REAL TWO-ZONE SOLUTIONS OF THE SINE-GORDON EQUATION

B. A. Dubrovin and S. M. Natanzon

UDC 517.9

Introduction

The goal of the present paper is to enumerate all smooth real solutions of the sine-Gordon equation $u_{xy} = \sin u$, expressed in terms of theta-functions of two variables. The appearance of this paper is due to S. P. Novikov, who turned the attention of the authors to the serious incompleteness of the investigations [1, 2] on the "finite-zone integration" of this nonlinear equation, important in geometric and physical applications (cf. [3]). (As Novikov informed the authors, results similar to those of [1] were also obtained by McKean in [14].) The main problem remained finding effective conditions for the reality of the "finite-zone" solutions constructed. This problem is completely solved in the present paper.*

The smooth real solutions constructed of the sine-Gordon equation are expressed in terms of theta-functions of real hyperelliptic curves (cf. below Secs. 1-3). Specially important is the case of curves of genus 2 (and the corresponding theta-functions of two variables), since here any theta-function (of general type) corresponds to some Riemann surface. Applying the method developed by one of the authors (cf. [4, 5]), one can eliminate Riemann surfaces from the calculations and express all the quantities appearing in the formulas for two-zone solutions only in terms of theta-functions. Thanks to this the formulas indicated acquire a specially simple and analytic effectiveness (cf. below Sec. 4).

1. Description of General Complex Algebro-Geometric ("Finite-Zone") Solutions of the Sine-Gordon Equation†

Let Γ be a hyperelliptic Riemann surface of genus g, the affine part of which is given in $\mathbf{C}^2$ by the equation

$$w^2 = P_{2g+1}(z) = \prod_{i=1}^{2g+1} (z - z_i); \tag{1.1}$$

the variables $z_1, \ldots, z_{2g+1}$ are pairwise distinct. The holomorphic differentials on this surface have the form

$$\eta_k = \frac{z^{k-1}\, dz}{\sqrt{P_{2g+1}(z)}}, \quad k = 1, \ldots, g. \tag{1.2}$$

We choose a basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ on Γ with matrix of intersection indices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. To it corresponds a normalized basis of holomorphic differentials $\omega_1, \ldots, \omega_g$, where

$$\omega_j = \sum_{k=1}^{g} c_{jk} \eta_k, \quad j = 1, \ldots, g; \tag{1.3}$$

the normalization condition has the form

$$\oint_{a_k} \omega_j = \delta_{jk}. \tag{1.4}$$

*The first discussion on this question with the participation of S. P. Novikov, Yu. Trubovits, and I. V. Cherednik arose at the Soviet–American Symposium on the Theory of Solitons (Kiev, 1979). Analysis of the results of [1,2] carried out later revealed the formulation of the problems solved in our paper.

†The goal of this section is to derive in the shortest way the formulas for (complex) solutions in terms of theta-functions. Hence we leave to the side a series of profound ideas (L–A-pairs, Baker–Akhiezer functions, etc., cf. [3, 5, 8]), constituting the basis of the method of algebro-geometric integration (or "finite-zone integration") of nonlinear equations.

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 16, No. 1, 27-43, January-March, 1982. Original article submitted June 10, 1981.

The matrix (c_{jk}) is calculated thus:

$$(c_{jk})=\left(\oint_{a_j}\eta_k\right)^{-1}. \tag{1.5}$$

We define the matrix of periods (B_{jk}) of the Riemann surface Γ:

$$B_{jk}=\oint_{b_k}\omega_j, \quad j, k=1, \ldots, g. \tag{1.6}$$

It is well known (cf. [9]) that (B_{jk}) is a symmetric matrix with positive-definite imaginary part, i.e., (B_{jk}) is a Riemann matrix. We define in the space $\mathbf{C}^g = \mathbf{R}^{2g}$ the integral lattice of periods, consisting of vectors of the form

$$M+BN, \quad M, N\in \mathbf{Z}^g. \tag{1.7}$$

The quotient of the space $\mathbf{C}^g$ by this lattice is a 2g-dimensional torus and is called the Jacobi manifold (or Jacobian) of the surface Γ; we denote it by $J(\Gamma)$.

