\alpha-1} dk]_+, \quad \alpha = 1, \ldots, n. \tag{4.68}$$

Proof. We have

$$-\partial_\alpha (\lambda\, dp)_{p=const} = \partial_\alpha (p\, d\lambda)_{\lambda=const} = \left(\frac{1}{n+1} \frac{1}{k^{n-\alpha+1}} + O\left(\frac{1}{k^{n+1}} \right) \right) dk^{n+1}$$

$$= k^{\alpha-1} dk + O\left(\frac{1}{k}\right) dk$$

since $k = const$ while $\lambda = const$. The very l.h.s. of this chain of equalities is a polynomial differential in p. And $[O(1/k)dk]_+ = 0$. Lemma is proved.

Corollary 4.6. *The variables t^1, ..., t^n are the flat coordinates for the metric (1.66b). The coefficients of the metric in these coordinates are*

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}. \tag{4.69}$$

Proof. From the previous lemma we have

$$\partial_\alpha (\lambda\, dp)_{p=const} = -k^{\alpha-1} dk + O(1/k) dk.$$

Substituting to the formula (4.43) we obtain

$$< \partial_\alpha, \partial_\beta >_\lambda = \operatorname*{res}_{p=\infty} \frac{k^{\alpha-1} dk\, k^{\beta-1} dk}{dk^{n+1}} = \frac{1}{n+1} \delta_{\alpha+\beta,n+1}$$

(the terms of the form $O(1/k)dk$ do not affect the residue). Corollary is proved.

Now we can easyly prove that the formulae of Example 1.7 describe a Frobenius structure on the space M of polynomials $\lambda(p)$. Indeed, the critical values of $\lambda(p)$ are the canonical coordinates u^i for the multiplication in the algebra of truncated polynomials $\mathbf{C}/(\lambda'(p)) = T_\lambda M$. The metric (1.66b) is flat on M and it is diagonal in the canonical coordinates. From the flatness and from (4.60) it follows that this is a Darboux - Egoroff metric on M. From Lemma 4.5 we conclude that M with the structure (1.66) is a Frobenius manifold. It also follows that the corresponding intersection form coincides with the A_n-invariant metric on $\mathbf{C}^n$. From the uniqueness part of Theorem 4.1 we conclude that the Frobenius structure (1.66) coincides (up to an equivalence) with the Frobenius structure of Theorem 4.1.

Remark 4.4. For the derivatives of the corresponding polynomial $F(t)$ in [42] the following formula was obtained

$$\partial_\alpha F = \frac{1}{(\alpha+1)(n+\alpha+2)} \operatorname*{res}_{p=\infty} \lambda^{\frac{n+\alpha+2}{n+1}} dp. \tag{4.70}$$

Example 4.3. In the three dimensional case there are three finite Coxeter groups: $W(A_3)$, $W(B_3)$, and $W(H_3)$. Applying to them our construction one obtains the three polynomial solutions (A.7), (A.8), and (A.9) resp. (this also follows from the uniqueness Theorem A.1).

Example 4.4. In four-dimensional case our construction produces the following five polynomial solutions of WDVV.

Group $W(A_4)$.

$$F = \frac{1}{2}t_1^2\, t_4 + t_1\, t_2\, t_3 + \frac{1}{2}t_2^3 + \frac{1}{3}t_3^4 + 6t_2t_3^2t_4 + 9t_2^2t_4^2 + 24t_3^2t_4^3 + \frac{216}{5}t_4^6.$$

Group $W(B_4)$.

$$F = \frac{1}{2}t_1^2\, t_4 + t_1\, t_2\, t_3 + {t_2}^3 + \frac{t_2\, {t_3}^3}{3} + 3\, {t_2}^2\, t_3\, t_4 + \frac{{t_3}^4\, t_4}{4}$$

$$+3\, t_2\, {t_3}^2\, {t_4}^2 + 6\, {t_2}^2\, {t_4}^3 + {t_3}^3\, {t_4}^3 + \frac{18\, {t_3}^2\, {t_4}^5}{5} + \frac{18\, {t_4}^9}{7}.$$

Group $W(D_4)$.

$$F = \frac{1}{2}t_1^2\, t_4 + t_1\, t_2\, t_3 + t_2^3 t_4 + t_3^3 t_4 + 6t_2t_3t_4^3 + \frac{54}{35}t_4^7.$$

Group $W(F_4)$.

$$F = \frac{1}{2}t_1^2\, t_4 + t_1\, t_2\, t_3 + \frac{t_2^3\, t_4}{18} + \frac{3\, t_3^4\, t_4}{4} + \frac{t_2\, t_3^2\, t_4^3}{2}$$

$$+\frac{t_2^2\, t_4^5}{60} + \frac{t_3^2\, t_4^7}{28} + \frac{t_4^{13}}{2^4 \cdot 3^2 \cdot 11 \cdot 13}.$$

Group $W(H_4)$

$$F = t_1\, t_2\, t_3 + \frac{{t_1}^2\, t_4}{2} + \frac{2\, {t_2}^3\, t_4}{3} + \frac{{t_3}^5\, t_4}{240} + \frac{t_2\, {t_3}^3\, {t_4}^3}{18} + \frac{{t_2}^2\, t_3\, {t_4}^5}{15} + \frac{{t_3}^4\, {t_4}^7}{2^3 \cdot 3^3 \cdot 5} +$$

$$\frac{t_2\, {t_3}^2\, {t_4}^9}{2 \cdot 3^4 \cdot 5} + \frac{8\, {t_2}^2\, {t_4}^{11}}{3^4 \cdot 5^2 \cdot 11} + \frac{{t_3}^3\, {t_4}^{13}}{2^2 \cdot 3^6 \cdot 5^2} + \frac{2\, {t_3}^2\, {t_4}^{19}}{3^8 \cdot 5^3 \cdot 19} + \frac{32\, {t_4}^{31}}{3^{13} \cdot 5^6 \cdot 29 \cdot 31}.$$

Other examples of polynomial solutions of WDVV associated with finite Coxeter groups can be found in [38, 165].

Remark 4.5. There are certain inclusions between the polynomial Frobenius manifolds of the form

$$M(W) := \mathbf{C}^n/W$$

(the orbit spaces) for a finite Coxeter group W acting in n-dimensional Euclidean space. These inclusions correspond to the operation of folding of Dynkin graphs [6]. As it is shown in [163], if the Dynkin graph of a Coxeter group W' is obtained by folding of the Dynkin graph of another Coxeter group W then the corresponding orbit space $M(W')$ is a (graded) linear subspace in $M(W)$ w.r.t. the Saito linear structure. From our construction we immediately conclude that the inclusion

$$M(W') \subset M(W)$$

is also an embedding of Frobenius manifolds. We obtain the following list of embeddings (they were obtained independently in [165] by a straightforward computation)

$$\begin{aligned} M(B_n) &\subset M(A_{2n-1}) \\ M(I_2(k)) &\subset M(A_{k-1}) \\ M(F_4) &\subset M(E_6) \\ M(H_3) &\subset M(D_6) \\ M(H_4) &\subset M(E_8). \end{aligned}$$

(The group $W(G_2)$ coincides with $W(I_2(6))$ and, therefore, $M(G_2) \subset M(A_5)$.)

The constructions of this Lecture can be generalized for the case when the monodromy group is an extension of affine Weyl groups. The simplest solution of this type is given by the quantum multiplication on CP^1 (see Example G.2 above). We will not describe here the general construction (to be published elsewhere) but we will give two examples of it. In these examples one obtains three-dimensional Frobenius manifolds.

Exercise 4.3. Prove that

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{1}{24}t_2^4 + t_2 e^{t_3} \tag{4.71}$$

is a solution of WDVV with the Euler vector field

$$E = t_1\partial_1 + \frac{1}{2}t_2\partial_2 + \frac{3}{2}\partial_3. \tag{4.72}$$

Prove that the monodromy group of the Frobenius manifold coinsides with an extension of the affine Weyl group $\widetilde{W}(A_2)$. Hint: Prove that the flat coordinates x, y, z of the intersection form are given by

$$\begin{aligned} t_1 &= 2^{-\frac{1}{3}} e^{\frac{2}{3}z}\left[e^{x+y} + e^{-x} + e^{-y}\right] \\ t_2 &= 2^{-\frac{2}{3}} e^{\frac{1}{3}z}\left[e^{-x-y} + e^{x} + e^{y}\right] \\ t_3 &= z. \end{aligned} \tag{4.73}$$

The intersection form in these coordinates is

$$ds^2 = -2(dx^2 + dxdy + dy^2) + \frac{2}{3}dz^2. \tag{4.74}$$

Exercise 4.4. Prove that

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{1}{48}t_2^4 + \frac{1}{4}t_2^2 e^{t_3} + \frac{1}{32}e^{2t_3} \tag{4.75}$$

is a solution of WDVV with the Euler vector field

$$E = t_1\partial_1 + \frac{1}{2}t_2\partial_2 + \partial_3. \tag{4.76}$$

Prove that the monodromy group of the Frobenius manifold is an extension of the affine Weyl group $\tilde{W}(B_2)$. Hint: Show that the flat coordinates of the intersection form are given by

$$\begin{aligned} t_1 &= e^{2\pi i z}\left[\cos 2\pi x \cos 2\pi y + \frac{1}{2}\right] \\ t_2 &= e^{\pi i z}\left[\cos 2\pi x + \cos 2\pi y\right] \\ t_3 &= 2\pi z. \end{aligned} \tag{4.77}$$

The intersection form in these coordinates is proportional to

$$ds^2 = dx^2 + dy^2 - \frac{1}{2}dz^2.$$

Remark 4.6. The corresponding extension of the dual affine Weyl group $\tilde{W}(C_2)$ gives an equivalent Frobenius 3-manifold.

In the appendix to this Lecture we outline a generalization of our constructions to the case of extended complex crystallographic groups.

We obtain now an integral representation of the solution of the Riemann - Hilbert b.v.p. of Lecture 3 for the polynomial Frobenius manifolds on the space of orbits of a finite Coxeter group W.

Let us fix a system of n reflections $T_1, \ldots, T_n$ generating the group W (the order of the reflections also will be fixed). Via $e_1, \ldots, e_n$ I denote the normal vectors to the mirrors of the reflections normalized as

$$T_i(e_i) = -e_i, \tag{4.78a}$$

$$(e_i, e_i) = 2, \quad i = 1, \ldots, n. \tag{4.78b}$$

Let $x_1, \ldots, x_n$ be the coordinates in $\mathbf{R}^n$ w.r.t. the basis (4.78).

Let us consider the system of equations for the unknowns $x_1, \ldots, x_n$

$$\begin{aligned} y_1(x_1, \ldots, x_n) &= y_1 - \lambda \\ y_2(x_1, \ldots, x_n &= y_2 \\ &\cdots \\ y_n(x_1, \ldots, x_n) &= y_n \end{aligned} \tag{4.79}$$

where $y_i(x)$ are basic homogeneous W-invariant polynomials for the group W, deg $y_1 = h = \max$ (see above). Let

$$x_1 = x_1(y, \lambda) \tag{4.80$_1$}$$
$$\dots$$
$$x_n = x_n(y, \lambda) \tag{4.80$_n$}$$

be the solution of this system (these are algebraic functions), $y = (y_1, \dots, y_n) \in M = \mathbf{C}/W$. (Note that these are the basic solutions of the Gauss - Manin equations on the Frobenius manifold.) By $\lambda = \lambda_1(x_1, y)$, ..., $\lambda = \lambda_n(x_n, y)$ we denote the inverse functions to (4.80$_1$), ..., (4.80$_n$) resp.

Proposition 4.2. *The functions*

$$h_a(y, z) := -z^{\frac{1}{h}} \int e^{z\lambda(x,y)} dx, \quad a = 1, \dots, n \tag{4.81}$$

are flat coordinates of the deformed connection (3.3) on the Frobenius manifold $\mathbf{C}/W$*. Taking*

$$\psi_{ia}(y, z) := \frac{\partial_i h_a(u, z)}{\sqrt{\eta_{ii}(y)}}, \; i = 1, \dots, n \tag{4.82}$$

where $\partial_i = \partial/\partial u^i$*,* u_i *are the roots of (4.29), we obtain the solution of the Riemann - Hilbert problem of Lecture 3 for the Frobenius manifold.*

Proof. The formula (4.81) follows from (H.51). The second statement is the inversion of Theorem H.2. Proposition is proved.

Corollary 4.7. *The nonzero off-diagonal entries of the Stokes matrix of the Frobenius manifold* $\mathbf{C}/W$ *for a finite Coxeter group* W *coincide with the entries of the Coxeter matrix of* W*.*

So the Stokes matrix of the Frobenius manifolds is "a half" of the corresponding Coxeter matrix. For the simply-laced groups (i.e., the A - D - E series) this was obtained from physical considerations in [29].

To obtain the LG superpotential for the Frobenius manifold $\mathbf{C}/W$ we are to find the inverse function $\lambda = \lambda(p, y)$ to

$$p = x_1(y, \lambda) + \dots + x_n(y, \lambda) \tag{4.83}$$

according to (I.10).

Example 4.5. We consider again the group A_n acting on the hyperplane (4.41) of the Euclidean space $\mathbf{R}^{n+1}$ with a standard basis f_0, f_1, ..., f_n. We chose the permutations

$$T_1 : \xi_0 \leftrightarrow \xi_1, \dots, T_n : \xi_0 \leftrightarrow \xi_n \tag{4.84}$$

as the generators of the reflection group. The corresponding root basis (in the hyperplane (4.41)) is

$$e_1 = f_1 - f_0, \dots, e_n = f_n - f_0. \tag{4.85}$$

The coordinates of a vector $\xi_0 f_0 + \xi_1 f_1 + \ldots + \xi_n f_n$ are

$$x_1 = \xi_1, \ldots, x_n = \xi_n.$$

Note that the sum

$$p = x_1 + \ldots + x_n = -\xi_0 \tag{4.86}$$

is one of the roots (up to a sign) of the equation

$$p^{n+1} + a_1 p^{n-1} + \ldots + a_n = 0.$$

The invariant polynomial of the highest degree is a_n. So to construct the LG superpotential we are to solve the equation

$$p^{n+1} + a_1 p^{n-1} + \ldots + a_n - \lambda = 0. \tag{4.87}$$

and then to invert it. It's clear that we obtain

$$\lambda = \lambda(p, a_1, \ldots, a_n) = p^{n+1} + a_1 p^{n-1} + \ldots + a_n. \tag{4.88}$$

We obtain a new proof of Lemma 4.5.

For other Coxeter groups W the above algorithm gives a universal construction of an analogue of the versal deformation of the corresponding simple singularity. But the calculations are more involved, and λ becomes an algebraic function of p.

Appendix J.
Extended complex crystallographic Coxeter groups and twisted Frobenius manifolds.

Complex crystallographic groups were introduced by Bernstein and Schwarzman in [16] (implicitly they had been already used by Looijenga in [102]). These are the groups of affine transformations of a complex affine n-dimensional space V with the linear part generated by reflections. The very important subclass is *complex crystallographic Coxeter groups* (CCC groups briefly). In this case by definition V is the complexification of a real space $V_{\mathbf{R}}$; it is required that the linear parts of the transformations of a CCC group form a Coxeter group acting in the real linear space of translations of $V_{\mathbf{R}}$.

CCC groups are labelled by Weyl groups of simple Lie algebras. For any fixed Weyl group W the corresponding CCC group $\tilde{W}$ depends on a complex number τ in the upper half-plane as on the parameter. Certain factorization w.r.t. a discrete group of Möbius transformations of the upper half-plane that we denote by Γ_W must be done to identify equivalent CCC groups with the given Weyl group W. Bernstein and Schvarzman found also an analogue of the Chevalley theorem for CCC groups. They proved that for a fixed τ the space of orbits of a CCC group $\tilde{W}$ is a weighted projective space. The weights coincide with the markings on the extended Dynkin graph of W. Note that the discriminant locus (i.e. the set of nonregular orbits) depends on τ.

We have a natural fiber bundle over the quotient $\{Im\tau > 0\}/\Gamma_W$ with the fiber $V/\tilde{W}$. It turns out that the space of this bundle (after adding of one more coordinate, see below the precise construction) carries a natural structure of a twisted Frobenius manifold in the sense of Apendix B (above).

For the simply-laced case $W = A_l,\ D_l,\ E_l$ the construction [16] of CCC groups is of special simplicity. The Weyl group W acts by integer linear transformations in the space $\mathbf{C}^l$ of the complexified root lattice $\mathbf{Z}^l$. This action preserves the lattice $\mathbf{Z}^l \oplus \tau\mathbf{Z}^l$. The CCC group $\tilde{W} = \tilde{W}(\tau)$ is the semidirect product of W and of the lattice $\mathbf{Z}^l \oplus \tau\mathbf{Z}^l$. The groups $\tilde{W}(\tau)$ and $\tilde{W}(\tau')$ are equivalent *iff*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}).$$

The space of orbits $\mathbf{C}^l/\tilde{W}(\tau)$ can be obtained in two steps. First we can factorize over the translations $\mathbf{Z}^l \oplus \tau\mathbf{Z}^l$. We obtain the direct product of l copies of identical elliptic curves $E_\tau = \mathbf{C}/\{\mathbf{Z} \oplus \tau\mathbf{Z}\}$. The Weyl group W acts on E_τ^l. After factorization E_τ^l/W we obtain the space of orbits.

To construct the Frobenius structure we need a $\tilde{W}$-invariant metric on the space of the fiber bundle over the modular curve $M = \{Im\tau > 0\}/SL(2, \mathbf{Z})$ with the fiber $\mathbf{C}^l/\tilde{W}(\tau)$. Unfortunately, such a metric does not exist.

To resolve the problem we will consider a certain central extension of the group $\tilde{W}(\tau)$ acting in an extended space $\mathbf{C}^l \oplus \mathbf{C}$. The invariant metric we need lives on the extended space. The modular group acts by conformal transformations of the metric. Factorizing over the action of all these groups we obtain the twisted Frobenius manifold that corresponds to the given CCC group.

I will explain here the basic ideas of the construction for the simplest example of the CCC group $\tilde{A}_1$ leaving more general considerations for a separate publication. The Weyl group A_1 acts on the complex line $\mathbf{C}$ by reflections

$$v \mapsto -v. \tag{J.1a}$$

The group $\tilde{A}_1(\tau)$ is the semidirect product of the reflections and of the translations

$$v \mapsto v + m + n\tau, \quad m, n \in \mathbf{Z}. \tag{J.1b}$$

The quotient

$$\mathbf{C}/\tilde{A}_1(\tau) = E_\tau/\{\pm 1\} \tag{J.2}$$

is the projective line. Indeed, the invariants of the group (J.1) are even elliptic functions on E_τ. It is well-known that any even elliptic function is a rational function of the Weierstrass $\wp$. This proves the Bernstein - Schwarzman's analogue of the Chevalley theorem for this very simple case.

Let us try to invent a metric on the space (v, τ) being invariant w.r.t. the transformations (J.1). We immediately see that the candidate dv^2 invariant w.r.t. the reflections does not help since under the transformations (J.1b)

$$dv^2 \mapsto (dv + nd\tau)^2 \neq dv^2.$$

The problem of constructing of an invariant metric can be solved by adding of one more auxiliary coordinate to the (v, τ)-space adjusting the transformation law of the new coordinate in order to preserve invariance of the metric. The following statement gives the solution of the problem.

Lemma J.1.

1. The metric

$$ds^2 := dv^2 + 2d\phi\, d\tau \tag{J.3}$$

remains invariant under the transformations $v \mapsto -v$ *and*

$$\begin{aligned} \phi &\mapsto \phi - nv - \frac{1}{2}n^2\tau + k \\ v &\mapsto v + m + n\tau \\ \tau &\mapsto \tau \end{aligned} \tag{J.4a}$$

for arbitrary m, n, k.

2. The formulae

$$\begin{aligned} \phi &\mapsto \tilde{\phi} = \phi + \frac{1}{2}\frac{cv^2}{c\tau + d} \\ v &\mapsto \tilde{v} = \frac{v}{c\tau + d} \\ \tau &\mapsto \tilde{\tau} = \frac{a\tau + b}{c\tau + d} \end{aligned} \tag{J.4b}$$

with $ad - bc = 1$ determine a conformal transformation of the metric ds^2

$$d\tilde{v}^2 + 2d\tilde{\phi}\, d\tilde{\tau} = \frac{dv^2 + 2d\phi\, d\tau}{(c\tau + d)^2}. \tag{J.5}$$

Proof is in a simple calculation.

We denote by $\hat{A}_1$ the group generated by the reflection $v \mapsto -v$ and by the transformations (J.4) with integer m, n, k, a, b, c, d.

Observe that the subgroup of the translations (J.1b) in $\tilde{A}_1$ becomes non-commutative as the subgroup in $\hat{A}_1$.

The group generated by the transformations (J.4) with integer parameters is called *Jacobi group.* This name was proposed by Eichler and Zagier [62]. The automorphic forms of subgroups of a finite index of Jacobi group are called *Jacobi forms* [*ibid.*]. They were systematically studied in [62]. An analogue $\hat{W}$ of the group $\hat{A}_1$ (the transformations (J.4) together with the Weyl group $v \mapsto -v$) can be constructed for any CCC group $\tilde{W}$ taking the Killing form of W instead of the squares v^2, n^2 and nv. Invariants of these Jacobi groups were studied by Saito [134] and Wirthmüller [154] (see also the relevant papers [103, 85, 7]). Particularly, Saito constructed flat coordinates for the so-called "codimension 1" case. (This means that there exists a unique maximum among the markings of the extended Dynkin graph. On the list of our examples only the E-groups are of codimension one.) Explicit formulae of the Jacobi forms for $\hat{E}_6$ that are the flat coordinates in the sense of [134] have been obtained recently in [136]. Theory of Jacobi forms for all the Jacobi groups but $\hat{E}_8$ was constructed in [154].

Let $\mathbf{C}^3 = \{(\phi, v, \tau), Im\tau > 0\}$. I will show that the space of orbits

$$\mathcal{M}_{\hat{A}_1} := \mathbf{C}^3/\hat{A}_1 \tag{J.6}$$

carries a natural structure of a twisted Frobenius manifold with the intersection form proportional to (J.3). It turns out that this coincides with the twisted Frobenius manifold of Appendix C. Furthermore, it will be shown that this twisted Frobenius manifold can be described by the LG superpotential

$$\lambda(p; \omega, \omega', c) := \wp(2\omega p; \omega, \omega') + c. \tag{J.7}$$

We will construct below the flat coordinates t^1, t^2, t^3 as functions of ω, ω', c.

The factorization over $\hat{A}_1$ will be done in two steps. First we construct a map

$$\mathbf{C}^3_0 = \{(\phi, v, \tau),\ Im\tau > 0,\ v \neq m + n\tau\} \to \mathbf{C}^3_0 = \{(z, \omega, \omega'),\ Im\frac{\omega'}{\omega} > 0,\ z \neq 2m\omega + 2n\omega'\} \tag{J.8}$$

such that the action of the group $\hat{A}_1$ transforms to the action of the group of translations

$$z \mapsto z + 2m\omega + 2n\omega' \tag{J.9a}$$

reflections

$$z \mapsto -z \tag{J.9b}$$

and changes of the basis of the lattice

$$\begin{matrix} \omega' \mapsto a\omega' + b\omega \\ \omega \mapsto c\omega' + d\omega \end{matrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}). \tag{J.9c}$$

Note that the subgroup generated by (J.9) look very similar to the CCC group $\tilde{A}_1$ but the non-normalized lattice $\{2m\omega + 2n\omega'\}$ is involved in the construction of this subgroup.

We will use the Weierstrass σ-function

$$\sigma(z;\omega,\omega') = z \prod_{m^2+n^2\neq 0} \left\{ \left(1 - \frac{z}{w}\right) \exp\left[\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right] \right\}, \tag{J.10a}$$

$$w := 2m\omega + 2n\omega'$$

$$\frac{d}{dz} \log \sigma(z;\omega,\omega') = \zeta(z;\omega,\omega'). \tag{J.10b}$$

It is not changed when changing the basis 2ω, $2\omega'$ of the lattice while for the translations (J.9a)

$$\begin{aligned} &\sigma(z + 2m\omega + 2n\omega'; \omega, \omega') \\ &= (-1)^{m+n+mn} \sigma(z;\omega,\omega') \exp\left[(z + m\omega + n\omega')(2m\eta + 2n\eta')\right] \end{aligned} \tag{J.11}$$

where $\eta = \eta(\omega,\omega')$, $\eta' = \eta'(\omega,\omega')$ are defined in (C.60)

Lemma J.2. *The equations*

$$\begin{aligned} \phi &= \frac{1}{2\pi i}\left[\log \sigma(z;\omega,\omega') - \frac{\eta}{2\omega} z^2\right] \\ v &= \frac{z}{2\omega} \\ \tau &= \frac{\omega'}{\omega} \end{aligned} \tag{J.12}$$

determine a map (J.8). This map locally is a bi-holomorphic equivalence.

Proof. Expressing σ-function via the Jacobi theta-functions we obtain

$$\phi = \frac{1}{2\pi i}\left[\log 2\omega + \log \frac{\theta_1(v;\tau)}{\theta_1'(0;\tau)}\right].$$

This can be uniquely solved for ω

$$\omega = \frac{1}{2} \frac{\theta_1'(0;\tau)}{\theta_1(v;\tau)} e^{2\pi i \phi}. \tag{J.13a}$$

Substituting to (J.12b) we obtain

$$z = v \frac{\theta_1'(0;\tau)}{\theta_1(v;\tau)} e^{2\pi i \phi} \tag{J.13b}$$

$$\omega' = \frac{\tau}{2} \frac{\theta_1'(0;\tau)}{\theta_1(v;\tau)} e^{2\pi i \phi}. \tag{J.13c}$$

Using the transformation law of σ-function and (J.4) we complete the proof of the lemma.

Corollary J.1. *The space of orbits $\mathbf{C}^3/\hat{A}_1$ coincides with the universal torus (C.51) factorized over the involution $z \mapsto -z$.*

Let us calculate the metric induced by (J.3) on the universal torus. To simplify this calculation I will use the following basis of vector fields on the universal torus (cf. Appendix C above)

$$\begin{aligned} D_1 &= \omega \frac{\partial}{\partial \omega} + \omega' \frac{\partial}{\partial \omega'} + z \frac{\partial}{\partial z} \\ D_2 &= \frac{\partial}{\partial z} \\ D_3 &= \eta \frac{\partial}{\partial \omega} + \eta' \frac{\partial}{\partial \omega'} + \zeta(z;\omega,\omega') \frac{\partial}{\partial z}. \end{aligned} \tag{J.14}$$

As it follows from Lemma the inner products of these vector fields are functions on the universal torus (invariant elliptic functions in the terminology of Appendix C).

Proposition J.1. *In the basis (J.14) the metric $(D_a, D_b) =: g_{ab}$ has the form*

$$(g_{ab}) = \frac{1}{4\omega^2} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -\wp(z;\omega,\omega') \end{pmatrix}. \tag{J.15}$$

Proof. By the definition we have

$$(D_a, D_b) = D_a v D_b v + D_a \phi D_b \tau + D_a \tau D_b \phi. \tag{J.16}$$

We have

$$D_1 \phi = \frac{1}{2\pi i}, \quad D_1 v = D_1 \tau = 0,$$

$$D_2 \phi = \frac{1}{2\pi i} \left[\zeta(z;\omega,\omega') - \frac{\eta}{\omega} z \right], \quad D_2 v = \frac{1}{2\omega}, \quad D_2 \tau = 0,$$

$$D_3 \phi = \frac{1}{4\pi i} \left[\left(\zeta(z;\omega,\omega') - \frac{\eta}{\omega} z \right)^2 + \wp(z;\omega,\omega') \right],$$

$$D_3 v = \frac{1}{2\omega} \left[\zeta(z;\omega,\omega') - \frac{\eta}{\omega} z \right], D_3 \tau = -\frac{\pi i}{2\omega^2}. \tag{J.17}$$

In the derivation of these formulae we used the formulae of Appendix C and also one more formula of Frobenius and Stickelberger

$$\eta \frac{\partial \log \sigma}{\partial \omega} + \eta' \frac{\partial \log \sigma}{\partial \omega'} = -\frac{1}{2}\zeta(z)^2 + \frac{1}{2}\wp(z) - \frac{1}{24} g_2 z \tag{J.18}$$

[67, formula 30.]. From (J.17) we easily obtain (J.15). Proposition is proved.

Remark J.1. The metric (J.15) on the universal torus has still its values in the line bundle ℓ. I recall (see above Appendix C) that ℓ is the pull-back of the tangent bundle of the modular curve w.r.t. the projection $(z, \omega, \omega') \mapsto \omega'/\omega$.

Remark J.2. The Euler vector field D_1 in the coordinates (ϕ, v, τ) has the form

$$D_1 = \frac{1}{2\pi i} \frac{\partial}{\partial \phi}. \tag{J.19}$$

Let

$$\hat{\mathcal{M}} := \left\{ (z, \omega, \omega'),\ Im \frac{\omega'}{\omega} > 0,\ z \neq 0 \right\} / \{ z \mapsto \pm z + 2m\omega + 2n\omega' \}. \tag{J.20}$$

The metric (J.15) is well-defined om $\hat{\mathcal{M}}$. The next step is to construct a Frobenius structure on $\hat{\mathcal{M}}$.

The functions on $\hat{\mathcal{M}}$ are rational combinations of ω, ω' and $\wp(z; \omega, \omega')$. We define a grading in the ring of these functions with only a pole at $z = 0$ allowed putting

$$\deg \omega = \deg \omega' = -\frac{1}{2}, \quad \deg \wp = 1. \tag{J.21a}$$

The corresponding Euler vector field for this grading is

$$E = -\frac{1}{2} D_1. \tag{J.21b}$$

Note the relation of the grading to that defined by the vector field $\partial / \partial \phi$.

Proposition J.2. *There exists a unique Frobenius structure on the manifold $\hat{\mathcal{M}}$ with the intersection form $-4\pi i\, ds^2$ (where ds^2 is defined in (J.3)), the Euler vector field (J.21) and the unity vector field $e = \partial / \partial \wp$. This Frobenius structure coincides with (C.87).*

Proof. I introduce coordinates t^1, t^2, t^3 on $\hat{\mathcal{M}}$ putting

$$\begin{aligned} t^1 &= -\frac{1}{\pi i} \left[\wp(z; \omega, \omega') + \frac{\eta}{\omega} \right] = -\frac{1}{\pi i} \left[\frac{\theta_1''(v; \tau)\theta_1(v; \tau) - {\theta_1'}^2(v; \tau)}{{\theta_1'}^2(0; \tau)} \right] e^{-4\pi i \phi} \\ t^2 &= \frac{1}{\omega} = 2 \frac{\theta_1(v; \tau)}{\theta_1'(0; \tau)} e^{-2\pi i \phi} \\ t^3 &= \frac{\omega'}{\omega} = \tau. \end{aligned} \tag{J.22}$$

Let us calculate the contravariant metric in these coordinates. To do this I use again the vector fields (J.14):

$$(dt^\alpha, dt^\beta) = g^{ab} D_a t^\alpha D_b t^\beta \tag{J.23}$$

where $g^{ab} = g^{ab}(z;\omega,\omega')$ is the inverse matrix to $-4\pi i g_{ab}$ (J.15)

$$\left(g^{ab}\right) = -\frac{\omega^2}{\pi i}\begin{pmatrix} \wp & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{J.24}$$

Using the formulae of Apppendix C we obtain

$$D_1 t^1 = \frac{2}{\pi i}\left[\wp + \frac{\eta}{\omega}\right], \quad D_1 t^2 = -\frac{1}{\omega}, \quad D_1 t^3 = 0$$

$$D_2 t^1 = -\frac{1}{\pi i}\wp', \quad D_2 t^2 = D_2 t^3 = 0$$

$$D_3 t^1 = \frac{1}{\pi i}\left[2\wp^2 - \frac{1}{4}g_2 + \frac{\eta^2}{\omega^2}\right], \quad D_3 t^2 = -\frac{\eta}{\omega^2}, \quad D_3 t^3 = -\frac{\pi i}{2\omega^2}. \tag{J.25}$$

From this it follows that the matrix $g^{\alpha\beta} := (dt^\alpha, dt^\beta)$ has the form

$$g^{11} = \frac{i}{\pi^3\omega^4}\left[\omega^6 g_3 - \eta\omega\,\omega^4 g_2 + 4(\eta\omega)^3\right],$$

$$g^{12} = \frac{3}{\pi^2\omega^3}\left[(\eta\omega)^2 - \frac{1}{12}\omega^4 g_2\right], \quad g^{13} = \frac{i}{\pi}\left[\wp + \frac{\eta}{\omega}\right],$$

$$g^{22} = \frac{i}{\pi}\left[\wp - 2\frac{\eta}{\omega}\right], \quad g^{23} = \frac{1}{2\omega}, \quad g^{33} = 0. \tag{J.26}$$

Differentiating $g^{\alpha\beta}$ w.r.t. e we obtain a constant matrix

$$\left(\eta^{\alpha\beta}\right) = \left(-\frac{1}{\pi i}\frac{\partial g^{\alpha\beta}}{\partial\wp}\right) \equiv \left(\frac{\partial g^{\alpha\beta}}{\partial t^1}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

So t^1, t^2, t^3 are the flat coordinates.

The next step is to calculate the matrix

$$F^{\alpha\beta} := \frac{g^{\alpha\beta}}{\deg g^{\alpha\beta}} \tag{J.27}$$

and then to find the function F from the condition

$$\frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} = \eta_{\alpha\alpha'}\eta_{\beta\beta'}F^{\alpha'\beta'}.$$

(We cannot do this for the F^{33} entry. But we know that F^{33} must be equal to $\partial^2 F/\partial {t^1}^2$.) It is straightforward to verify that the function F of the form (C.87) satisfies (J.27). Proposition is proved.

Remark J.3. Another way to prove the proposition is close to the proof of the main theorem of Lecture 4. We verify that the entries of the contravariant metric on the space of orbits $\hat{\mathcal{M}}$ (J.6) and the contravariant Christoffel symbols (3.25) of it are elliptic functions with at most second order pole in the point $z = 0$. From this it immediately follows that the metrics $g^{\alpha\beta}$ and $\partial g^{\alpha\beta}/\partial\wp$ form a flat pencil. This gives a Frobenius structure on $\hat{\mathcal{M}}$. The last step is to prove analyticity of the components of $g^{\alpha\beta}$ in the point $\omega'/\omega = i\infty$. This completes the alternative proof of the proposition.

Factorizing the Frobenius manifold we have obtained over the action of the modular group we obtain the twisted Frobenius structure on the space of orbits of the extended CCC group $\hat{A}_1$ coinciding with this of Appendix C. Observe that the flat coordinates t^α are not globally single-valued functions on the orbit space due to the transformation law (B.19) determining the structure of the twisted Frobenius manifold.

We will consider briefly the twisted Frobenius manifolds for the CCC groups $\hat{A}_n$ in Lecture 5. The twisted Frobenius structures for the groups $\hat{E}_6$ and $\hat{E}_7$ can be obtained using the results of [149] and [106].

Lecture 5.

Differential geometry of Hurwitz spaces.

Hurwitz spaces are the moduli spaces of Riemann surfaces of a given genus g with a given number of sheets $n+1$. In other words, these are the moduli spaces of pairs (C, λ) where C is a smooth algebraic curve of the genus g and λ is a meromorphic function on C of the degree $n+1$. Just this function realizes C as a $n+1$-sheet covering over CP^1 (i.e. as a $n+1$-sheet Riemann surface). I will consider Hurwitz spaces with an additional assumption that the type of ramification of C over the infinite point is fixed.

Our aim is to construct a structure of twisted Frobenius manifold on a Hurwitz space for any g and n. The idea is to take the function $\lambda = \lambda(p)$ on the Riemann surface $p \in C$ depending on the moduli of the Riemann surface as the superpotential in the sense of Appendix I. But we are to be more precise to specify what is the argument of the superpotential to be kept unchanged whith the differentiation along the moduli (see (I.11) - (I.13)).

I will construct first a Frobenius structure on a covering of these Hurwitz spaces corresponding to fixation of a symplectic basis of cycles in the homologies $H_1(C, \mathbf{Z})$. Factorization over the group $Sp(2g, \mathbf{Z})$ of changes of the basis gives us a twisted Frobenius structure on the Hurwitz space. It will carry a g-dimensional family of metrics being sections of certain bundle over the Hurwitz space.

We start with explicit description of the Hurwitz spaces.