With respect to the matrix of periods $B = (B_{jk})$ we construct the theta-function with characteristics $[\alpha] = [\alpha'; \alpha''] \in \mathbf{R}^{2g}$:

$$\theta[\alpha'; \alpha''](z)\equiv\theta[\alpha'; \alpha''](z\,|\,B) = \sum_{k\in\mathbf{Z}^g}\exp\{\pi i\langle B(k+\alpha'), k+\alpha'\rangle+2\pi i\langle k+\alpha', z+\alpha''\rangle\}. \tag{1.8}$$

Here $z = (z_1, \ldots, z_g) \in \mathbf{C}^g$; the summation is carried out over the integral lattice $\mathbf{Z}^g$; the angular brackets denote the Euclidean scalar product. Specially important is the function $\theta(z) = \theta[0; 0](z)$. Characteristics for which all coordinates of the vector $[\alpha]$ are equal to 0 or 1/2 are called semiperiods. The semiperiod $[\alpha'; \alpha'']$ is even if $4\langle \alpha', \alpha''\rangle \equiv 0 \pmod 2$, and odd otherwise. To even periods correspond even theta-functions, to odd, odd. Upon translation by vectors of the lattice of periods (1.7), the theta-functions transform according to the law

$$\theta[\alpha'; \alpha''](z+M+BN) = \exp\{-\pi i\langle BN, N\rangle-2\pi i\langle N, z\rangle+2\pi i(\langle\alpha', M\rangle-\langle\alpha'', N\rangle)\}\,\theta[\alpha'; \alpha''](z). \tag{1.9}$$

For the construction of solutions of the sine-Gordon equation we use the following identity (cf. [10, (39)]; this identity can also be derived in the realms of the theory of nonlinear equations, cf. [5]), valid for any Riemann surface Γ and any pair of its points P, Q:

$$\frac{\theta(z+\Delta)\,\theta(z-\Delta)}{\varepsilon^2(P, Q)\,\theta^2(z)}=\alpha+\sum_{i,j}U_iV_j\frac{\partial^2\ln\theta(z)}{\partial z_i\,\partial z_j}. \tag{1.10}$$

Here $z\in \mathbf{C}^g$ is an arbitrary vector,

$$\Delta=\left(\int_Q^P\omega_1, \ldots, \int_Q^P\omega_g\right), \tag{1.11}$$

the vectors $U = (U_1, \ldots, U_g)$, $V = (V_1, \ldots, V_g)$ have the form

$$U_i=\frac{\omega_i(P)}{dp}, \quad V_i=\frac{\omega_i(Q)}{dq}, \tag{1.12}$$

where p, q are local parameters in neighborhoods of the points P, Q, respectively;

$$\varepsilon^2(P, Q)=\frac{\theta^2[\nu](\Delta)}{\sum U_i\frac{\partial\theta[\nu](0)}{\partial z_i}\sum V_j\frac{\partial\theta[\nu](0)}{\partial z_j}}, \tag{1.13}$$

where $[\nu]$ is any odd nondegenerate (i.e., $\operatorname{grad}\theta[\nu](0) \neq 0$) semiperiod; the explicit form of the quantity $\alpha = \alpha(P, Q)$ is inessential for us. The identity (1.10) is a generalization of the "addition theorem" for Weierstrass σ-functions

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)}=\wp(v)-\wp(u) \tag{1.14}$$

(cf. [11]).