Let $M = M_{g;n_0,\ldots,n_m}$ be a moduli space of dimension

$$n = 2g + n_0 + \ldots + n_m + 2m \tag{5.1}$$

of pairs

$$(C; \lambda) \in M_{g;n_0,\ldots,n_m} \tag{5.2}$$

where C is a Riemann surface with marked meromorphic function

$$\lambda : C \to CP^1, \quad \lambda^{-1}(\infty) = \infty_0 \cup \ldots \cup \infty_m \tag{5.3}$$

where ∞_0, ..., ∞_m are distinct points of the curve C. We require that the function λ has the degree $n_i + 1$ near the point ∞_i. (This is a connected manifold as it follows from [117].) We need the critical values of λ

$$u^j = \lambda(P_j), \ d\lambda|_{P_j} = 0, \ j = 1, \ldots, n \tag{5.4}$$

(i.e. the ramification points of the Riemann surface (5.3)) to be local coordinates in open domains in $\hat{M}$ where

$$u^i \neq u^j \text{ for } i \neq j \tag{5.5}$$

(due to the Riemann existence theorem). Note that P_j in (5.4) are the branch points of the Riemann surface. Another assumption is that the one-dimensional affine group acts on $\hat{M}$ as

$$(C; \lambda) \mapsto (C; a\lambda + b) \tag{5.6a}$$

$$u^i \mapsto a u^i + b,\ i = 1, \ldots, n. \tag{5.6b}$$

Example 5.1. For the case $g = 0$, $m = 0$, $n_0 = n$ the Hurwitz space consists of all the polynomials of the form

$$\lambda(p) = p^{n+1} + a_n p^{n-1} + \ldots + a_1, \quad a_1, \ldots, a_n \in \mathbf{C}. \tag{5.7}$$

We are to supress the term with p^n in the polynomial $\lambda(p)$ to provide a possibility to coordinatize (5.7) by the n critical values of λ. The affine transformations $\lambda \mapsto a\lambda + b$ act on (5.7) as

$$p \mapsto a^{\frac{1}{n+1}} p, \quad a_i \mapsto a_i a^{\frac{n-i+1}{n+2}} \text{ for } i > 1,\ a_1 \mapsto a\, a_1 + b. \tag{5.8}$$

Example 5.2. For $g = 0$, $m = n$, $n_0 = \ldots = n_m = 0$ the Hurwitz space consists of all rational functions of the form

$$\lambda(p) = p + \sum_{i=1}^{n} \frac{q_i}{p - p_i}. \tag{5.9}$$

The affine group $\lambda \mapsto a\lambda + b$ acts on (5.9) as

$$p \mapsto ap + b, \quad p_i \mapsto p_i + \frac{b}{a}, \quad q_i \mapsto a q_i. \tag{5.10}$$

Example 5.3. For a positive genus g, $m = 0$, $n_0 = 1$ the Hurwitz space consists of all hyperelliptic curves

$$\mu^2 = \prod_{j=1}^{2g+1} (\lambda - u^j).$$

The critical values u^1, $\ldots$, u^{2g+1} of the projection $(\lambda, \mu) \mapsto \lambda$ are the local coordinates on the moduli space. Globally they are well-defined up to a permutation.

Example 5.4. $g > 0$, $m = 0$, $n := n_0 \geq q$. In this case the quotient of the Hurwitz space over the affine group (5.6) is obtained from the moduli space of *all* smooth algebraic curves of the genus g by fixation of a non-Weierstrass point $\infty_0 \in C$.

I describe first the basic idea of introducing of a Frobenius structure in the covering of a Hurwitz space. We define the multiplication of the tangent vector fields declaring that the ramification points u^1, $\ldots$, u^n are the canonical coordinates for the multiplication, i.e.

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i \tag{5.11}$$

for

$$\partial_i := \frac{\partial}{\partial u^i}.$$

This definition works for Riemann surfaces with pairwise distinct ramification points of the minimal order two. Below we will extend the multiplication onto all the Hurwitz space.

The unity vector field e and the Euler vector field E are the generators of the action (5.6) of the affine group

$$e = \sum_{i=1}^{n} \partial_i, \quad E = \sum_{i=1}^{n} u^i \partial_i. \tag{5.12}$$

To complete the description of the Frobenius structure we are to describe admissible one-forms on the Hurwitz space. I recall that a one-form Ω on the manifold with a Frobenius algebra structure in the tangent planes is called admissible if the invariant inner product

$$< a, b >_\Omega := \Omega(a \cdot b) \tag{5.13}$$

(for any two vector-fields a and b) determines on the manifold a Frobenius structure (see Lecture 3).

Any quadratic differential Q on C holomorphic for $|\lambda| < \infty$ determines a one-form Ω_Q on the Hurwitz space by the formula

$$\Omega_Q := \sum_{i=1}^{n} du^i \operatorname*{res}_{P_i} \frac{Q}{d\lambda}. \tag{5.14}$$

A quadratic differential Q is called $d\lambda$-*divisible* if it has the form

$$Q = q d\lambda \tag{5.15}$$

where the differential q has no poles in the branch points of C. For a $d\lambda$-divisible quadratic differential Q the corresponding one-form $\Omega_Q = 0$. Using this observation we extend the construction of the one-form Ω_Q to *multivalued* quadratic differentials. By the definition this is a quadratic differential Q on the universal covering of the curve C such that the monodromy transformation along any cycle γ acts on Q by

$$Q \mapsto Q + q_\gamma d\lambda \tag{5.16}$$

for a differential q_λ. Multivalued quadratic differentials determine one-forms on the Hurwitz space by the same formula (5.14).

We go now to an appropriate covering of a Hurwitz space in order to describe multivalued quadratic differentials for which the one-forms (5.14) will define metrics on the Hurwitz space according to (5.13).

The covering $\hat{M} = \hat{M}_{g;n_0,\dots,n_m}$ will consist of the sets

$$(C; \lambda; k_0, \dots, k_m; a_1, \dots, a_g, b_1, \dots, b_g) \in \hat{M}_{g;n_0,\dots,n_m} \tag{5.17}$$

with the same C, λ as above and with a marked symplectic basis $a_1, \dots, a_g$, $b_1, \dots, b_g \in H_1(C, \mathbf{Z})$, and marked branches k_0, ..., k_m of roots of λ near ∞_0, ..., ∞_m of the orders $n_0 + 1$, ..., $n_m + 1$ resp.,

$$k_i^{n_i+1}(P) = \lambda(P), \; P \text{ near } \infty_i. \tag{5.18}$$

(This is still a connected manifold.)

The admissible quadratic differentials on the Hurwitz space will be constructed as squares $Q = \phi^2$ of certain differentials ϕ on C (or on a covering of C). I will call them *primary* differentials. I give now the list of primary differentials.

Type 1. One of the normalized Abelian differentials of the second kind on C with poles only at ∞_0, ..., ∞_m of the orders less then the corresponding orders of the differential $d\lambda$. More explicitly,

$$\phi = \phi_{t^{i;\alpha}}, \quad i = 0, \dots, m, \quad \alpha = 1, \dots, n_i \tag{5.19a}$$

is the normalized Abelian differential of the second kind with a pole in ∞_i,

$$\phi_{t^{i;\alpha}} = -\frac{1}{\alpha} dk_i^\alpha + \text{ regular terms} \quad \text{near } \infty_i, \tag{5.19b}$$

$$\oint_{a_j} \phi_{t^{i;\alpha}} = 0; \tag{5.19c}$$

Type 2.

$$\phi = \sum_{i=1}^m \delta_i \phi_{v^i} \text{ for } i = 1, \dots, m \tag{5.20a}$$

with the coefficients δ_1, ..., δ_i independent on the point of $\hat{M}$. Here ϕ_{v^i} is one of the normalized Abelian differentials of the second kind on C with a pole only at ∞_i with the principal part of the form

$$\phi_{v^i} = -d\lambda + \text{ regular terms} \quad \text{near } \infty_i, \tag{5.20b}$$

$$\oint_{a_j} \phi_{v^i} = 0; \tag{5.20c}$$

Type 3.

$$\phi = \sum_{i=1}^m \alpha_i \phi_{w^i} \tag{5.21a}$$

$$\oint_{a_j} \phi = 0 \tag{5.21b}$$

with the coefficients $\alpha_1, \dots, \alpha_m$ independent on the point of $\hat{M}$. Here ϕ_{w^i} is the normalized Abelian differential of the third kind with simple poles at ∞_0 and ∞_i with residues -1 and $+1$ resp.;

Type 4.

$$\phi = \sum_{i=1}^g \beta_i \phi_{r^i} \tag{5.22a}$$

with the coefficients $\beta_1, \ldots, \beta_g$ independent on the point of $\hat{M}$. Here ϕ_{r^i} is the normalized multivalued differential on C with increments along the cycles b_j of the form

$$\phi_{r^i}(P + b_j) - \phi_{r^i}(P) = -\delta_{ij} d\lambda, \tag{5.22b}$$

$$\oint_{a_j} \phi_{r^i} = 0 \tag{5.22c}$$

without singularities but those prescribed by (5.22b);

Type 5.

$$\phi = \sum_{i=1}^{g} \gamma_i \phi_{s^i} \tag{5.23a}$$

with the coefficients $\gamma_1, \ldots, \gamma_g$ independent on the point of $\hat{M}$. Here ϕ_{s^i} is the holomorphic differentials on C normalized by the condition

$$\oint_{a_j} \phi_{s^i} = \delta_{ij}. \tag{5.23b}$$

Exercise 5.1. Prove that the one-forms (5.14) where Q = square of any of the primary differentials (5.19) - (5.23) span the cotangent space to $\hat{M}$ at any point with pairwise distinct ramification points.

Let ϕ be one of the primary differentials of the above list. We put

$$Q = \phi^2 \tag{5.24}$$

and we will show that the corresponding one-form Ω_Q on the Hurwitz space is admissible. This will give a Frobenius structure on the covering $\hat{M}$ of the Hurwitz space for any of the primary differentials. We recall that the metric corresponding to the one-form Ω_{ϕ^2} has by definition the form

$$< \partial', \partial'' >_\phi := \Omega_{\phi^2}(\partial' \cdot \partial'') \tag{5.25}$$

for any two tangent fields ∂', ∂'' on $\hat{M}$.

To construct the superpotential of this Frobenius structure we introduce a multivalued function p on C taking the integral of ϕ

$$p(P) := \text{v.p.} \int_{\infty_0}^{P} \phi. \tag{5.26}$$

The principal value is defined by the subtraction of the divergent part of the integral as the corresponding function on k_0. So

$$\phi = dp. \tag{5.27}$$

Now we can consider the function λ on C locally as the function $\lambda = \lambda(p)$ of the complex variable p. This function also depends on the point of the space $\hat{M}$ as on parameter.

Let $\hat{M}_\phi$ be the open domain in $\hat{M}$ specifying by the condition

$$\phi(P_i) \neq 0, \quad i = 1, \ldots, n. \tag{5.28}$$

Theorem 5.1.

1. *For any primary differential ϕ of the list (5.19) - (5.23) the multiplication (5.11), the unity and the Euler vector field (5.12), and the one-form Ω_{ϕ^2} determine on $\hat{M}_\phi$ a structure of Frobenius manifold. The corresponding flat coordinates t^A, $A = 1, \ldots, N$ consist of the five parts*

$$t^A = \left(t^{i;\alpha},\, i = 0, \ldots, m,\, \alpha = 1, \ldots, n_i;\; p^i, q^i,\, i = 1, \ldots, m;\; r^i, s^i,\, i = 1, \ldots, g\right) \tag{5.29}$$

where

$$t^{i;\alpha} = \operatorname*{res}_{\infty_i} k_i^{-\alpha} p \, d\lambda, \;\; i = 0, \ldots, m, \;\; \alpha = 1, \ldots, n_i; \tag{5.30a}$$

$$p^i = \text{v.p.} \int_{\infty_0}^{\infty_i} dp, \;\; i = 1, \ldots, m; \tag{5.30b}$$

$$q^i = -\operatorname*{res}_{\infty_i} \lambda dp, \;\; i = 1, \ldots, m; \tag{5.30c}$$

$$r^i = \oint_{b_i} dp, \tag{5.30d}$$

$$s^i = -\frac{1}{2\pi i} \oint_{a_i} \lambda dp, \;\; i = 1, \ldots, g. \tag{5.30e}$$

The metric (5.22) in the coordinates has the following form

$$\eta_{t^{i;\alpha} t^{i;\beta}} = \frac{1}{n_i + 1} \delta_{ij} \delta_{\alpha+\beta, n_i+1} \tag{5.31a}$$

$$\eta_{v^i w^j} = \frac{1}{n_i + 1} \delta_{ij} \tag{5.31b}$$

$$\eta_{r^i s^j} = \frac{1}{2\pi i} \delta_{ij}, \tag{5.31c}$$

other components of the η vanish.

The function $\lambda = \lambda(p)$ is the superpotential of this Frobenius manifold in the sense of Appendix I.

2. *For any other primary differential φ the one-form Ω_φ is an admissible one-form on the Frobenius manifold.*

Proof. From (5.11) it follows that the metric (5.25) which we will denote also by ds^2_ϕ is diagonal in the coordinates $u^1, \ldots, u^N$

$$ds^2_\phi = \sum_{i=1}^N \eta_{ii}(u) {du^i}^2 \tag{5.32a}$$

with

$$\eta_{ii}(u) = \operatorname*{res}_{P_i} \frac{\phi^2}{d\lambda}. \tag{5.32b}$$

We first prove that this is a Darboux - Egoroff metric for any primary differential.

Main lemma. *For any primary differential ϕ on the list (5.19) - (5.23) the metric (5.32) is a Darboux - Egoroff metric satisfying also the invariance conditions*

$$\mathcal{L}_e ds^2_\phi = 0, \tag{5.33a}$$

$$\mathcal{L}_E ds^2_\phi \text{ is proportional to } ds^2_\phi. \tag{5.33b}$$

The rotation coefficients of the metric do not depend on the choice of the primary differential ϕ.

Proof. We prove first some identity relating a bilinear combination of periods and principal parts at $\lambda = \infty$ of differentials on C as functions on the moduli to the residues in the branch points.

We introduce a local parameter z_a near the point ∞_a putting

$$z_a = k_a^{-1}. \tag{5.34}$$

Let $\omega^{(1)}$, $\omega^{(2)}$ be two differentials on the universal covering of $C \setminus (\infty_0 \cup \ldots \cup \infty_m)$ holomorphic outside of the infinity with the following properties at the infinite points

$$\omega^{(i)} = \sum_k c^{(i)}_{ka} z_a^k dz_a + d \sum_{k>0} r^{(i)}_{ka} \lambda^k \log \lambda, \quad P \to \infty_a \tag{5.35a}$$

$$\oint_{a_\alpha} \omega^{(i)} = A^{(i)}_\alpha \tag{5.35b}$$

$$\omega^{(i)}\left(P + a_\alpha\right) - \omega^{(i)}\left(P\right) = dp^{(i)}_\alpha(\lambda), \quad p^{(i)}_\alpha(\lambda) = \sum_{s>0} p^{(i)}_{s\alpha} \lambda^s \tag{5.35c}$$

$$\omega^{(i)}\left(P + b_\alpha\right) - \omega^{(i)}\left(P\right) = dq^{(i)}_\alpha(\lambda), \quad q^{(i)}_\alpha(\lambda) = \sum_{s>0} q^{(i)}_{s\alpha} \lambda^s, \tag{5.35d}$$

$i = 1, 2$, where $c^{(i)}_{ka}$, $r^{(i)}_{ka}$, $A^{(i)}_\alpha$, $p^{(i)}_{s\alpha}$, $q^{(i)}_{s\alpha}$ are some constants (i.e. independent on the curve). We introduce also a bilinear pairing of such differentials putting

$$< \omega^{(1)} \omega^{(2)} >:=$$

$$= -\sum_{a=0}^m \left[\sum_{k \geq 0} \frac{c^{(1)}_{-k-2,a}}{k+1} c^{(2)}_{k,a} + c_{-1,a} \text{v.p.} \int_{P_0}^{\infty_a} \omega^{(2)} + 2\pi i \text{v.p.} \int_{P_0}^{\infty_a} r^{(1)}_{k,a} \lambda^k \omega^{(2)} \right]$$

$$+ \frac{1}{2\pi i} \sum_{\alpha=1}^g \left[-\oint_{a_\alpha} q^{(1)}_\alpha(\lambda) \omega^{(2)} + \oint_{b_\alpha} p^{(1)}_\alpha(\lambda) \omega^{(2)} + A^{(1)}_\alpha \oint_{b_\alpha} \omega^{(2)} \right]. \tag{5.36}$$

The principal values, as above, are obtained by subtraction of the divergent parts of the integrals as the corresponding functions on $k_0, \ldots, k_m$; P_0 is a marked point of the curve C with

$$\lambda(P_0) = 0. \tag{5.37}$$

We will use also a natural connection in the tautological bundle

$$\begin{array}{cc} \downarrow & C \\ \hat{M} & \end{array}. \tag{5.38}$$

The connection is uniquely determined by the requirement that for the horizontal lifts of the vector fields ∂_i

$$\partial_i \lambda = 0. \tag{5.39}$$

Lemma 5.1. *The following identity holds*

$$\operatorname*{res}_{P_j} \frac{\omega^{(1)}\omega^{(2)}}{d\lambda} = \partial_j < \omega^{(1)}\, \omega^{(2)} > . \tag{5.40}$$

Proof. We realize the symplectic basis $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ by oriented cycles passing through the point P_0. After cutting C along these cycles we obtain a $4g$-gon. We connect one of the vertices of the $4g$-gon (denoting this again by P_0) with the infinite points $\infty_0, \ldots, \infty_m$ by pairwise nonintersecting paths running inside of the $4g$-gon. Adding cuttings along these paths we obtain a domain $\tilde{C}$. We assume that the λ-images of all the cuttings do not depend on the moduli $u \in \hat{M}$. This can be done locally. Then we have an identity

$$\frac{1}{2\pi i} \oint_{\partial \tilde{C}} \left(\omega^{(1)}(P) \int_{P_0}^{P} \partial_j \omega^{(2)} \right) = - \operatorname*{res}_{P_j} \frac{\omega^{(i)}\omega^{(2)}}{d\lambda}. \tag{5.41}$$

After calculation of all the residues and of all contour integrals we obtain (5.40). Lemma is proved.

Corollary 5.1. *The pairing (5.36) of the differentials of the form (5.35) is symmetric up to an additive constant not depending on the moduli.*

Proof of Main Lemma. From Lemma 5.1 we obtain

$$\eta_{jj}(u) = \partial_j < \phi\, \phi >, \quad j = 1, \ldots, N. \tag{5.42}$$

From this we obtain the symmetry of the rotation coefficients of the metric (5.32). To prove the identity (3.70a) for the rotation coeffients let us consider the differential

$$\partial_i \partial_j \phi \int \partial_k \phi$$

for distinct i, j, k. It has poles only in the branch points P_i, P_j, P_k. The contour integral of the differential along $\partial \tilde{C}$ (see above the proof of Lemma 5.1) equals zero. Hence the sum of the residues vansish. This reads

$$\partial_j \sqrt{\eta_{ii}} \partial_k \sqrt{\eta_{ii}} + \partial_i \sqrt{\eta_{jj}} \partial_k \sqrt{\eta_{jj}} = \sqrt{\eta_{kk}} \partial_i \partial_j \sqrt{\eta_{kk}}.$$

This can be written in the form (3.70a) due to the symmetry $\gamma_{ji} = \gamma_{ij}$.

Let us prove now that the rotation coefficients do not depend on ϕ. Let φ be another primary differential. We consider the differential

$$\partial_i \phi \int \partial_j \varphi$$

for $i \neq j$. From vanishing of the sum of the residues we obtain

$$\sqrt{\eta^{\varphi}_{jj}}\partial_i \sqrt{\eta^{\phi}_{jj}} = \sqrt{\eta^{\phi}_{ii}}\partial_j \sqrt{\eta^{\varphi}_{ii}}.$$

Using the symmetry $\gamma_{ji} = \gamma_{ij}$ we immediately obtain that the rotation coefficients of the two metrics coincide.