LEMMA 1. Let P and Q be branch points of the Riemann surface (1.1). Then the vector Δ of the form (1.11) is a semiperiod,

$$\Delta=\frac{1}{2}M+\frac{1}{2}BN, \quad M, N\in Z^g. \tag{1.15}$$

Proof. Let l be a path joining the points Q and P. We consider a second copy $\tilde{l}$ of this path, going from P to Q on another sheet of the Riemann surface. We have

$$\Delta_j=\int_l \omega_j=\int_{\tilde{l}} \omega_j, \quad j=1, \ldots, g. \tag{1.16}$$

But the integrals of $\omega_1, \ldots, \omega_g$ along the closed cycle $\gamma = l \cup \tilde{l}$ give some vector of the lattice of periods of the form M + BN. By virtue of (1.16) this proves the assertion of the lemma.

In what follows it will be assumed that the local parameters $p = c_1\sqrt{z - z_i}$, $q = c_2\sqrt{z - z_j}$ (c_1, c_2 are constants) in neighborhoods of these branch points $P = \{z = z_i\}$, $Q = \{z = z_j\}$ are compatible so that $\varepsilon^2(P, Q) = 1$.

THEOREM 1. The function

$$u(x, y)=\frac{1}{i}\left[\ln\frac{\theta(xU+yV+\Delta+\zeta)\,\theta(xU+yV+\zeta-\Delta)}{\theta^2(xU+yV+\zeta)}+\pi i\langle N, \Delta\rangle\right] \tag{1.17}$$

is, under the hypotheses listed above, a solution of the equation

$$u_{xy} = -4\varkappa \sin u, \tag{1.18}$$

where

$$\varkappa = \exp(-\pi i\langle N, \Delta\rangle). \tag{1.18'}$$

The proof consists of direct substitution, using (1.10) and the transformation law (1.9).

Definition. Solutions of the form (1.17) of Eq. (1.18), constructed from a hyperelliptic Riemann surface Γ of genus g and its pair of branch points P, Q, will be called g-zone solutions of the sine-Gordon equation.

2. Real Algebraic Curves of Genus 2 and Their Theta-Functions

Let Eq. (1.1) of the Riemann surface Γ have real coefficients. Then on Γ there acts the anti-involution (anti-holomorphic automorphism)

$$\tau(z, w) = (\bar{z}, \bar{w}), \quad \tau^2 = 1. \tag{2.1}$$

The topological properties of such "pairs," a Riemann surface (1.1) together with an anti-involution (2.1) given on it, are determined in the hyperelliptic case by the collection of real roots of the polynomial $P_{2g+1}(z)$ [cf. (1.1)]. First we analyze the example of a surface of genus 2 which is basic for us

$$w^2=P_5(z)=\prod_{i=1}^{5}(z-z_i). \tag{2.2}$$

Type I. All roots $z_1 < \ldots < z_5$ of the polynomial $P_5(z)$ are real. In this case the anti-involution τ has on the surface (2.2) three fixed ovals [three real components of the curve (2.2)]:

$$\begin{aligned} &A_1: \{z_1 \leqslant z \leqslant z_2,\ w = \pm\sqrt{P_5(z)}\},\\ &A_2: \{z_3 \leqslant z \leqslant z_4,\ w = \pm\sqrt{P_5(z)}\},\\ &A_3: \{z_5 \leqslant z \leqslant \infty,\ w = \pm\sqrt{P_5(z)}\}. \end{aligned} \tag{2.3}$$

We note that the union of the real ovals divides the Riemann surface Γ into two disjoint components. A basis of cycles on such a Riemann surface is represented in Fig. 1. The anti-involution acts on these cycles thus:

$$\tau a_1 = a_1, \quad \tau a_2 = a_2, \quad \tau b_1 = -b_1, \quad \tau b_2 = -b_2 \tag{2.4}$$

[equality in the homology group $H_1(\Gamma; \mathbf{Z})$]. The normalized holomorphic differentials

$$\omega_1=\frac{c_{11}+c_{12}z}{\sqrt{P_5(z)}}dz, \quad \omega_2=\frac{c_{21}+c_{22}z}{\sqrt{P_5(z)}}dz \tag{2.5}$$

on Γ have hence real coefficients. In other words, under the action of the anti-involution τ these differentials are transformed according to the law (here $\tau^*[f(z)dz] = f(\tau(z))d\tau(z)$)

$$\tau^*\omega_k = \overline{\omega}_k, \quad k = 1, 2. \tag{2.6}$$

The matrix of periods is purely imaginary:

$$\overline{B}_{kj} = \oint_{b_k} \overline{\omega}_j = \oint_{b_k} \tau^*\omega_j = \oint_{\tau b_k} \omega_j = -B_{kj}. \tag{2.7}$$