Now we are to prove the identity (5.33a). Let us define an operator D_e on functions $f = f(P, u)$ by the formula

$$D_e f = \frac{\partial f}{\partial \lambda} + \partial_e f. \tag{5.43}$$

The operator D_e can be extended to differentials as the Lie derivative (i.e. requiring $dD_e = D_e d$). We have

$$D_e \phi = 0 \tag{5.44}$$

for any of the primary differentials ϕ. Indeed, from the definition of these differentials it folows that these do not change with the transformations

$$\lambda \mapsto \lambda + b, \quad u^j \mapsto u^j + b, \quad j = 1, \ldots, N.$$

Such invariance is equivalent to (3.70b). From (5.42) we immediately obtain

$$\partial_e \eta_{jj} = 0$$

for the metric (5.32). Note that this implies also (3.70b).

Doing in a similar way we prove also (5.33b). Introduce the operator

$$D_E := \lambda \frac{\partial}{\partial \lambda} + \partial_E \tag{5.45}$$

we have

$$D_E \phi = [\phi]\phi. \tag{5.46}$$

Here the numbers $[\phi]$ for the differentials (5.19) - (5.23) have the form

$$\begin{aligned} [\phi_{t^{i;\alpha}}] &= \frac{\alpha}{n_i + 1} \\ [\phi_{v^i}] &= 1 \\ [\phi_{w^i}] &= 0 \\ [\phi_{r^i}] &= 1 \\ [\phi_{s^i}] &= 0. \end{aligned} \tag{5.47}$$

From this we obtain

$$\partial_E \eta^{\phi}_{ii} = (2[\phi]-1)\eta^{\phi}_{ii}. \tag{5.48}$$

The equation (3.70c) also follows from (5.48). Lemma is proved.

So we have obtained for any primary differential ϕ a flat Darboux - Egoroff metric in an open domain of $\hat{M}$ being invariant w.r.t. the multiplication (5.11). It is easy to see that the unity vector field is covariantly constant w.r.t. the Levi-Civita connection for the metric. According to the results of Lecture 3, this determine a Frobenius structure on the domain in $\hat{M}$. Any other metric for another primary differential will be admissible for this Frobenius structure due to Lemma 5.1 and Proposition 3.6. So all the Frobenius structures are obtained one from another by the Legendre-type transformations (B.2).

The open domains $\hat{M}_\phi$ cover all the universal covering of the Hurwitz space under consideration.

We are to prove the formulae for the flat coordinates of the metric and to establish that $\lambda = \lambda(p)$ is the superpotential for this Frobenius manifold.

We start from the superpotential.

Lemma 5.2. *Derivatives of $\lambda(p)dp$ along the variables (5.19) - (5.23) have the form*

$$\begin{aligned}
\partial_{t^{i;\alpha}}\lambda(p)dp &= -\phi_{t^{i;\alpha}}\\
\partial_{v^i}\lambda(p)dp &= -\phi_{v^i}\\
\partial_{w^i}\lambda(p)dp &= -\phi_{w^i}\\
\partial_{r^i}\lambda(p)dp &= -\phi_{r^i}\\
\partial_{s^i}\lambda(p)dp &= -\phi_{s^i}
\end{aligned} \tag{5.49}$$

I recall that the differentiation in (5.49) is to be done with $p = const$.

Proof. Using the "thermodynamic identity" (4.67) we can rewrite any of the derivatives (5.49) as

$$\partial_{t^A}\left(\lambda(p)dp\right)_{p=const} = -\partial_{t^A}\left(p\, d\lambda\right)_{\lambda=const}. \tag{5.50}$$

The derivatives like

$$\partial_{t^A} p(\lambda)_{\lambda=const}$$

are holomorphic on the finite part of the curve C outside the branch points P_j. In the branch point these derivatives have simple poles. But the differential $d\lambda$ vanishes precisely at the branch points. So (5.50) is holomorphic everywhere.

Let us consider now behaviour of the derivatives (5.50) at the infinity. We have

$$p = \text{ singular part } -$$

$$-(1-\delta_{i0})v^i - \frac{1}{n_i+1}\sum_{\alpha=1}^{n_i} t^{i;\alpha}k_i^{-(n_i-\alpha+1)} - \frac{1}{n_i+1}w^i k_i^{-(n_i+1)} + O\left(k_i^{-(n_i+2)}\right) \tag{5.51}$$

near the point ∞_i. The singular part by the construction does not depend on the moduli. Also we have

$$p(P+b_\alpha) - p(P) = r^\alpha, \quad \alpha = 1, \ldots, g. \tag{5.52}$$

For the differential $\omega := p\,d\lambda$ we will obtain from (5.51), (5.52) and from

$$d\lambda = (n_i+1)k_i^{n_i} dk_i \quad \text{near } \infty_i \tag{5.53}$$

the following analytic properties

$$\omega = \text{singular part } -(1-\delta_{i0})v^i d\lambda - \sum_{\alpha=1}^{n_i} t^{i;\alpha} k_i^{\alpha-1} dk_i - w^i \frac{dk_i}{k_i} + O\left(k_i^{-2}\right) dk_i \tag{5.54a}$$

$$\omega(P+b_\alpha) - \omega(P) = r^\alpha d\lambda \tag{5.54b}$$

$$\oint_{a_\alpha} \omega = -s^\alpha. \tag{5.54c}$$

Differentiating these formulae w.r.t. one of the variables (5.30) we obtain precisely one of the differentials with the analytic properties (5.19) - (5.23). This proves the lemma.

To complete the proof of Theorem we need to prove that (5.30) are the flat coordinates*) for the metric $<\ ,\ >_\phi$ and that the matrix of this metric in the coordinates (5.30) has the form (5.31).

Let t^A be one of the coordinates (5.30). We denote

$$\phi_A := -\partial_{t^A}\lambda\, dp. \tag{5.55}$$

By the definition and using Lemma 5.1 we obtain

$$< \partial_{t^A}, \partial_{t^B} >_\phi = \sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\phi_A \phi_B}{d\lambda} = \partial_e < \phi_A\, \phi_B > . \tag{5.56}$$

Note that in the formula (5.56) for $< \phi_A\, \phi_B >$ only the contribution of the second differential ϕ_B depends on the moduli. Let us define the coefficients $c_{ka}^{(A,\,B)}$, $A_\alpha^{(A,\,B)}$, $q_{1\alpha}^{(A,\,B)}$ for the differentials $\phi_{A,\,B}$ as in the formula (5.35). Note that by the construction of the primary differentials all the coefficients $r_{ka}^{(A,\,B)}$, $p_{s\alpha}^{(A,\,B)}$ vanish; the same is true for $q_{s\alpha}^{(A,\,B)}$ for $s>1$. From the equation $D_e\phi_B = 0$ (see (5.44) above) we obtain

$$\partial_e c_{ka}^{(B)} = \frac{k+1}{n_a+1} c_{k-n_a-1,a}^{(B)},$$

$$\partial_e \int_{P_0}^{\infty_a} \phi_B = \frac{c_{-n_a-2,a}^{(B)}}{n_a+1} + \left(\frac{\phi_B}{d\lambda}\right)_{P_0}$$

*) The statement of my papers [48, 49] that these are global coordinates on the Hurwitz space is wrong. They are coordinates on $\hat{M}_\phi$ only. Changing ϕ we obtain a coordinate system of the type (5.30) in a neighbourhood of any point of the Hurwitz space.

$$\partial_e \oint_{a_\alpha} \lambda\phi_B = \oint_{a_\alpha} \phi_B,$$

$$\partial_e \oint_{b_\alpha} \phi_B = -\frac{\phi_B}{d\lambda}(P+b_\alpha) + \frac{\phi_B}{d\lambda}(P) \tag{5.57}$$

(this does not depend on the point P).

The proof of all of these formulae is essentially the same. I will prove for example the last one.

Let us represent locally the differential ϕ_B as

$$\phi_B = d\Phi(\lambda; u^1, \dots, u^N).$$

We have for arbitrary a, b

$$\partial_e \int_a^b \phi_B = \partial_e \int_a^b d\Phi(\lambda; u^1, \dots, u^N) = \frac{d}{d\epsilon}\int_a^b d\Phi(\lambda; u^1+\epsilon, \dots, u^N+\epsilon)|_{\epsilon=0}. \tag{5.58}$$

Doing the change of the variable $\lambda \mapsto \lambda - \epsilon$ we rewrite (5.58) as

$$\frac{d}{d\epsilon}\int_{a-\epsilon}^{b-\epsilon} d\Phi(\lambda+\epsilon; u^1+\epsilon, \dots, u^N+\epsilon)|_{\epsilon=0} = -\left(\frac{d\Phi(\lambda; u^1, \dots, u^N)}{d\lambda}\right)_{\lambda=a}^{\lambda=b} \tag{5.59}$$

since

$$\frac{d}{d\epsilon} d\Phi(\lambda+\epsilon; u^1+\epsilon, \dots, u^N+\epsilon) \equiv D_e\phi_B = 0$$

(see (5.44) above). Using (5.59) for the contour integral we obtain (5.57).

From this and from (5.36) we obtain

$$\partial_e < \phi_A\, \phi_B >= -\sum_{a=0}^m \left(\frac{1}{n_a+1}\sum_{k=0}^{n_a-1} c^{(A)}_{-k-2,a} c^{(B)}_{k-n_a-1,a}\right) -$$

$$-\sum_{a=1}^m \frac{1}{n_a+1}\left(c^{(A)}_{-1,a} c^{(B)}_{-n_a-2,a} + c^{(B)}_{-1,a} c^{(A)}_{-n_a-2,a}\right) - \frac{1}{2\pi i}\sum_{\alpha=1}^g \left(q^{(A)}_{1\alpha} A^{(B)}_\alpha + A^{(A)}_\alpha q^{(B)}_{1\alpha}\right). \tag{5.60}$$

The r.h.s. of this expression is a constant that can be easyly calculated using the explicit form of the differentials ϕ_A, ϕ_B. This gives (5.31).

Observe now that the structure functions $c_{ABC}(t)$ of the Frobenius manifold can be calculated as

$$c_{ABC}(t) = \sum_{i=1}^n \operatorname*{res}_{P_i} \frac{\phi_{t^A}\phi_{t^B}\phi_{t^C}}{d\lambda dp}. \tag{5.61}$$

Extension of the Frobenius structure on *all* the moduli space $\hat{M}$ is given by the condition that the differential

$$\frac{\phi_{t^A}\phi_{t^B} - c^C_{AB}\phi_{t^C}\, dp}{d\lambda} \tag{5.62}$$

is holomorphic for $|w| < \infty$. (The Frobenius algebra on $T_t M$ will be nilpotent for Riemann surfaces $\lambda : C \to CP^1$ with more than double branch points.) Theorem is proved.

Remark 5.1. Interesting algebraic-geometrical examples of solutions of equations of associativity (1.14) (not satisfying the scaling invariance) generalizing ours were constructed in [95]. In these examples M is a moduli space of Riemann surfaces of genus g with marked points, marked germs of local parameters near these points, and with a marked normalized Abelian differential of the second kind $d\lambda$ with poles at marked points and with fixed b-periods

$$\oint_{b_i} d\lambda = B_i, \ \ i = 1, \dots, g. \tag{5.63}$$

For $B_i = 0$ $d\lambda$ is a perfect differential of a function λ. So one obtains the above Frobenius structures on $M_{g;n_0,\dots,n_m}$.

Exercise 5.2. Prove that the function F for the Frobenius structure constructed in Theorem has the form

$$F = -\frac{1}{2} < p\, d\lambda \; p\, d\lambda > \tag{5.64}$$

where the pairing $< \ \ >$ is defined in (5.36).

Exercise 5.3. Prove the formula

$$\partial_{t^A} \partial_{t^B} F = - < \phi_A \, \phi_B > . \tag{5.65}$$

Note, particularly, that for the t^A-variables of the fifth type (5.30e) the formula (5.65) reads

$$\partial_{s^\alpha} \partial_{s^\beta} F = -\tau_{\alpha\beta} \tag{5.66a}$$

where

$$\tau_{\alpha\beta} := \oint_{b_\beta} \phi_{s^\alpha} \tag{5.66b}$$

is the period matrix of holomorphic differentials on the curve C.

Remark 5.2. The formula (5.66) means that the Jacobians $J(C)$ of the curve C

$$J(C) := \mathbf{C}^g / \{m + \tau n\}, \quad m, \, n \in \mathbf{Z}^g$$

are Lagrangian manifolds for the symplectic structure

$$\sum_{\alpha=1}^{g} ds^\alpha \wedge dz_\alpha \tag{5.67}$$

where $z_1, \dots, z_g$ are the natural coordinates on $J(C)$ (i.e., coming from the linear coordinates in $\mathbf{C}^g$). Indeed, the shifts

$$z \mapsto z + m + \tau n \tag{5.68}$$

preserve the symplectic form (5.67) due to (5.66). Conversely, if the shifts (5.68) preserve (5.67) then the matrix τ can be presented in the form (5.66a). So the representation

(5.66) is a manifestation of the phenomenon that the Jacobians are *complex Liouville tori* and the coordinates z_α, s^α are the *complex action-angle variables* on the tori. The origin of the symplectic structure (5.67) in the geometry of Hurwitz spaces is in realization of these spaces as the moduli spaces [93] of the algebraic-geometrical solutions of integrable hierarchies of the KdV type (see also the next Lecture).

In the recent paper [45] an interesting symplectic structure has been constructed on the fiber bundle of the intermediate Jacobians of a Calabi - Yau three-folds X over the moduli space of pairs (X, Ω), $\Omega \in H^{3,0}(X)$. The intermediate Jacobians are also complex Liouville tori (i.e., Lagrangian manifolds) although they are not Abelian varieties. Their period matrix thus also can be represented in the form (5.66a) for appropriate coordinates on the moduli space.

Remark 5.3. A part of the flat coordinates of the intersection form of the Frobenius manifold can be obtained by the formulae similar to (5.30b) - (5.30d) with the substitution $\lambda \mapsto \log \lambda$. Another part is given, instead of (5.30a), by the formula

$$\tilde{t}^a := p(Q_a), \quad a = 1, \ldots, n \tag{5.69a}$$

where

$$n + 1 := n_0 + 1 + n_1 + 1 + \ldots + n_m + 1$$

is the number of sheets of the Riemann surface $\lambda : C \to CP^1$, Q_0, Q_1, $\ldots$, Q_n are the zeroes of λ on C,

$$\lambda(Q_a) = 0. \tag{5.69b}$$

The last step in our construction is to factorize over the group $Sp(g, \mathbf{Z})$ of changes of the symplectic basis $a_1, \ldots, a_g$, $b_1, \ldots, b_g$

$$\begin{aligned} a_i &\mapsto \sum_{j=1}^{g} (C_{ij} b_j + D_{ij} a_j) \\ b_i &\mapsto \sum_{j=1}^{g} (A_{ij} b_j + B_{ij} a_j) \end{aligned} \tag{5.70}$$

where the matrices $A = (A_{ij}),\ B = (B_{ij}),\ C = (C_{ij}),\ D = (D_{ij})$ are integer-valued matrices satisfying

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.71}$$

and changes of the branches k_j of the roots of λ near ∞_j

$$k_j \mapsto e^{\frac{2\pi i l}{n_j+1}} k_j, \quad l = 1, \ldots, n_j + 1. \tag{5.72}$$

We can calculate the transformation law of the primary differentials (5.19) - (5.23) with the transformations (5.70) - (5.72). This determines the transformation law of the metrics

(5.32). They transform like squares of the primary differentials. All this gives a messy picture on the quotient of the twisted Frobenius manifold coinciding with the Hurwitz space. An important simplification we have is that the multiplication law of tangent vectors to the Hurwitz space stays invariant w.r.t. the transformations.

The picture is simplified drastically if the genus of the curves equals zero. In this case the modular group (5.70) disappears and the action of the group of roots of unity (5.72) is very simple.

Exercise 5.4. Verify that the Frobenius structure of Theorem 5.1 on the Hurwitz space (5.7) for the primary differential $\phi = dp$ coincides with the structure of Example 1.7. Check also that the formulae (5.30a) for the flat coordinates in this case coincide with the formulae (4.61).

For the positive genus case the above transformation law of the invariant metrics splits into five blocks corresponding to the five types of the primary differentials (5.19) - (5.23). Let us consider only the invariant metrics and the corresponding Frobenius structures that correspond to the holomorphic primary differentials (the type five in the previous notations). Any of these structures for a given symplectic basis of cycles is parametrized by g constants $\gamma_1, \ldots, \gamma_g$ in such a way that the corresponding primary differential is

$$\phi = \sum_{i=1}^{g} \gamma_i \phi_{s^i}. \tag{5.73}$$

I recall that ϕ_{s^i} are the basic normalized holomorphic differentials on the curve C w.r.t. the given symplectic basis of cycles. The corresponding metric (5.32) we denote by $< \, , \, >_\gamma$. A change (5.70) of the symplectic basis of cycles determines the transformation law of the metric:

$$< \, , \, >_\gamma \mapsto < \, , \, >_{(C\tau+D)^{-1}\gamma} . \tag{5.74}$$

Here

$$\gamma = (\gamma_1, \ldots, \gamma_g)^T.$$

This is an example of twisted Frobenius manifold: g-parameter family of Frobenius structures on the Hurwitz space with the action (5.74) of the Siegel modular group (5.70), (5.71).

We stop now the general considerations postponing for further publications, and we consider examples.

Example 5.5. $M_{0;\,1,0}$. This is the space of rational functions of the form

$$\lambda = \frac{1}{2}p^2 + a + \frac{b}{p-c} \tag{5.75}$$

where a, b, c are arbitrary complex parameters. We take first dp as the basic primary differential. So $\lambda(p)$ is the LG superpotential (I.11). Using the definition (I.11) we immediately obtain for the metric

$$< \partial_a, \partial_a > = < \partial_b, \partial_c > = 1 \tag{5.76}$$

other components vanish, and for the trilinear tensor (I.12)

$$\begin{aligned} < \partial_a, \partial_a, \partial_a > &= 1 \\ < \partial_a, \partial_b, \partial_c > &= 1 \\ < \partial_b, \partial_b, \partial_b > &= b^{-1} \\ < \partial_b, \partial_c, \partial_c > &= c \\ < \partial_c, \partial_c \partial_c > &= b \end{aligned} \tag{5.77}$$

otherwise zero. The free energy and the Euler operator read

$$F = \frac{1}{6}a^3 + abc + \frac{1}{2}b^2 \log b + \frac{1}{6}bc^3 - \frac{3}{4}b^2 \tag{5.78}$$

(we add the quadratic term for a convenience later on)

$$E = a\partial_a + \frac{3}{2}b\partial_b + \frac{1}{2}c\partial_c. \tag{5.79}$$

This is the solution of WDVV of the second type (i.e. $< e, e > \neq 0$).