Consequently, the lattice of periods (1.7) in $\mathbf{C}^2 = \mathbf{R}^4$ is invariant with respect to complex conjugation

$$(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2). \tag{2.8}$$

Thus there is defined an anti-involution on the Jacobian $J(\Gamma)$. We shall find its real

$$\bar{z} \equiv z \tag{2.9}$$

and imaginary

$$\bar{z} \equiv -z \tag{2.10}$$

components on the Jacobian $J(\Gamma)$. (The sign $\equiv$ here and later will be used to denote the equality of vectors from $\mathbf{C}^g$ modulo the period lattice.) To find these components we expand any vector $z = (z_1, z_2)$ in terms of a basis for the period lattice:

$$z = (z_1, z_2) \equiv B\alpha + \beta, \tag{2.11}$$

where for the real vectors $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ all coordinates lie between zero and one. Then the reality condition has the form

$$B\alpha + \beta = -B\alpha + \beta + m + Bn, \tag{2.12}$$

where $m, n \in \mathbf{Z}^2$. We get four real components (of the real two-dimensional torus)

$$z \equiv \beta + (1/2)\, Bn, \tag{2.13}$$

where $\beta \in \mathbf{R}^2$, $n = (0, 0), (1, 0), (0, 1), (1, 1)$. Analogously the imaginary components have the form

$$z \equiv i\alpha + (1/2)\, n, \tag{2.14}$$

where again $\alpha \in \mathbf{R}^2$, the vector n assumes the same four values. This is again four two-dimensional real tori.

Type II. The roots $z_1 < z_2 < z_3$ are real, and $z_4 = \bar{z}_5$ are complex. The anti-involution τ has only two fixed ovals

$$A_1 : \{z_1 \leqslant z \leqslant z_2,\ w = \pm\sqrt{P_5(z)}\}, \quad A_2 : \{z_3 \leqslant z \leqslant \infty,\ w = \pm\sqrt{P_5(z)}\}, \tag{2.15}$$

while their union no longer divides the surface Γ into two components. A basis of cycles is represented in Fig. 2. The action of the anti-involution τ on these cycles is as follows:

$$\tau a_1 = a_1, \quad \tau a_2 = a_2, \quad \tau b_1 = a_1 + a_2 - b_1, \quad \tau b_2 = a_1 + a_2 - b_2. \tag{2.16}$$

The law of transformation of holomorphic differentials again has the form (2.6). But for the period matrix B one has the following relation, which follows from (2.6) and (2.16):

$$\overline{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - B. \tag{2.17}$$

As before the period lattice in $\mathbf{C}^2$ is invariant with respect to complex conjugation $(z_1, z_2) \to (\bar{z}_1, \bar{z}_2)$, which defines an anti-involution on the Jacobian $J(\Gamma)$. But this anti-involution has only two real and two imaginary components on $J(\Gamma)$, each of which is a two-dimensional torus. The real components have the form

$$z \equiv \beta + \frac{1}{2} Bn; \quad \beta \in \mathbf{R}^2, \quad n = (0, 0), (1, 1). \tag{2.18}$$

The imaginary components have the form

$$z \equiv i\alpha + \frac{1}{2} n, \quad \alpha \in \mathbf{R}^2, \quad n = (0, 0), (1, 0). \tag{2.19}$$

Type III. The root z_5 is real and the rest are complex: $z_1 = \bar{z}_2$, $z_3 = \bar{z}_4$. There is only one real oval:

$$A_1 : \{z_5 \leqslant z \leqslant \infty,\ w = \pm\sqrt{P_5(z)}\}; \tag{2.20}$$