To obtain a more interesting solution of WDVV let us do the Legendre transform S_b (in the notations of Appendix B). The new flat coordinates read

$$\begin{aligned} \hat{t}^a &= c \\ \hat{t}^b &= a + \frac{1}{2}c^2 \\ \hat{t}^c &= \log b. \end{aligned} \tag{5.80}$$

The new free energy $\hat{F}$ is to be determined from the equations (B.2b). After simple calculations we obtain

$$\hat{F} = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{1}{24}t_2^4 + t_2 e^{t_3} \tag{5.81a}$$

where

$$\begin{aligned} t_1 &:= \hat{t}^b \\ t_2 &:= \hat{t}^a \\ t_3 &:= \hat{t}^c. \end{aligned} \tag{5.81b}$$

The Euler vector field of the new solution is

$$E = t_1\partial_1 + \frac{1}{2}t_2\partial_2 + \frac{3}{2}\partial_3. \tag{5.82}$$

So for the solution (5.81) of WDVV $d = 1$, $r = 3/2$. This is just the Frobenius manifold of Exercise 4.3. The corresponding primary differential is

$$d\hat{p} := \partial_b(\lambda dp) = \frac{dp}{p - c}.$$

From (5.26) we obtain

$$\hat{p} = \log \frac{p-c}{\sqrt{2}}.$$

Substituting to (5.75) we derive the superpotential for the Frobenius manifold (5.81)

$$\lambda = \lambda(\hat{p}) = e^{2\hat{p}} + t_2\sqrt{2}e^{\hat{p}} + t_1 + \frac{1}{\sqrt{2}}e^{t_3-\hat{p}}. \tag{5.84}$$

The intersection form of the Frobenius manifold (5.81) is given by the matrix

$$\left(g^{\alpha\beta}(t)\right) = \begin{pmatrix} 2t_2e^{t_3} & \frac{3}{2}e^{t_3} & t_1 \\ \frac{3}{2}e^{t_3} & t_1 - \frac{1}{2}t_2^2 & \frac{1}{2}t_2 \\ t_1 & \frac{1}{2}t_2 & \frac{3}{2} \end{pmatrix}. \tag{5.85}$$

The flat coordinates x, y, z of the intersection form are obtained using Remark 5.3 in the form (4.73) where

$$\hat{p} = -\frac{1}{6}\log 2 + \frac{z}{3} + x + (2m+1)\pi i \text{ and } \hat{p} = -\frac{1}{6}\log 2 + \frac{z}{3} + y + (2n+1)\pi i \tag{5.86}$$

are two of the roots of the equation $\lambda(\hat{p}) = 0$ (m, n are arbitrary integers). As in Exercise 4.3 the monodromy around the discriminant of the Frobenius manifold is an affine Weyl group (this time of the type $A_2^{(1)}$). It is generated, say, by the transformations

$$\begin{aligned} x &\mapsto y \\ y &\mapsto x \\ z &\mapsto z \end{aligned} \tag{5.87a}$$

and

$$\begin{aligned} x &\mapsto x \\ y &\mapsto -x-y \\ z &\mapsto z \end{aligned} \tag{5.87b}$$

and by translations

$$\begin{aligned} x &\mapsto x + 2m\pi i \\ y &\mapsto y + 2n\pi i \\ z &\mapsto z. \end{aligned} \tag{5.87c}$$

The monodromy around the obvious closed loop along t_3 gives an extension of the affine Weyl group by means of the transformation

$$\begin{aligned} x &\mapsto y - \frac{2\pi i}{3} \\ y &\mapsto -x - y - \frac{2\pi i}{3} \\ z &\mapsto z + 2\pi i. \end{aligned} \tag{5.87d}$$

This is the analogue of the gliding reflection (G.24b) since the cube of it is just a translation

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y \\ z &\mapsto z + 6\pi i. \end{aligned} \tag{5.87e}$$

We conclude that the monodromy group of the Frobenius manifold (5.81) coincides with the extension of the affine Weyl group of the type $A_2^{(1)}$ by means of the group of cubic roots of the translation (5.87e) (cf. Exercise 4.3 above).

Exercise 5.5. Using an appropriate generalization of the above example to the Hurwitz space $M_{0;n-1,\,0}$ show, that the monodromy group of the Frobenius manifold with the superpotential

$$\lambda(p) = e^{np} + a_1 e^{(n-1)p} + \ldots + a_n + a_{n+1} e^{-p} \tag{5.88a}$$

($a_1, \ldots, a_{n+1}$ are arbitrary complex numbers) is an extension of the affine Weyl group of the type $A_n^{(1)}$ by means of the group of roots of the order $n+1$ of the translation

$$\log a_{n+1} \mapsto \log a_{n+1} + 2(n+1)\pi i. \tag{5.88b}$$

Example 5.6. $M_{1;1}$. By the definition this is the space of elliptic curves of the form

$$\mu^2 = 4\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{5.89}$$

with arbitrary coefficients a_1, a_2, a_3 providing that the polynomial in the r.h.s. of (5.89) has no multiple roots. I will show that in this case the twisted Frobenius structure of Theorem 5.1 coincides with the twisted Frobenius structure on the space of orbits of the extended CCC group $\hat{A}_1$ (see above Appendix C and Appendix J).

It is convenient to use elliptic uniformization of the curves. I will use the Weierstrass uniformization. For this I rewrite the curve in the form

$$\mu^2 = 4(\lambda - c)^2 - g_2(\lambda - c) - g_3 = 4(\lambda - c - e_1)(\lambda - c - e_2)(\lambda - c - e_3) \tag{5.90}$$

where c and the parameters g_2, g_3 of the Weierstrass normal form are uniquely specified by a_1, a_2, a_3. The Weierstrass uniformization of (5.90) reads

$$\begin{aligned} \lambda &= \wp(z) + c \\ \mu &= \wp'(z) \end{aligned} \tag{5.91}$$

where $\wp = \wp(z; g_2, g_3)$ is the Weierstrass function. The infinite point is $z = 0$. Fixation of a basis of cycles on the curve (5.90) corresponds to fixation of an ordering of the roots e_1, e_2, e_3. The corresponding basis of the lattice of periods is 2ω, $2\omega'$ where

$$\wp(\omega) = e_1, \quad \wp(\omega') = e_3. \tag{5.92}$$

Let us use the holomorphic primary differential

$$dp = \frac{dz}{2\omega}$$

to construct a Frobenius structure on M. The corresponding superpotential is

$$\lambda(p) := \wp(2\omega p; \omega, \omega') + c. \tag{5.93}$$

$$p \simeq p + m + n\tau. \tag{5.94}$$

The flat coordinates (5.30) read

$$t^1 := \frac{1}{\pi i}\oint_a \lambda dp = \frac{1}{2\pi i\omega}\int_0^{2\omega} \left[\wp(z;\omega,\omega') + c\right] dz = \frac{1}{\pi i}\left[-c + \frac{\eta}{\omega}\right] \tag{5.95a}$$

$$t^2 := -\operatorname*{res}_{z=0} \lambda^{-1/2} p\, d\lambda = 1/\omega \tag{5.95b}$$

$$t^3 := \oint_b dp = \tau \quad \text{where } \tau = \omega'/\omega. \tag{5.95c}$$

(I have changed slightly the normalization of the flat coordinates (5.30).) The metric $< \, , \, >_{dp}$ in these coordinates has the form

$$ds^2 = dt^{2^2} + 2dt^1 dt^3. \tag{5.96}$$

Changes of the basis of cycles on the elliptic curve determined by the action of $SL(2, \mathbf{Z})$

$$\begin{aligned} \omega' &\mapsto a\omega' + b\omega \\ \omega &\mapsto c\omega' + d\omega \end{aligned} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) \tag{5.97}$$

gives rise to the following transformation of the metric

$$ds^2 \mapsto \frac{ds^2}{(c\tau + d)^2}. \tag{5.98}$$

These formulae determine a twisted Frobenius structure on the moduli space $M_{1;1}$ of elliptic curves (5.89).

The coincidence of the formulae for the flat coordinates and for the action (5.97) of the modular group with the above formulae in the constructions in the theory of the extended CCC group $\hat{A}_1$ is not accidental. We will prove now the following statement.

Theorem 5.2. *The space of orbits of $\hat{A}_1$ and the manifold $M_{1;1}$ are isomorphic as twisted Frobenius manifolds.*

Proof. Let us consider the intersection form of the Frobenius manifold $M_{1;1}$. Substituting

$$c = -\wp(z; \omega, \omega') \tag{5.99}$$

we obtain a flat metric on the $(z;\omega,\omega')$-space. Calculating this metric in the basis of the vector fields D_1, D_2, D_3 (J.14) we obtain the metric (J.24). It is clear that the unity and the Euler vector fields of $M_{1;1}$ coincide with $e = \partial/\partial\wp$ and (J.21) resp. So Theorem 5.1 implies the isomorphism of the Frobenius manifolds. The action of the modular group on these manifolds is given by the same formulae. Theorem is proved.

Example 5.7. We consider very briefly the Hurwitz space $M_{1;n}$ coinciding, as we will see, with the space of orbits of $\hat{A}_n$.

This is the moduli space of all elliptic curves

$$E_L := \mathbf{C}/L, \quad L = \{2m\omega + 2n\omega'\} \tag{5.100}$$

with a marked meromorphic function of degree $n+1$ with only one pole. We can assume this pole to coincide with $z = 0$.

To realize the space of orbits of the extended CCC group $\hat{A}_n$ we consider the family of the Abelian manifolds E_L^{n+1} fibered over the space $\mathcal{L}$ of all lattices L in $\mathbf{C}$. The symmetric group acts on the space of the fiber bundle by permutations $(z_0, z_1, \dots, z_n) \mapsto (z_{i_0}, z_{i_1}, \dots, z_{i_n})$ (we denote by z_k the coordinate on the k-th copy of the elliptic curve). We must restrict this onto the hyperplane

$$z_0 + z_1 + \ldots + z_n = 0 \tag{5.101}$$

After factorization over the permuations we obtain the space of orbits of $\hat{A}_n$.

Note that the conformal invariant metric for $\hat{A}_n$ on the space $(z_0, z_1, \dots, z_n; \omega, \omega')$ can be obtained as the restriction of the direct sum of the invariant metrics (J.15) onto the hyperplane (5.101).

We assign now to any point of the space of orbits of $\hat{A}_n$ a point (E_L, λ) in the Hurwitz space $M_{1;n}$ where E_L is the same elliptic curve and the function λ is defined by the formula

$$\lambda := (-1)^{n-1} \frac{\prod_{k=0}^n \sigma(z - z_k)}{\sigma^{n+1}(z) \prod_{k=0}^n \sigma(z_k)} = \frac{1}{n!} \frac{\det \begin{pmatrix} 1 & \wp(z) & \wp'(z) & \dots & \wp^{(n-1)}(z) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(n-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_n) & \wp'(z_n) & \dots & \wp^{(n-1)}(z_n) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(n-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(n-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_n) & \wp'(z_n) & \dots & \wp^{(n-2)}(z_n) \end{pmatrix}}$$

$$= \frac{(-1)^{n-1}}{n!} \wp^{(n-1)}(z) + c_n(z_0, z_1, \ldots z_n) \wp^{(n-2)}(z) + \ldots + c_1(z_0, z_1, \ldots, z_n) \tag{5.102}$$

where the coefficients $c_1, \dots, c_n$ are defined by this equation (I have used the classical addition formula [153, p.458] for sigma-functions). Taking, as above, the normalized holomorphic differential

$$dp := \frac{dz}{2\omega}$$

we obtain the twisted Frobenius structure on the space of functions (5.102). By the construction, the superpotential of this Frobenius manifold has the form (5.102) where one must substitute

$$z \mapsto 2\omega p.$$

It is easy to check that the intersection form of this Frobenius structure coincides with the conformal invariant metric of $\hat{A}_n$. Indeed, the flat coordinates (5.69) of the intersection form are the zeroes of $\lambda(p)$

$$\tilde{t}^0 := \frac{z_0}{2\omega}, \dots, \ \tilde{t}^n := \frac{z_n}{2\omega} \tag{5.103a}$$

related by the linear constraint

$$\tilde{t}^0 + \dots + \tilde{t}^n = 0. \tag{5.103b}$$

These coordinates are not well-defined. The ambiguity in their definition comes from two origins. The first one is the monodromy group of the Riemann surface $\lambda : C \to CP^1$. It acts by permutations on the set of the zeroes of the function λ. Another origin of the ambiguity is in the multivaluedness of the function p (Abelian integral) on the Riemann surface. So the translations of the flat coordinates

$$\tilde{t}^a \mapsto m + n\tau, \quad m, n \in \mathbf{Z} \tag{5.104}$$

give the same point of the Hurwitz space. The permutations and the translations just give the action of the CCC group $\tilde{A}_n$. To see the extended CCC group $\hat{A}_n$ one should consider all the flat coordinates of the intersection form. They are obtained by adding two more coordinates (5.30d) and (5.30e) (with the substitution $\lambda \mapsto \log \lambda$) to (5.103)

$$\tau = \frac{\omega'}{\omega} = \oint_b dp, \quad \phi := -\frac{1}{2\pi i} \oint_a \log \lambda \, dp \tag{5.105}$$

(cf. (J.12)).

For the general case $g > 1$ on the twisted Fobenius manifold $M_{g;n_0,\dots,n_m}$ with the g-dimensional family $< \ , \ >_\gamma$ (see (5.73) above) of the metrics corresponding to the holomorphic primary differentials the ambiguity in the definition of the flat coordinates (5.69) can be described by an action of the semidirect product of the monodromy group of the Riemann surfaces by the lattice of periods of holomorphic differentials. This can be considered as a higher genus generalization of CCC groups.

Our construction even for $g = 0$ does not cover the Frobenius structures of Lecture 4 on the spaces of orbits of finite Coxeter groups besides A_n. To include this class of examples into the general scheme of geometry on Hurwitz spaces and also to cover the orbit spaces of the extended CCC groups but $\hat{A}_n$ we are to consider equivariant Hurwitz spaces. These by definition consist of the pairs (C, λ) as above where a finite group acts on C preserving invariant the function λ. We are going to do it in further publications.

Lecture 6.

Frobenius manifolds and integrable hierarchies.
Coupling to topological gravity.

We start with explanation of the following observation: all the examples of Frobenius manifolds constructed in Lecture 5 are finite dimensional invariant manifolds of integrable hierarchies of the KdV type. To explain this relation we are to explain briefly the notion of semi-classical limit of an integrable hierarchy. In physical language the semi-classical (more particularly, the dispersionless) limit will correspond to the description of the quantum field theory after coupling to topological gravity considered in the genus zero (i.e., the tree-level) approximation.

Let

$$\partial_{t_k} y^a = f_k^a(y, \partial_x y, \partial_x^2 y, \ldots), \ a = 1, \ldots, l, \ k = 0, 1, \ldots \tag{6.1}$$

be a commutative hierarchy of Hamiltonian integrable systems of the KdV type. "Hierarchy" means that the systems are ordered, say, by action of a recursion operator. Number of recursions determine a level of a system in the hierarchy. Systems of the level zero form a primary part of the hierarchy (these correspond to the primary operators in TFT); others can be obtained from the primaries by recursions.

Example 6.1. The *nKdV* (or Gelfand - Dickey) hierarchy is an infinite system of commuting evolutionary PDEs for functions $a_1(x), \ldots, a_n(x)$. To construct the equations of the hierarchy we consider the operator

$$L = \partial^{n+1} + a_1(x)\partial^{n-1} + \ldots + a_n(x), \tag{6.2}$$

$$\partial = d/dx.$$

For any pair

$$(\alpha, p), \quad \alpha = 1, \ldots, n, \quad p = 0, 1, \ldots \tag{6.3}$$

we consider the evolutionary system of the Lax form

$$\partial_{t^{\alpha,p}} L = \left[L, \left[L^{\frac{\alpha}{n+1}+p}\right]_+\right]. \tag{6.4}$$

The brackets [,] stand for commutator of the operators, $\left[L^{\frac{\alpha}{n+1}+p}\right]_+$ denotes the differential part of the pseudodifferential operator $L^{\frac{\alpha}{n+1}+p}$. The commutator in the r.h.s. is an ordinary differential operator (in x) of the order at most $n-1$. So (6.4) is a system of PDE for the functions $a_1(x,t), \ldots, a_n(x,t)$, the time variable is $t = t^{\alpha,p}$. For example, for $\alpha = 1, \ p = 0$ we have $\left[L^{\frac{\alpha}{n+1}+p}\right]_+ = \partial$, so the corresponding PDE is the x-translations

$$\partial_{t^{1,0}} a_i(x) + \partial_x a_i(x) = 0. \tag{6.5}$$

The equations (6.4) have a bihamiltonian structure: they can be represented in the form

$$\partial_{t^{\alpha,p}} a_i(x) = \{a_i(x), H_{\alpha,p}\} = \{a_i(x), H_{\alpha,p-1}\}_1 \tag{6.6}$$

for some family of local Hamiltonians $H_{\alpha,p} = H_{\alpha,p}[a(x)]$ and w.r.t. a pair of Poisson brackets $\{\ ,\ \}$ and $\{\ ,\ \}_1$ on an appropriate space of functionals of $a_i(x)$. There is no symmetry between the Poisson brackets: the Casimirs (i.e., generators of the annihilator) of the *first* Poisson bracket $\{\ ,\ \}$ only are local functionals $H_{\alpha,-1}[a(x)]$. For the example of the KdV hierarchy ($n = 1$) the two Poisson brackets are

$$\{u(x), u(y)\} = \delta'(x-y) \tag{6.7a}$$

$$\{u(x), u(y)\}_1 = -\frac{1}{2}\delta'''(x-y) + 2u(x)\delta'(x-y) + u'(x)\delta(x-y). \tag{6.7b}$$

Here $u(x) = -a_1(x)$ in the previous notations. The annihilator of the first Poisson bracket is generated by the local Casimir

$$H_{-1} = \int u(x)\, dx. \tag{6.8}$$

The Poisson brackets satisfy a very important property of *compatibility*: any linear combination

$$\{\ ,\ \}_1 - \lambda\{\ ,\ \} \tag{6.9}$$

for an arbitrary parameter λ is again a Poisson bracket. This gives a possibility to construct an infinite sequence of the commuting Hamiltonians $H_{\alpha,p}$ starting from the Casimirs $H_{\alpha,-1}$, $\alpha = 1, \ldots, n$ of the first Poisson bracket $\{\ ,\ \}$ using (6.6) as the recursion relations (see [108] for details). For any *primary* Hamiltonian $H_{\alpha,-1}$ (a Casimir of the first Poisson structure) we obtain an infinite chain of its *descendants* $H_{\alpha,p}$, $p \geq 0$ determined from the recursion relations

$$\{a_i(x), H_{\alpha,p+1}\} = \{a_i(x), H_{\alpha,p}\}_1, \quad p = -1, 0, 1, 2, \ldots. \tag{6.10}$$

They all are local functionals (i.e. integrals of polynomials of $a_i(x)$ and of their derivatives).