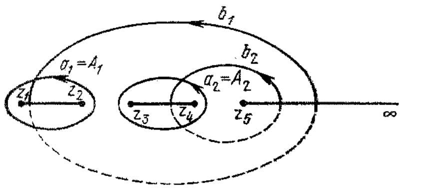

Fig. 1

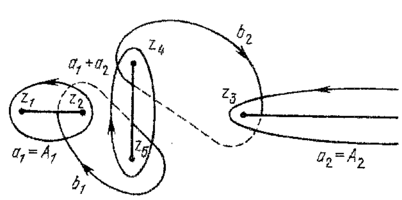

Fig. 2

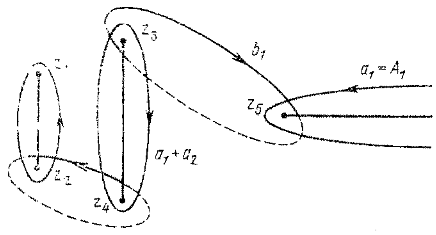

Fig. 3

Fig. 1. Basis of cycles on a Riemann surface of type I. Dashed lines represent parts of cycles lying on the "lower" sheet of the Riemann surface.

Fig. 2. Cycles on a Riemann surface of type II.

Fig. 3. Cycles on a Riemann surface of type III.

it does not separate the Riemann surface Γ. A basis of cycles is represented in Fig. 3. The real anti-involution on these cycles is as follows:

$$\begin{gathered}\tau a_1 = a_1, \quad \tau a_2 = a_2,\\ \tau b_1 = a_1 + a_2 - b_1,\\ \tau b_2 = a_1 + 2a_2 - b_2.\end{gathered} \tag{2.21}$$

The holomorphic differentials satisfy (2.6); for the period matrix B one has the relation

$$\overline{B} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} - B. \tag{2.22}$$

The anti-involution on the Jacobian J(Γ) has one real

$$z \equiv \beta, \qquad \beta \in \mathbf{R}^2, \tag{2.23}$$

and one imaginary

$$z \equiv i\alpha, \qquad \alpha \in \mathbf{R}^2, \tag{2.24}$$

component (each of them is a two-dimensional torus).

Real algebraic curves of genus 2, which are not of types I-III, have the form $w^2 = P_6(z)$, where $P_6(z)$ is a polynomial with nonreal zeros. Such curves either have no real points (type IV), or have exactly one separating oval (type V). These curves are not suitable for integration of the sine-Gordon equation, and we shall not consider them. We note only that a suitable basis of cycles a_1, a_2, b_1, b_2 on such Riemann surfaces transforms under the action of the anti-involution according to the law

$$\tau a_1 = a_2, \quad \tau a_2 = a_1, \quad \tau b_1 = -b_2, \quad \tau b_2 = -b_1. \tag{2.25}$$

The Jacobian J(Γ) with the anti-involution $(z_1, z_2) \to (\bar{z}_2, \bar{z}_1)$ has one real and one imaginary component.

Now we clarify the question about the symmetry of the theta-functions of Riemann surfaces of types I-III.

LEMMA 2. Theta-functions of Riemann surfaces of types I-III with the basis of cycles indicated above have the following symmetries:

$$\overline{\theta(z)} = \theta(\bar{z} + \lambda), \tag{2.26}$$

where λ is a real semiperiod of the form $\lambda = 0$ for type I, $\lambda = (1/2, 1/2)$ for type II, $\lambda = (1/2, 0)$ for type III.

The proof follows quickly from the definition of the function $\theta(z)$ and the symmetries (2.7), (2.17), and (2.22) for the period matrices of the enumerated Riemann surfaces.