The hierarchy posesses a rich family of finite-dimensional invariant manifolds. Some of them can be found in a straightforward way; one needs to apply algebraic geometry methods [56] to construct more wide class of invariant manifolds. Any of these manifolds after an extension to complex domain turns out to be fibered over some base M (a complex manifold of some dimension n) with m-dimensional tori as the fibers (common invariant tori of the hierarchy). For $m = 0$ M is nothing but the family of common stationary points of the hierarchy. For any $m \geq 0$ M is a moduli space of Riemann surfaces of some genus g with certain additional structures: marked points and a marked meromorphic function with poles of a prescribed order in these points [93]. This is just a Hurwitz space considered in Lecture 5. Therefore they are the families of parameters of the finite-gap ("g-gap") solutions of the hierarchy. Our main observation is that any such M carries a natural structure of a Frobenius manifold.

Example 6.2. For $nKdV$ the family of stationary solutions of the hierarchy consists of all operators L with constant coefficients (any two such operators commute pairwise). This coincides with Frobenius manifold of Example 1.7.

Example 6.3. For the same $nKdV$ the family of "g-gap" solutions is the Hurwitz space $M_{g,n}$ in the notations of Lecture 5.

To give an idea how an integrable Hamiltonian hierarchy of the above form induces tensors $c_{\alpha\beta}^{\gamma}$, $\eta_{\alpha\beta}$ on a finite dimensional invariant manifold M I need to introduce the notion of semiclassical limit of a hierarchy near a family M of invariant tori (sometimes it is called also a *dispersionless limit* or *Whitham averaging* of the hierarchy; see details in [57, 58]. In the simplest case of the family of stationary solutions the semiclassical limit is defined as follows: one should substitute in the equations of the hierarchy

$$x \mapsto \epsilon x = X, \ t_k \mapsto \epsilon t_k = T_k \tag{6.11}$$

and tend ϵ to zero. For more general M (family of invariant tori) one should add averaging over the tori. As a result one obtains a new integrable Hamiltonian hierarchy where the dependent variables are coordinates v^1, ..., v^n on M and the independent variables are the slow variables X and T_0, T_1, ... This new hierarchy always has a form of a quasilinear system of PDE of the first order

$$\partial_{T_k} v^p = c_{k}{}_{q}^{p}(v) \partial_X v^q, \ k = 0, 1, \ldots \tag{6.12}$$

for some matrices of coefficients $c_{k}{}_{q}^{p}(v)$. One can keep in mind the simplest example of a semiclassical limit (just the dispersionless limit) of the KdV hierarchy. Here M is the one-dimensional family of constant solutions of the KdV hierarchy. For example, rescaling the KdV one obtains

$$u_T = u u_X + \epsilon^2 u_{XXX} \tag{6.13}$$

(KdV with small dispersion). After $\epsilon \to 0$ one obtains

$$u_T = u u_X. \tag{6.14}$$

The semiclassical limit of all the KdV hierarchy has the form

$$\partial_{T_k} u = \frac{u^k}{k!} \partial_X u, \ k = 0, 1, \ldots. \tag{6.15}$$

A semiclassical limit of spatialy discretized hierarchies (like Toda system) is obtained by a similar way. It is still a system of quasilinear PDE of the first order.

It is important to note that the commutation representation (6.4) in the semiclassical limit takes the form

$$\partial_{T^{\alpha,p}} \lambda(X, p) = \{\lambda(X, p), \rho_{\alpha,p}(X, p)\}. \tag{6.16}$$

In the r.h.s. $\{\ ,\ \}$ stands for the standard Poisson bracket on the (X, p)-plane

$$\{\lambda(x, p), \rho(x, p)\} = \frac{\partial \lambda}{\partial p} \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial p} \frac{\partial \lambda}{\partial x}. \tag{6.17}$$

We will call (6.16), (6.17) *semiclassical Lax representation.* For the dispersionless limit the function $\lambda(x, p)$ is just the symbol of the L-operator obtained by the substitution

$d/dx \to p$. The function $\rho(x,p)$ can be computed using fractional powers as in (6.4). For the case of the semiclassical limits on a family of oscillating finite-gap solutions the construction of the functions λ and ρ is more complicated (roughly speaking, $\lambda(X,p)$ is the Bloch dispersion law, i.e. the dependence of the eigenvalue λ of the operator L with periodic or quasiperiodic coefficients on the quasimomentum p and on the slow spatial variable $X = \epsilon x$).

Let us come back to determination of tensors $\eta_{\alpha\beta}$, $c_{\alpha\beta}^{\gamma}$ on M. Let $v_1, \ldots, v^n$ be arbitrary coordinates on M. A semiclassical limit (or "averaging") of both the Hamiltonian structures in the sense of general construction of S.P.Novikov and the author induces a compatible pair of Hamiltonian structures of the semiclassical hierarchy: a pair of Poisson brackets of the form

$$\{v^p(X), v^q(Y)\}_{\text{semiclassical}} = \eta^{ps}(v(X))[\delta_s^q \partial_X \delta(X-Y) - \gamma_{sr}^q(v) v_X^r \delta(X-Y)] \tag{6.18a}$$

$$\{v^p(X), v^q(Y)\}_{\text{semiclassical}}^1 = g^{ps}(v(X))[\delta_s^q \partial_X \delta(X-Y) - \Gamma_{sr}^q(v) v_X^r \delta(X-Y)] \tag{6.18b}$$

where $\eta^{pq}(v)$ and $g^{pq}(v)$ are contravariant components of two metrics on M and $\gamma_{pr}^q(v)$ and $\Gamma_{pr}^q(v)$ are the Christoffel symbols of the corresponding Levi-Civitá connections for the metrics $\eta^{pq}(v)$ and $g^{pq}(v)$ resp. (the so-called *Poisson brackets of hydrodynamic type*). Observe that the metric $\eta_{\alpha\beta}$ is obtained from the semiclassical limit of the first Hamiltonian structure of the original hierarchy. From the general theory of Poisson brackets of hydrodynamic type [57, 58] one concludes that both the metrics on M have zero curvature. [In fact, from the compatibility of the Poisson brackets it follows that the metrics $\eta^{pq}(v)$ and $g^{pq}(v)$ form a flat pencil in the sense of Lecture 3.] So local flat coordinates t^1, ..., t^n on M exist such that the metric $\eta^{pq}(v)$ in this coordinates is constant

$$\frac{\partial t^\alpha}{\partial v^p} \frac{\partial t^\beta}{\partial v^q} \eta^{pq}(v) = \eta_{\alpha\beta} = \text{const}.$$

The Poisson bracket $\{\ ,\ \}_{\text{semiclassical}}$ in these coordinates has the form

$$\{t^\alpha(X), t^\beta(Y)\}_{\text{semiclassical}} = \eta^{\alpha\beta} \delta'(X-Y). \tag{6.19}$$

The tensor $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1}$ together with the flat coordinates t^α is the first part of a structure we want to construct. (The flat coordinates t^1, ..., t^n can be expressed via Casimirs of the original Poisson bracket and action variables and wave numbers along the invariant tori - see details in [57, 58, 120].)

To define a tensor $c_{\alpha\beta}^{\gamma}(t)$ on M (or, equivalently, the "primary free energy" $F(t)$) we need to use a semiclassical limit of the τ-function of the original hierarchy [94, 95, 48, 49, 141]. For the dispersionless limit the definition of the semiclassical τ-function reads

$$\log \tau_{\text{semiclassical}}(T_0, T_1, \ldots) = \lim_{\epsilon \to 0} \epsilon^{-2} \log \tau(\epsilon t_0, \epsilon t_1, \ldots). \tag{6.20}$$

Then

$$F = \log \tau_{\text{semiclassical}} \tag{6.21}$$

for a particular τ-function of the hierarchy. Here $\tau_{\mathrm{semiclassical}}$ should be considered as a function only of the n primary slow variables. The semiclassical τ-function as the function of all slow variables coincides with the tree-level partition function of the matter sector $\eta_{\alpha\beta}$, $c_{\alpha\beta}^{\gamma}$ coupled to topological gravity.

Summarizing, we can say that a structure of Frobenius manifold (i.e., a solution of WDVV) on an invariant manifold M of an integrable Hamiltonian hierarchy is induced by a semiclassical limit of the first Poisson bracket of the hierarchy and of a particular τ-function of the hierarchy.

Now we will try to solve the inverse problem: starting from a Frobenius manifold M to construct a bi-hamiltonian hierarchy and to realize M as an invariant manifold of the hierarchy in the sense of the previous construction. By now we are able only to construct the semiclassical limit of the unknown hierarchy corresponding to any Frobenius manifold M. The problem of recovering of the complete hierarchy looks to be more complicated. Probably, this can be done not for arbitrary Frobenius manifold - see an interesting discussion of this problem in the recent preprint [61].

We describe now briefly the corresponding construction. After this we explain why this is equivalent to coupling of the matter sector of TFT described by the Frobenius manifold to topological gravity in the tree-level approximation.

Let us fix a Frobenius manifold M. Considering this as the matter sector of a 2D TFT model, let us try to calculate the tree-level (i.e., the zero-genus) approximation of the complete model obtained by coupling of the matter sector to topological gravity. The idea to use hierarchies of Hamiltonian systems of hydrodynamic type for such a calculation was proposed by E.Witten [157] for the case of topological sigma-models. An advantage of my approach is in effective construction of these hierarchies for *any* solution of WDVV. The tree-level free energy of the model will be identified with τ-function of a particular solution of the hierarchy. The hierarchy carries a bihamiltonian structure under a non-resonance assumption for scaling dimensions of the model.

So let $c_{\alpha\beta}^{\gamma}(t)$, $\eta_{\alpha\beta}$ be a solution of WDVV, $t = (t^1, \ldots, t^n)$. I will construct a hierarchy of the first order PDE systems linear in derivatives (*systems of hydrodynamic type*) for functions $t^\alpha(T)$, T is an infinite vector of times

$$T = (T^{\alpha,p}), \quad \alpha = 1, \ \ldots, \ n, \quad p = 0, \ 1, \ \ldots; \ T^{1,0} = X, \tag{6.22}$$

$$\partial_{T^{\alpha,p}} t^\beta = c_{(\alpha,p)}{}_{\gamma}^{\beta}(t) \partial_X t^\gamma \tag{6.23}$$

for some matrices of coefficients $c_{(\alpha,p)}{}_{\gamma}^{\beta}(t)$. The marked variable $X = T^{1,0}$ usualy is called *cosmological constant*.

I will consider the equations (6.23) as dynamical systems (for any (α, p)) on the loop space $\mathcal{L}(M)$ of functions $t = t(X)$ with values in the Frobenius manifold M.

A. Construction of the systems. I define a Poisson bracket on the space of functions $t = t(X)$ (i.e. on the loop space $\mathcal{L}(M)$) by the formula

$$\{t^\alpha(X), t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X - Y). \tag{6.24}$$

All the systems (6.23) have hamiltonian form

$$\partial_{T^{\alpha,p}} t^\beta = \{t^\beta(X), H_{\alpha,p}\} \tag{6.25}$$

with the Hamiltonians of the form

$$H_{\alpha,p} = \int h_{\alpha,p+1}(t(X))dX. \tag{6.26}$$

The generating functions of densities of the Hamiltonians

$$h_\alpha(t,z) = \sum_{p=0}^{\infty} h_{\alpha,p}(t) z^p, \ \alpha = 1, \dots, n \tag{6.27}$$

coincide with the flat coordinates $\tilde{t}(t,z)$ of the perturbed connection $\tilde{\nabla}(z)$ (see (3.5) - (3.7)). That means that they are determined by the system (cf. (3.5))

$$\partial_\beta \partial_\gamma h_\alpha(t,z) = z c^\epsilon_{\beta\gamma}(t) \partial_\epsilon h_\alpha(t,z). \tag{6.28}$$

This gives simple recurrence relations for the densities $h_{\alpha,p}$. Solutions of (6.28) can be normalized in such a way that

$$h_\alpha(t,0) = t_\alpha = \eta_{\alpha\beta} t^\beta, \tag{6.29a}$$

$$< \nabla h_\alpha(t,z), \nabla h_\beta(t,-z) >= \eta_{\alpha\beta} \tag{6.29b}$$

$$\partial_1 h_\alpha(t,z) = z h_\alpha(t,z) + \eta_{1\alpha}. \tag{6.29c}$$

Here ∇ is the gradient (in t) w.r.t. the metric $< \, , \, >$.

Example 6.4. For a massive Frobenius manifold with $d < 1$ the functions $h_\alpha(t,z)$ can be determined uniquely from the equation

$$\partial_i h_\alpha = z^{-\mu_\alpha} \sqrt{\eta_{ii}(u)} \psi^0_{i\alpha}(u,z) \tag{6.30}$$

where the solution $\psi^0_{i\alpha}(u,z)$ of the system (3.122) has the form (3.124). Thus these functions can be continued analytically in any sector of the complex z-plane.

Exercise 6.1. Prove the following identities for the gradients of the generating functions $h_\alpha(t,z)$:

$$\nabla < \nabla h_\alpha(t,z), \nabla h_\beta(t,w) >= (z+w) \nabla h_\alpha(t,z) \cdot \nabla h_\beta(t,w) \tag{6.31}$$

$$[\nabla h_\alpha(t,z), \nabla h_\beta(t,w)] = (w-z) \nabla h_\alpha(t,z) \cdot \nabla h_\beta(t,w). \tag{6.32}$$

There is the product of the vector fields on the Frobenius manifold in the r.h.s. of the formulae.

Let us show that the Hamiltonians (6.26) are in involution. So all the systems of the hierarchy (6.23) commute pairwise.

Lemma 6.1. *The Poisson brackets (6.24) of the functionals $h_\alpha(t(X),z)$ for any fixed z have the form*

$$\{h_\alpha(t(X),z_1), h_\beta(t(Y),z_2)\} = [q_{\alpha\beta}(t(Y),z_1,z_2) + q_{\beta\alpha}(t(X),z_2,z_1)]\, \delta'(X-Y) \tag{6.33a}$$

where

$$q_{\alpha\beta}(t, z_1, z_2) := \frac{z_2}{z_1 + z_2} < \nabla h_\alpha(t, z_1), \nabla h_\beta(t, z_2) > -\frac{1}{2}\frac{z_1 - z_2}{z_1 + z_2}\eta_{\alpha\beta}. \tag{6.33b}$$

Proof. For the derivatives of $q_{\alpha\beta}(t, z_1, z_2)$ one has from (6.31)

$$\nabla q_{\alpha\beta}(t, z_1, z_2) = z_2 \nabla h_\alpha(t, z_1) \cdot \nabla h_\beta(t, z_2).$$

The l.h.s. of (6.33a) has the form

$$\begin{aligned}\{h_\alpha(t(X), z_1), h_\beta(t(Y), z_2)\} &=< \nabla h_\alpha(t(X), z_1), \nabla h_\beta(t(Y), z_2) > \delta'(X - Y)\\ &=< \nabla h_\alpha(t(X), z_1), \nabla h_\beta(t(X), z_2) > \delta'(X - Y)\\ &+ z_2 < \nabla h_\alpha(t, z_1) \cdot \nabla h_\beta(t, z_2), \partial_X t > \delta(X - Y).\end{aligned}$$

This completes the proof.

Exercise 6.2. For any solution $h(t, z)$ of the equation (6.28) prove that

$$\partial_{T^{\alpha,k}} h(t, z) = \partial_X \operatorname*{res}_{w=0} \frac{w^{-k-1}}{z + w} < \nabla h(t, z), \nabla h_\alpha(t, w) > . \tag{6.34}$$

(The denominator $z + w$ was lost in the formula (3.53) of [50].)

Corollary 6.1. *The Hamiltonians (6.26) commute pairwise.*

Observe that the functionals

$$H_{\alpha,-1} = \int t_\alpha(X)\, dX$$

span the annihilator of the Poisson bracket (6.24).

Exercise 6.3. Show that the equations (6.25) for $p = 0$ (the primary part of the hierarchy) have the form

$$\partial_{T^{\alpha,0}} t^\gamma = c^\gamma_{\alpha\beta}(t) \partial_X t^\beta. \tag{6.35}$$

For the equations with $p > 0$ obtain the recursion relation

$$\partial_{T^{\alpha,p}} = \nabla^\epsilon h_{\alpha,p} \partial_{T^{\epsilon,0}}. \tag{6.36}$$

Exercise 6.4. Prove that for a massive Frobenius manifold all the systems of the hierarchy (6.25) are diagonal in the canonical coordinates $u^1, \ldots, u^n$.

In this case from the results of Tsarev [144] it follows completeness of the family of the conservation laws (6.26) for any of the systems in the hierarchy (6.25).

B. Specification of a solution $t = t(T)$. The hierarchy (6.25) admits an obvious scaling group

$$T^{\alpha,p} \mapsto cT^{\alpha,p}, \quad t \mapsto t. \tag{6.37}$$

Let us take the nonconstant invariant solution for the symmetry

$$(\partial_{T^{1,1}} - \sum T^{\alpha,p}\partial_{T^{\alpha,p}})t(T) = 0 \tag{6.38}$$

(I identify $T^{1,0}$ and X. So the variable X is supressed in the formulae.) This solution can be found without quadratures from a fixed point equation for the gradient map

$$t = \nabla\Phi_T(t), \tag{6.39}$$

$$\Phi_T(t) = \sum_{\alpha,p} T^{\alpha,p} h_{\alpha,p}(t). \tag{6.40}$$

Proposition 6.1. *The hierarchy (6.25) has a unique solution*

$$t(X,T) := t(T)_{T^{1,0}\mapsto T^{1,0}+X}$$

where

$$t(T) = \left\{ T_0 + \sum_{q>0} T^{\beta,q}\nabla h_{\beta,q}(T_0) + \sum_{p,q>0} T^{\beta,q}T^{\gamma,p}\nabla h_{\beta,q-1}\cdot\nabla h_{\gamma,p} + \dots \right\} \tag{6.41}$$

satisfying (6.38) in the form of a power series in $T^{\alpha,p>0}$ with coefficients depending on $T_0 := \left(T^{1,0},\dots,T^{n,0}\right)$. This series can be found from the variational equations

$$\operatorname{grad}_t \left[\Phi_T(t) - \frac{1}{2}\eta_{\alpha\beta}t^\alpha t^\beta\right] = 0 \tag{6.42}$$

or, equivalently, as the fixed point of the gradient map

$$t = \nabla\Phi_T(t) \tag{6.43}$$

where

$$\Phi_T(t) = \sum_{\alpha,\, p} T^{\alpha,p} h_{\alpha,p}(t). \tag{6.44}$$

Proof. For the invariant solutions of (6.23) one has

$$\left(\sum c^{\beta}_{(\alpha,p)\,\gamma}(t) - t^\epsilon c^\beta_{\epsilon\gamma}(t)\right)\partial_X t^\gamma = 0, \quad \beta = 1,\dots,n. \tag{6.45}$$

Using the recursion (6.36) this system can be recasted in the form (6.42). In the points $T^{\alpha,p>0} = 0$, $T^{\alpha,0}$ arbitrary, $\alpha = 1,\dots,n$ one has

$$\partial_\mu\partial_\nu\left[\Phi_T(t) - \frac{1}{2}\eta_{\alpha\beta}t^\alpha t^\beta\right] = -\eta_{\mu\nu}.$$

Hence the solution of (6.38) is locally unique. Therefore it satisfies (6.25). The last equation makes it possible to apply the implicit function theorem to the system (6.43). Solving the stationary point equation (6.43) by iterations we obtain the needed power series. Proposition is proved.

C. τ-function. Let us define functions $< \phi_{\alpha,p}\phi_{\beta,q} > (t)$ from the expansion

$$(z+w)^{-1}(< \nabla h_\alpha(t,z), \nabla h_\beta(t,w) > -\eta_{\alpha\beta})$$

$$= \sum_{p,q=0}^{\infty} < \phi_{\alpha,p}\phi_{\beta,q} > (t) z^p w^q =: < \phi_\alpha(z)\phi_\beta(w) > (t). \tag{6.46}$$

The infinite matrix of the coefficients $< \phi_{\alpha,p}\phi_{\beta,q} > (t)$ has a simple meaning: it is the energy-momentum tensor of the commutative Hamiltonian hierarchy (6.25). That means that the matrix entry $< \phi_{\alpha,p}\phi_{\beta,q} > (t)$ is the density of flux of the Hamiltonian $H_{\alpha,p-1}$ along the flow $T^{\beta,q}$:

$$\partial_{T^{\beta,q}} h_{\alpha,p}(t) = \partial_X < \phi_{\alpha,p}\phi_{\beta,q} > (t). \tag{6.47}$$

Then we define

$$\log\tau(T) = \frac{1}{2}\sum < \phi_{\alpha,p}\phi_{\beta,q} > (t(T)) T^{\alpha,p}T^{\beta,q}$$

$$-\sum < \phi_{\alpha,p}\phi_{1,1} > (t(T)) T^{\alpha,p} + \frac{1}{2} < \phi_{1,1}\phi_{1,1} > (t(T)). \tag{6.48}$$

Here $t = t(T)$ is the solution (6.41).

Exercise 6.5. Prove that the τ-function satisfies the identity

$$\partial_{T^{\alpha,p}}\partial_{T^{\beta,q}} \log\tau = < \phi_{\alpha,p}\phi_{\beta,q} > . \tag{6.49}$$

The equations (6.47), (6.49) are the main motivation for the name "τ-function" (cf. [135, 84]).

Exercise 6.6. Derive the following formulae

$$\begin{aligned}
< \phi_{\alpha,0}\phi_{\beta,0} > &= \partial_\alpha\partial_\beta F \\
< \phi_{\alpha,p}\phi_{1,0} > &= h_{\alpha,p} \\
< \phi_{\alpha,p}\phi_{\beta,0} > &= \partial_\beta h_{\alpha,p+1} \\
< \phi_{\alpha,p}\phi_{1,1} > &= (t^\gamma\partial_\gamma - 1)\, h_{\alpha,p+1} \\
< \phi_{\alpha,0}\phi_{1,1} > &= t^\lambda\partial_\alpha\partial_\lambda F - \partial_\alpha F \\
< \phi_{1,1}\phi_{1,1} > &= t^\alpha t^\beta\partial_\alpha\partial_\beta F - 2t^\alpha\partial_\alpha F + 2F.
\end{aligned} \tag{6.50}$$

Exercise 6.7. Show that

$$< \phi_\alpha(z)\phi_1(w) > = h_\alpha(t,z) + O(w). \tag{6.51}$$

Remark 6.1. More general solutions of (6.25) has the form

$$\nabla[\Phi_T(t) - \Phi_{T_0}(t)] = 0 \tag{6.52}$$

for arbitrary constant infinite vector $T_0 = (T_0^{\alpha,p})$. For massive Frobenius manifolds these form a dense subset in the space of all solutions of (6.25) (see [144, 50]). Formally they can be obtained from the solution (6.42) by a shift of the arguments $T^{\alpha,p}$. τ-function of the solution (6.52) can be formaly obtained from (6.48) by the same shift. For the example of topological gravity [156, 157] such a shift is just the operation that relates the tree-level free energies of the topological phase of 2D gravity and of the matrix model. It should be taken into account that the operation of such a time shift in systems of hydrodynamic type is a subtle one: it can pass through a point of gradient catastrophe where derivatives become infinite. The corresponding solution of the KdV hierarchy has no gradient catastrophes but oscillating zones arise (see [74] for details).

Theorem 6.1. *Let*

$$\mathcal{F}(T) = \log \tau(T), \tag{6.53a}$$

$$< \phi_{\alpha,p}\phi_{\beta,q} \ldots >_0 = \partial_{T^{\alpha,p}}\partial_{T^{\beta,q}} \ldots \mathcal{F}(T). \tag{6.53b}$$

Then the following relations hold

$$\mathcal{F}(T)|_{T^{\alpha,p}=0 \text{ for } p>0,\ T^{\alpha,0}=t^\alpha} = F(t) \tag{6.54a}$$

$$\partial_X \mathcal{F}(T) = \sum T^{\alpha,p}\partial_{T^{\alpha,p-1}}\mathcal{F}(T) + \frac{1}{2}\eta_{\alpha\beta}T^{\alpha,0}T^{\beta,0} \tag{6.54b}$$

$$< \phi_{\alpha,p}\phi_{\beta,q}\phi_{\gamma,r} >_0 = < \phi_{\alpha,p-1}\phi_{\lambda,0} >_0 \eta^{\lambda\mu} < \phi_{\mu,0}\phi_{\beta,q}\phi_{\gamma,r} >_0 . \tag{6.54c}$$

Proof can be obtained using the results of the Exercises (6.5) - (6.7) (see [50] for details).

Let me establish now a 1-1 correspondence between the statements of the theorem and the axioms of Dijkgraaf and Witten of coupling to topological gravity. In a complete model of 2D TFT (i.e. a matter sector coupled to topological gravity) there are infinite number of operators. They usualy are denoted by $\phi_{\alpha,p}$ or $\sigma_p(\phi_\alpha)$. The operators $\phi_{\alpha,0}$ can be identified with the primary operators ϕ_α; the operators $\phi_{\alpha,p}$ for $p > 0$ are called *gravitational descendants* of ϕ_α. Respectively one has infinite number of coupling constants $T^{\alpha,p}$. For example, for the A_n-topological minimal models (see Lecture 2 above) the gravitational descendants come from the Mumford - Morita - Miller classes of the moduli spaces $\mathcal{M}_{g,s}$. A similar description of gravitational descendants can be done also for topological sigma-models [157, 41].

The formula (6.53a) expresses the tree-level (i.e. genus zero) partition function of the model of 2D TFT via logarythm of the τ-function (6.48). Equation (6.53b) is the standard relation between the correlators (of genus zero) in the model and the free energy. Equation (6.54a) manifests that before coupling to gravity the partition function (6.53a) coincides with the primary partition function of the given matter sector. Equation (6.54b) is the string equation for the free energy [156, 157, 41]. And equations (6.54c) coincide with the genus zero recursion relations for correlators of a TFT [43, 157].

We conclude with another formulation of Theorem 6.1.

Theorem 6.2. *For any Frobenius manifold the formulae give a solution of the Dijkgraaf - Witten relations defining coupling to topological gravity at the tree level.*

I recall that the solution of these relations for a given matter sector of a TFT (i.e., for a given Frobenius manifold) is unique, according to [43].

Remark 6.2. According to Lecture 2, in topological sigma-models the genus zero correlators of the primary fields can be computed as the values on the fundamental class $[M_{0,n}]$ of the products $\phi_\alpha \phi_\beta \dots$ of some cohomology classes $\phi_\alpha,\ \phi_\beta, \dots \in H^*(M_{0,n})$ of the moduli space $M_{0,n}$ of rational curves with n punctures. Coupling to topological gravity is given in terms of the values of the same products on the cycles dual to Mumford - Morita - Miller classes. So, our Theorem gives an algorythm of reconstructing of the generating function of these values starting from a solution of the equations of associativity (note that the scaling (1.5) has not been used in the construction). Recently Kontsevich and Manin proved that the cohomology classes $\phi_\alpha,\ \phi_\beta, \dots$ themselves can be uniquely reconstructed starting from a solution of the associativity equation (the second reconstruction theorem of [92]).

Particularly, from (6.53) one obtains

$$< \phi_{\alpha,p}\phi_{\beta,q} >_0 = < \phi_{\alpha,p}\phi_{\beta,q} > \tag{6.55a}$$

where the r.h.s. is defined in (6.46),

$$< \phi_{\alpha,p}\phi_{1,0} >_0 = h_{\alpha,p}(t(T)), \tag{6.55b}$$

$$< \phi_{\alpha,p}\phi_{\beta,q}\phi_{\gamma,r} >_0 = < \nabla h_{\alpha,p} \cdot \nabla h_{\beta,q} \cdot \nabla h_{\gamma,r}, [e - \sum T^{\alpha,p} \nabla h_{\alpha,p-1}]^{-1} > . \tag{6.55c}$$

The second factor of the inner product in the r.h.s. of (6.55c) is an invertible element (in the Frobenius algebra of vector fields on M) for sufficiently small $T^{\alpha,p}$, $p > 0$. From the last formula one obtains

Proposition 6.2. *The coefficients*

$$c_{p,}{}^{\gamma}_{\alpha\beta}(T) = \eta^{\gamma\mu} \partial_{T^{\alpha,p}} \partial_{T^{\beta,p}} \partial_{T^{\mu,p}} \log \tau(T) \tag{6.56}$$

for any p and any T are structure constants of a commutative associative algebra with the invariant inner product $\eta_{\alpha\beta}$.

As a rule such an algebra has no unity.

In fact the Proposition holds also for a τ-function of an arbitrary solution of the form (6.52).

We see that the hierarchy (6.25) determines a family of Bäcklund transforms of the associativity equations (1.14)

$$F(t) \mapsto \tilde{F}(\tilde{t}),$$

$$\tilde{F} = \log \tau, \ \tilde{t}^\alpha = T^{\alpha,p} \tag{6.57}$$

for a fixed p and for arbitrary τ-function of (6.25). So it is natural to consider equations of the hierarchy as Lie – Bäcklund symmetries of WDVV.

Up to now I even did not use the scaling invariance (1.9). It turns out that this gives rise to a bihamiltonian structure of the hierarchy (6.25) under certain nonresonancy conditions.

We say that a pair α, p is *resonant* if

$$\frac{d+1}{2} - q_\alpha + p = 0. \tag{6.58}$$

Here p is a nonnegative integer. The Frobenius manifold with the scaling dimensions q_α, d is *nonresonant* if all pairs α, p are nonresonant. For example, manifolds satisfying the inequalities

$$0 = q_1 \le q_2 \le \ldots \le q_n = d < 1 \tag{6.59}$$

all are nonresonant.

Theorem 6.3. *1) For a Frobenius manifold with the scaling dimensions q_α and d the formula*

$$\{t^\alpha(X), t^\beta(Y)\}_1 = g^{\alpha\beta}(t(X))\delta'(X-Y) + \left(\frac{d+1}{2} - q_\beta\right) c_\gamma^{\alpha\beta}(t(X))\partial_X t^\gamma(X)\delta(X-Y) \tag{6.60}$$

where $g^{\alpha\beta}(t)$ is the intersection form of the Frobenius manifold determines a Poisson bracket compatible with the Poisson bracket (6.24). 2) For a nonresonant TCFT model all the equations of the hierarchy (6.25) are Hamiltonian equations also with respect to the Poisson bracket (6.60).

Proof follows from (H.12).

The nonresonancy condition is essential: equations (6.25) with resonant numbers (α, p) do not admit another Poisson structure.

Example 6.5. Trivial Frobenius manifold corresponding to a graded n-dimensional Frobenius algebra A. Let

$$\mathbf{t} = t^\alpha e_\alpha \in A. \tag{6.61}$$

The linear system (6.28) can be solved easily:

$$h_\alpha(t,z) = z^{-1} < e_\alpha, e^{z\mathbf{t}} - 1 > . \tag{6.62}$$

This gives the following form of the hierarchy (6.25)

$$\partial_{T^{\alpha,p}} \mathbf{t} = \frac{1}{p!} e_\alpha \mathbf{t}^p \partial_X \mathbf{t}. \tag{6.63}$$

The solution (6.43) is specified as the fixed point

$$G(\mathbf{t}) = \mathbf{t}, \tag{6.64a}$$

$$G(\mathbf{t}) = \sum_{p=0}^{\infty} \frac{\mathbf{T}_p}{p!} \mathbf{t}^p. \tag{6.64b}$$

Here I introduce A-valued coupling constants

$$\mathbf{T}_p = T^{\alpha,p} e_\alpha \in A, \ p = 0, \ 1, \ldots. \tag{6.65}$$

The solution of (6.64a) has the well-known form

$$\mathbf{t} = G(G(G(\ldots))) \tag{6.66}$$

(infinite number of iterations). The τ-function of the solution (6.66) has the form

$$\log \tau = \frac{1}{6} < 1, \mathbf{t}^3 > - \sum_p \frac{< \mathbf{T}_p, \mathbf{t}^{p+2} >}{(p+2)p!} + \frac{1}{2} \sum_{p,q} \frac{< \mathbf{T}_p \mathbf{T}_q, \mathbf{t}^{p+q+1} >}{(p+q+1)p!q!}. \tag{6.67}$$

For the tree-level correlation functions of a TFT-model with constant primary correlators one immediately obtains

$$< \phi_{\alpha,p} \phi_{\beta,q} >_0 = \frac{< e_\alpha e_\beta, \mathbf{t}^{p+q+1} >}{(p+q+1)p!q!}, \tag{6.68a}$$

$$< \phi_{\alpha,p} \phi_{\beta,q} \phi_{\gamma,r} >_0 = \frac{1}{p!q!r!} < e_\alpha e_\beta e_\gamma, \frac{\mathbf{t}^{p+q+r}}{1 - \sum_{s \geq 1} \frac{\mathbf{T}_s \mathbf{t}^{s-1}}{(s-1)!}} > . \tag{6.68b}$$

Let us consider now the second hamiltonian structure (6.60). I start with the most elementary case $n = 1$ (the pure gravity). Let me redenote the coupling constant

$$u = t^1.$$

The Poisson bracket (6.60) for this case reads

$$\{u(X), u(Y)\}_1 = \frac{1}{2}(u(X) + u(Y))\delta'(X - Y). \tag{6.69}$$

This is nothing but the Lie – Poisson bracket on the dual space to the Lie algebra of one-dimensional vector fields.

For arbitrary graded Frobenius algebra A the Poisson bracket (6.60) also is linear in the coordinates t^α

$$\{t^\alpha(X), t^\beta(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha) c_\gamma^{\alpha\beta} t^\gamma(X) + (\frac{d+1}{2} - q_\beta) c_\gamma^{\alpha\beta} t^\gamma(Y)]\delta'(X - Y). \tag{6.70}$$

It determines therefore a structure of an infinite dimensional Lie algebra on the loop space $\mathcal{L}(A^*)$ where A^* is the dual space to the graded Frobenius algebra A. Theory of linear Poisson brackets of hydrodynamic type and of corresponding infinite dimensional

Lie algebras was constructed in [12] (see also [58]). But the class of examples (6.70) is a new one.

Let us come back to a general (i.e. nontrivial) Frobenius manifold. I will assume that the scaling dimensions are ordered in such a way that

$$0 = q_1 < q_2 \leq \ldots \leq q_{n-1} < q_n = d. \tag{6.71}$$

Then from (6.60) one obtains

$$\{t^n(X), t^n(Y)\}_1 = \frac{1-d}{2}(t^n(X) + t^n(Y))\delta'(X-Y). \tag{6.72}$$

Since

$$\{t^\alpha(X), t^n(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)t^\alpha(X) + \frac{1-d}{2}t^\alpha(Y)]\delta'(X-Y), \tag{6.73}$$

the functional

$$P = \frac{2}{1-d}\int t^n(X)dX \tag{6.74}$$

generates spatial translations. We see that for $d \neq 1$ the Poisson bracket (6.60) can be considered as a nonlinear extension of the Lie algebra of one-dimensional vector fields. An interesting question is to find an analogue of the Gelfand – Fuchs cocycle for this bracket. I found such a cocycle for a more particular class of Frobenius manifolds. We say that a Frobenius manifold is *graded* if for any t the Frobenius algebra $c_{\alpha\beta}^\gamma(t)$, $\eta_{\alpha\beta}$ is graded.

Theorem 6.4. *For a graded Frobenius manifold the formula*

$$\{t^\alpha(X), t^\beta(Y)\}\hat{}_1 = \{t^\alpha(X), t^\beta(Y)\}_1 + \epsilon^2\eta^{1\alpha}\eta^{1\beta}\delta'''(X-Y) \tag{6.75}$$

determines a Poisson bracket compatible with (6.24) and (6.60) for arbitrary ϵ^2 (the central charge). For a generic graded Frobenius manifold this is the only one deformation of the Poisson bracket (6.60) proportional to $\delta'''(X-Y)$.

For $n = 1$ (6.75) determines nothing but the Lie – Poisson bracket on the dual space to the Virasoro algebra

$$\{u(X), u(Y)\}\hat{}_1 = \frac{1}{2}[u(X) + u(Y)]\delta'(X-Y) + \epsilon^2\delta'''(X-Y) \tag{6.76}$$

(the second Poisson structure of the KdV hierarchy). For $n > 1$ and constant primary correlators (i.e. for a constant graded Frobenius algebra A) the Poisson bracket (6.75) can be considered as a vector-valued extension (for $d \neq 1$) of the Virasoro.