To conclude this section we give some general information about Riemann surfaces of genus $g \geq 2$ with an anti-involution τ. Let the anti-involution τ on Γ have n fixed ovals $(0 \leq n \leq g + 1)$. Two cases are possible: a) the union of the real ovals separates Γ into two components; b) the union of ovals does not separate Γ. The properties of "separating" Riemann surfaces [case a)] and their theta-functions are well studied; see, e.g., [10, Chap. 6]. Theta-functions of nonseparating surfaces did not arise in applications (as far as is known to the authors). Hence we give here information about theta-functions of nonseparating surfaces. On such a surface one can always choose a basis of cycles $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ with intersection matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, transforming under the action of the anti-involution according to the law

$$\tau a_i = a_i, \quad i = 1, \ldots, g, \quad \tau b_i = \begin{cases} a - b_i, & 1 \leqslant i \leqslant n, \\ a + a_i - b_i, & n+1 \leqslant i \leqslant g, \end{cases} \tag{2.27}$$

where $a = \sum_{i=1}^{g} a_i$ (cf. [6]). For g = 2 such a basis was produced for types II (n = 2) and III (n = 1). The period matrix of holomorphic differentials, calculated in this basis, has the following symmetry:

$$\bar{B} = \left(\begin{array}{cccc|cccc} 1 & 1 & \ldots & 1 & 1 & & \ldots & 1 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 1 & & \ldots & 1 & 1 & & \ldots & 1 \\ \hline 1 & & \ldots & 1 & 2 & 1 & \ldots & 1 \\ \vdots & & & \vdots & 1 & 2 & \ldots & 1 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 1 & & \ldots & 1 & 1 & & \ldots & 2 \end{array}\right) - B, \tag{2.28}$$

where on the right side the square blocks have sizes n × n and (g − n) × (g − n). The theta-function $\theta(z) = \theta(z \mid B)$ has the symmetry

$$\overline{\theta(z)} = \theta(\bar{z} + \lambda), \tag{2.29}$$

where the semiperiod λ has the form

$$\lambda = (1/2)(1, \ldots, 1, 0, \ldots, 0) \tag{2.30}$$

(ones in the first n places). It is easy to verify that the anti-involution $z \to \bar{z}$ on the Jacobian $J(\Gamma)$ has 2^{n-1} pairwise disjoint real components and 2^{n-1} imaginary components for n > 0, each of which is a real g-dimensional torus. For n = 0 there is one component if g is even and two if g is odd. The function $\theta(z)$ is real on the real tori of the form

$$\begin{aligned} & z \equiv i\alpha + \tfrac{1}{2}(\varepsilon_1, \ldots, \varepsilon_{n-1}, 0, \ldots, 0) + \tfrac{1}{2}\lambda, \quad \alpha \in \mathbf{R}^g, \quad n > 0; \\ & z \equiv i\alpha \quad \text{for} \quad n = 0, \quad g = 2p+1; \\ & z \equiv i\alpha + \tfrac{1}{2}(\varepsilon_1, 0, \ldots, 0) \quad \text{for} \quad n = 0, \quad g = 2p, \end{aligned} \tag{2.31}$$

where $\varepsilon_1, \ldots, \varepsilon_{n-1}$ assume the values 0, 1. (It is easy to see that the condition of reality of the function $\theta(z)$ depends only on the class of the vector z modulo the period lattice.) We note that on all these real tori the function $\theta(z)$ has zeros (see the Supplement below), so nonseparating Riemann surfaces, as a rule, cannot be used for the construction of smooth "finite-zone" solutions of nonlinear equations, integrable by the method of the inverse problem.* However the sine-Gordon equation is an exception to this rule: in the following section we shall show that any real hyperelliptic curve with at least one real branch point gives a smooth real solution of the sine-Gordin equation. With the exception of the simplest case, where all branch points are real, such Riemann surfaces are always nonseparating.

3. Selection of Real Solutions of the Sine-Gordon Equation

In this section we shall show that for the reality and smoothness of the finite-zone solutions constructed in Sec. 1 of the sine-Gordon equation it is necessary and sufficient that the following conditions on the Riemann surface Γ of the form (1.1) and its branch points P, Q hold:

1) the hyperelliptic curve (1.1) must be real (i.e., a Riemann surface with anti-involution);

2) the branch points P, Q must lie on one real oval of the curve (1.1).