The compatible pair of the Poisson brackets (6.24) and (6.75) generates an integrable hierarchy of PDE for a non-resonant graded Frobenius manifold using the standard machinery of the bihamiltonian formalism (see above)

$$\partial_{T^{\alpha,p}}t^\beta = \{t^\beta(X), \hat{H}_{\alpha,p}\} = \{t^\beta(X), \hat{H}_{\alpha,p-1}\}\hat{}_1. \tag{6.77}$$

Here the Hamiltonians have the form

$$\hat{H}_{\alpha,p} = \int \hat{h}_{\alpha,p+1} dX, \tag{6.78a}$$

$$\hat{h}_{\alpha,p+1} = [\frac{d+1}{2} - q_\alpha + p]^{-1} h_{\alpha,p+1}(t) + \epsilon^2 \Delta\hat{h}_{\alpha,p+1}(t, \partial_X t, \ldots, \partial_X^p t; \epsilon^2) \tag{6.78b}$$

where $\Delta\hat{h}_{\alpha,p+1}$ are some polynomials determined by (6.77). They are graded-homogeneous of degree 2 where $\deg \partial_X^k t = k$, $\deg \epsilon = -1$. The small dispersion parameter ϵ also plays the role of the string coupling constant. It is clear that the hierarchy (6.25) is the zero-dispersion limit of this hierarchy. For $n = 1$ using the pair (6.24) and (6.76) one immediately recover the KdV hierarchy. Note that this describes the topological gravity. For a trivial manifold (i.e. for a graded Frobenius algebra A) the first nontrivial equations of the hierarchy are

$$\partial_{T^{\alpha,1}} \mathbf{t} = e_\alpha \mathbf{t}\mathbf{t}_X + \frac{2\epsilon^2}{3-d} e_\alpha e_n \mathbf{t}_{XXX}. \tag{6.79}$$

For non-graded Frobenius manifolds it could be of interest to find nonlinear analogues of the cocycle (6.75). These should be differential geometric Poisson brackets of the third order [58] of the form

$$\begin{gathered} \{t^\alpha(X), t^\beta(Y)\}\hat{}_1 = \{t^\alpha(X), t^\beta(Y)\}_1 + \\ \epsilon^2 \{g^{\alpha\beta}(t(X))\delta'''(X-Y) + b_\gamma^{\alpha\beta}(t(X)) t_X^\gamma \delta''(X-Y) + \\ [f_\gamma^{\alpha\beta}(t(X)) t_{XX}^\gamma + h_{\gamma\delta}^{\alpha\beta}(t(X)) t_X^\gamma t_X^\delta] \delta'(X-Y) + \\ [p_\gamma^{\alpha\beta}(t) t_{XXX}^\gamma + q_{\gamma\delta}^{\alpha\beta}(t) t_{XX}^\gamma t_X^\delta + r_{\gamma\delta\lambda}^{\alpha\beta}(t) t_X^\gamma t_X^\delta t_X^\lambda] \delta(X-Y)\}. \end{gathered} \tag{6.80}$$

I recall (see [58]) that the form (6.80) of the Poisson bracket should be invariant with respect to nonlinear changes of coordinates in the manifold M. This implies that the leading term $g^{\alpha\beta}(t)$ transforms like a metric (may be, degenerate) on the cotangent bundle T_*M, $b_\gamma^{\alpha\beta}(t)$ are contravariant components of a connection on M etc. The Poisson bracket (6.80) is assumed to be compatible with (6.24). Then the compatible pair (6.24), (6.80) of the Poisson brackets generates an integrable hierarchy of the same structure (6.77), (6.78). The hierarchy (6.25) will be the dispersionless limit of (6.77).

Example 6.6. Let me describe the hierarchies (6.25) for two-dimensional Frobenius manifolds. Let us redenote the coupling constants

$$t^1 = u, \ t^2 = \rho. \tag{6.81}$$

For $d \neq -1, 1, 3$ the primary free energy F has the form

$$F = \frac{1}{2}\rho u^2 + \frac{g}{a(a+2)}\rho^{a+2}, \tag{6.82}$$

$$a = \frac{1+d}{1-d} \tag{6.83}$$

where we introduce an arbitrary constant g. Let me give an example of equations of the hierarchy (6.25) (the $T = T^{1,1}$-flow)

$$u_T + uu_X + g\rho^a \rho_X = 0 \tag{6.84a}$$

$$\rho_T + (\rho u)_X = 0. \tag{6.84b}$$

These are the equations of isentropic motion of one-dimensional fluid with the dependence of the pressure on the density of the form $p = \frac{g}{a+2}\rho^{a+2}$. The Poisson structure (6.24) for these equations was proposed in [123]. For $a = 0$ (equivalently $d = -1$) the system coincides with the equations of waves on shallow water (the dispersionless limit [164] of the nonlinear Schrödinger equation (NLS)).

For $d = 1$ the primary free energy has the form

$$F = \frac{1}{2}\rho u^2 + g e^{\rho}. \tag{6.85}$$

This coincides with the free energy of the topological sigma-model with CP^1 as the target space. The corresponding $T = T^{2,0}$-system of the hierarchy (6.25) reads

$$u_T = g(e^{\rho})_X$$

$$\rho_T = u_X.$$

Eliminating u one obtains the long wave limit

$$\rho_{TT} = g(e^{\rho})_{XX} \tag{6.86}$$

of the Toda system

$$\rho_{n\,tt} = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}}. \tag{6.87}$$

Example 6.7. For the Frobenius manifold of Example 1.7 the hierarchy (6.25) is just the dispersionless limit of the $nKdV$-hierarchy. This was essentially obtained in [42] and elucidated by Krichever in [94]. The two metrics on the Frobenius manifold (I recall that this coincides with the space of orbits of the group A_n) are just obtained from the two hamiltonian structures of the $nKdV$ hierarchy: the Saito metric is obtained by the semiclassical limit of from the first Gelfand-Dickey Poisson bracket of $nKdV$ and the Euclidean metric is obtained by the same semiclassical limit from the second Gelfand-Dickey Poisson bracket. The Saito and the Euclidean coordinates on the orbit space are the Casimirs for the corresponding Poisson brackets. The factorization map $V \to M = V/W$ is the semiclassical limit of the Miura transformation.

Example 6.8. The Hurwitz spaces $M_{g;n_0,\dots,n_m}$ parametrize "g-gap" solutions of certain integrable hierarchies. The Lax operator L for these hierarchies must have a form of a $(m+1) \times (m+1)$-matrix. The equation $L\psi = 0$ for a $(m+1)$-component vector-function ψ must read as a system of ODE (in x) of the orders $n_0 + 1, \dots, n_m + 1$ resp. Particularly, for $m = 0$ one obtains the scalar operator L of the order $n = n_0$. So the

Hurwitz spaces $M_{g;n}$ parametrize algebraic-geometrical solutions [93] (of the genus g) of the $nKdV$ hierarchy.

To describe the hierarchy (6.25) we are to solve the recursion system (6.28).

Proposition 6.3. *The generating functions $h_{t^A}(t;z)$ (6.27) (where t^A is one of the flat coordinates (5.30) on the Hurwitz space) have the form*

$$\begin{aligned} h_{t^{i;\alpha}}(t;z) &= -\frac{n_i+1}{\alpha}\operatorname*{res}_{p=\infty_i} k_i^{\alpha}\,{}_1F_1(1;1+\frac{\alpha}{n_i+1};z\lambda(p))dp \\ h_{p^i} &= \text{v.p.}\int_{\infty_0}^{\infty_i} e^{\lambda(p)z}dp \\ h_{q^i} &= \operatorname*{res}_{\infty_i}\frac{e^{\lambda z}-1}{z}dp \\ h_{r^i} &= \oint_{b_i} e^{\lambda z}dp \\ h_{s^i} &= \frac{1}{2\pi i}\oint_{a_i} pe^{\lambda z}d\lambda. \end{aligned} \tag{6.88}$$

Here ${}_1F_1(a,b,z)$ is the Kummer confluent hypergeometric function (see [107]).

We leave the proof as an exercise for the reader (see [48]).

Remark 6.3. Integrals of the form (6.88) seem to be interesting functions on the moduli space of the form $M = M_{g;n_0,\dots,n_m}$. A simplest example of such an integral for a family of elliptic curves reads

$$\int_0^{\omega} e^{\lambda\wp(z)}dz \tag{6.89}$$

where $\wp(z)$ is the Weierstrass function with periods 2ω, $2\omega'$. For real negative λ a degeneration of the elliptic curve ($\omega \to \infty$) reduces (6.89) to the standard probability integral $\int_0^\infty e^{\lambda x^2}dx$. So the integral (6.89) is an analogue of the probability integral as a function on λ and on moduli of the elliptic curve. I recall that dependence on these parameters is specified by the equations (6.28).

Gradients of this functions on the Hurwitz space M have the form

$$\begin{aligned} \partial_{t^A}h_{t^{i;\alpha}} &= \operatorname*{res}_{\infty_i} k_i^{\alpha-n_i-1}\,{}_1F_1(1;\frac{\alpha}{n_i+1};z\lambda(p))\phi_{t^A} \\ \partial_{t^A}h_{p^i} &= \eta_{t^A p^i} - z\,\text{v.p.}\int_{\infty_0}^{\infty_i} e^{\lambda z}\phi_{t^A} \\ \partial_{t^A}h_{q^i} &= \operatorname*{res}_{\infty_i} e^{\lambda z}\phi_{t^A} \\ \partial_{t^A}h_{r^i} &= \eta_{t^A r^i} - z\oint_{b_i} e^{\lambda z}\phi_{t^A} \\ \partial_{t^A}h_{s^i} &= \frac{1}{2\pi i}\oint_{a_i} e^{\lambda z}\phi_{t^A}. \end{aligned} \tag{6.90}$$

The pairing (6.46) $<\phi_\alpha(z)\phi_\beta(w)>$ involved in the definition of the τ-function (6.48) coincides with (5.36).

Remark 6.4. For any Hamiltonian $H_{A,k}$ of the form (6.26), (6.88) one can construct a differential $\Omega_{A,k}$ on C or on a covering $\tilde{C}$ with singulariries only at the marked infinite points such that

$$\frac{\partial}{\partial u^i} h_{t^A,k} = \operatorname*{res}_{P_i} \frac{\Omega_{A,k} dp}{d\lambda}, \;\; i = 1, \ldots, n. \tag{6.91}$$

We give here, following [48], an explicit form of these differentials (for $m = 0$ also see [49]). All these will be normalized (i.e. with vanishing a-periods) differentials on C or on the universal covering of C with no other singularities or multivaluedness but those indicated in (6.92) - (6.97)

$$\Omega_{t^{i;\alpha},k} = -\frac{1}{n_i+1}\left[\left(\frac{\alpha}{n_i+1}\right)_{k+1}\right]^{-1} d\lambda^{\frac{\alpha}{n_i+1}+k} + \text{regular terms} \tag{6.92}$$

$$\Omega_{v^i,k} = -d\left(\frac{\lambda^{k+1}}{(k+1)!}\right) + \text{regular terms}, \;\; i = 1, \ldots, m \tag{6.93}$$

$$\Omega_{w^i,k} = \begin{cases} -\frac{1}{n_i+1} d\psi_k(\lambda) + \text{reg. terms} & \text{near } \infty_i \\ \frac{1}{n_0+1} d\psi_k(\lambda) + \text{reg. terms} & \text{near } \infty_0 \end{cases} \tag{6.94}$$

where

$$\psi_k(\lambda) := \frac{\lambda^k}{k!}\left[\log\lambda - \left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right)\right], \;\; k > 0 \tag{6.95}$$

$$\Omega_{r^i,k}(P + b_j) - \Omega_{r^i,k}(P) = -\delta_{ij}\frac{\lambda^k}{k!} d\lambda \tag{6.96}$$

$$\Omega_{s^i,k}(P + a_j) - \Omega_{s^i,k}(P) = \delta_{ij}\frac{\lambda^{k-1}}{(k-1)!} d\lambda. \tag{6.97}$$

Using these differentials the hierarchy (6.25) can be written in the Flaschka – Forest –McLaughlin form [63]

$$\partial_{T^{A,p}} dp = \partial_X \Omega_{A,p} \tag{6.98}$$

(derivatives of the differentials are to be calculated with λ =const.).

Integrating (6.98) along the Riemann surface C one obtains for the the Abelian integrals

$$q_{A,k} := \int \Omega_{A,k}$$

a similar representation

$$\partial_{T^{A,p}} p(\lambda) = \partial_X q_{A,p}(\lambda).$$

Rewriting this for the inverse function $\lambda = \lambda(p)$ we obtain the semiclassical Lax representation of the hierarchy (6.25) for the Hurwitz space (see details below in a more general setting)

$$\partial_{T^{A,p}} \lambda(p) = \{\lambda, \rho_{A,k}\}$$

where

$$\rho_{A,k} := q_{A,k}(\lambda(p)).$$

The matrix $< \phi_{A,p}\phi_{B,q} >$ determines a pairing of these differentials with values in functions on the moduli space

$$< \Omega_{A,p}\,\Omega_{B,q} >=< \phi_{A,p}\phi_{B,q} > (t) \tag{6.99}$$

This pairing coincides with the two-point correlators (6.55a). Particularly, the primary free energy F as a function on M can be written in the form

$$F = -\frac{1}{2} < pd\lambda\, pd\lambda > . \tag{6.100}$$

Note that the differential $pd\lambda$ can be written in the form

$$pd\lambda = \sum \frac{n_i+1}{n_i+2}\Omega_{\infty_i}^{(n_i+2)} + \sum t^A \phi_{t^A} \tag{6.101}$$

where $\Omega_{\infty_i}^{(n_i+2)}$ is the Abelian differential of the second kind with a pole at ∞_i of the form

$$\Omega_{\infty_i}^{(n_i+2)} = dk_i^{n_i+2} + \text{ regular terms} \quad \text{near } \infty_i. \tag{6.102}$$

For the pairing (6.99) one can obtain from [95] the following formula

$$< f_1 d\lambda\, f_2 d\lambda >= \frac{1}{2}\int\int_C (\bar\partial f_1 \partial f_2 + \partial f_1 \bar\partial f_2) \tag{6.103}$$

where the differentials ∂ and $\bar\partial$ along the Riemann surface should be understood in the distribution sense. The meromorphic differentials $f_1 dw$ and $f_2 dw$ on the covering $\tilde C$ should be considered as piecewise meromorphic differentials on C with jumps on some cuts.

Exercise 6.8. Prove that the three-point correlators $\partial_{T^{A,p}}\partial_{T^{B,q}}\partial_{T^{C,r}}\mathcal{F}$ as functions on $T^{\alpha,0} = t^\alpha$ with $T^{\alpha,p} = 0$ for $p > 0$ can be written in the form

$$< \phi_{A,p}\phi_{B,q}\phi_{C,r} >= \sum \operatorname*{res}_{d\lambda=0} \frac{\Omega_{A,p}\Omega_{B,q}\Omega_{C,r}}{d\lambda dp}. \tag{6.104}$$

Here A, B, C denote the labels of one of the flat coordinates (5.29), the numbers p, q, r take values $0, 1, 2, \ldots$. [Hint: use (6.55c) and (6.91).]

For the space of polynomials $M_{0;n}$ (the Frobenius manifold of the A_n-topological minimal model) the formula (6.104) was obtained in [97, 60].

The corresponding hierarchy (6.25) is obtained by averaging along invariant tori of a family of g-gap solutions of a KdV-type hierarchy related to a matrix operator L of the matrix order $m+1$ and of orders n_0, ..., n_m in $\partial/\partial x$. The example $m = 0$ (the averaged Gelfand – Dickey hierarchy) was considered in more details in [49].

Also for $g+m>0$ one needs to extend the KdV-type hierarchy to obtain (6.25) (see [49]). To explain the nature of such an extension let us consider the simplest example of $m=0$, $n_0=1$. The moduli space M consists of hyperelliptic curves of genus g with marked homology basis

$$y^2=\prod_{i=1}^{2g+1}(\lambda-u_i). \tag{6.105}$$

This parametrizes the family of g-gap solutions of the KdV. The L operator has the well-known form

$$L=-\partial_x^2+u. \tag{6.106}$$

In real smooth periodic case $u(x+T)=u(x)$ the quasimomentum $p(\lambda)$ is defined by the formula

$$\psi(x+T,\lambda)=e^{ip(\lambda)T}\psi(x,\lambda) \tag{6.107}$$

for a solution $\psi(x,\lambda)$ of the equation

$$L\psi=\lambda\psi \tag{6.108}$$

(the Bloch – Floquet eigenfunction). The differential dp can be extended onto the family of all (i.e. quasiperiodic complex meromorphic) g-gap operators (6.105) as a normalized Abelian differential of the second kind with a double pole at the infinity $\lambda=\infty$. (So the superpotential $\lambda=\lambda(p)$ has the sense of the Bloch dispersion law, i.e. the dependence of the energy λ on the quasimomentum p.) The Hamiltonians of the KdV hierarchy can be obtained as coefficients of expansion of dp near the infinity. To obtain a complete family of conservation laws of the averaged hierarchy (6.25) one needs to extend the family of the KdV integrals by adding nonlocal functionals of u of the form

$$\oint_{a_i}\lambda^k dp,\quad \oint_{b_i}\lambda^{k-1}dp,\ \ k=1,2,\dots. \tag{6.109}$$

We will obtain now the semiclassical Lax representation for the equations of the hierarchy (6.25) for *arbitrary* Frobenius manifold.

Let $h(t,z)$ be any solution of the equation

$$\partial_\alpha\partial_\beta h(t,z)=zc_{\alpha\beta}^\gamma(t)\partial_\gamma h(t,z) \tag{6.110}$$

normalized by the homogeneity condition

$$z\partial_z h=\mathcal{L}_E h. \tag{6.111}$$

By $p(t,\lambda)$ I will denote the corresponding flat coordinate of the pencil (H.1) given by the integral (H.11)

$$p(t,\lambda)=\oint z^{\frac{d-3}{2}}e^{-\lambda z}h(t,z)\,dz. \tag{6.112}$$

We introduce the functions

$$q_{\alpha,k}(t,\lambda) := \operatorname*{res}_{w=0} \oint \frac{z^{\frac{d-3}{2}} w^{-k-1}}{z+w} e^{-\lambda z} < \nabla h(t,z), \nabla h_\alpha(t,w) > \, dz. \tag{6.113}$$

Lemma 6.2. *The following identity holds*

$$\partial_{T^{\alpha,k}} p(t,\lambda) = \partial_X q_{\alpha,k}(t,\lambda). \tag{6.114}$$

Proof. Integrating the formula (6.34) with the weight $z^{\frac{d-3}{2}} e^{-\lambda z}$ we obtain (6.114). Lemma is proved.

Exercise 6.9. Let $p(\lambda)$, $q(\lambda)$ be two functions of λ depending also on parameters x and t in such a way that

$$\partial_t p(\lambda)_{\lambda=const} = \partial_x q(\lambda)_{\lambda=const}. \tag{6.115}$$

Let $\lambda = \lambda(p)$ be the function inverse to $p = p(\lambda)$ and

$$\rho(p) := q(\lambda(p)). \tag{6.116}$$

Prove that

$$\partial_t \lambda(p)_{p=const} = \{\lambda, \rho\} := \frac{\partial \lambda}{\partial x}\frac{\partial \rho}{\partial p} - \frac{\partial \rho}{\partial x}\frac{\partial \lambda}{\partial p}. \tag{6.117}$$

Theorem 6.5. *The hierarchy (6.25) admits the semiclassical Lax representation*

$$\partial_{T^{\alpha,k}} \lambda = \{\lambda, \rho_{\alpha,k}\} \tag{6.118}$$

where

$$\rho_{\alpha,k} := q_{\alpha,k}(t, \lambda(p,t)) \tag{6.119}$$

and the functions $\lambda = \lambda(p,t)$ is the inverse to (6.112).

Proof follows from (6.114) and (6.117).

In fact we obtain many semiclassical Lax representations of the hierarchy (6.25): one can take any solution of (6.114) and the corresponding flat coordinate of the intersection form and apply the above procedure. The example of A_n Frobenius manifold suggests that for $d < 1$ one should take in (6.118) the flat coordinate $p = p(\lambda, t)$ (I.10) inverse to the LG superpotential constructed in Appendix I for any massive Frobenius manifold with $d < 1$.

The next chapter in our story about Frobenius manifolds could be a quantization of the dispersionless Lax pairs (6.118). We are to substitute back $p \to d/dx$ and to obtain a hierarchy of the KdV type. We hope to address the problem of the quantization in subsequent publications.

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