For the case of genus 2 the surface Γ must be of one of the types I–III, listed in the previous section. Moreover, it is also necessary to impose a restriction on the parameter ζ, determining the solution [cf. (1.17)]; we shall do this a little later. The condition on the branch points P, Q means that Δ is a real vector in $\mathbf{C}^g$,

$$\Delta = \tfrac{1}{2} M, \quad M \in \mathbf{Z}^g. \tag{3.1}$$

*Using the technique developed here, one of the authors (Dubrovin) proved that the nonlinear Schrödinger equation $i\psi_t = \psi_{xx} - |\psi|^2\psi$ (case of repulsion) has no smooth finite-zone solutions.

For definiteness one can assume that the points P, Q lie on the cycle a_1. Then $M = (1, 0, \ldots, 0)$. Hence (1.17) and (1.18) can be simplified and we get: the function

$$u(x, y) = \frac{1}{i} \ln \left[\frac{\theta\left(xU + yV - \frac{1}{2}M + \zeta\right)}{\theta(xU + y \quad + \zeta)} \right]^2 \tag{3.2}$$

is a solution of the equation

$$u_{xy} = -4 \sin u. \tag{3.3}$$

Making the substitution $x, y \to ix/2, iy/2$, one can reduce this equation to the standard form $u_{xy} = \sin u$. We consider the function

$$\varphi(z) = \frac{\theta^2\left(z - \frac{1}{2}M\right)}{\theta^2(z)}. \tag{3.4}$$

By virtue of the transformation law (1.9), this is a single-valued meromorphic function on the torus $J(\Gamma)$. Now we impose the restriction on the vector ξ:

$$\bar{\zeta} \equiv -\frac{1}{2}M + \lambda - \zeta, \tag{3.5}$$

where $\lambda = 0$ for type I, $\lambda = (1/2, 1/2)$ for type II, and $\lambda = (1/2, 0)$ for type III. It is clear that such vectors ξ are in one-to-one correspondence with imaginary points of the Jacobian:

$$\xi = -\frac{1}{4}M + \frac{1}{2}\lambda + z, \text{ where } \bar{z} \equiv -z. \tag{3.6}$$

LEMMA 3. The equation $|\varphi(\xi)|^2 = 1$ is valid if and only if (3.5) holds.

Proof. Suppose for the vector ξ, (3.5) holds. Then

$$\overline{\varphi(\zeta)} = \overline{\left[\frac{\theta^2\left(\zeta - \frac{1}{2}M\right)}{\theta^2(\zeta)}\right]} = \frac{\theta^2\left(\bar{\zeta} - \frac{1}{2}M + \lambda\right)}{\theta^2(\bar{\zeta} + \lambda)} = \frac{1}{\varphi(\zeta)}. \tag{3.7}$$

Conversely, if $\varphi(\xi)\overline{\varphi(\xi)} = 1$, then by (3.7),

$$\bar{\zeta} + \lambda = \pm\left(\zeta - \frac{1}{2}M\right). \tag{3.8}$$

The plus sign in this equation is impossible, since the vector $\bar{\xi} - \xi$ is purely imaginary, and the vector $\lambda + (1/2)M$ is real. The lemma is proved.

We choose local parameters p, q in neighborhoods of the branch points P, Q so that $\tau(p) = \bar{p}$, $\tau(q) = \bar{q}$. Then the vectors U, V are real: $\bar{U} = U$, $\bar{V} = V$. This is obvious from their definition (1.12) and the form of the holomorphic differentials (2.6). Hence if (3.5) holds for the vector ξ, then the relation

$$\bar{\xi} = -\frac{1}{2}M + \lambda - \xi$$

holds for the vector

$$\xi = \frac{1}{2}ixU + \frac{1}{2}iyV - \frac{1}{2}M + \zeta \tag{3.9}$$

for real x, y.

THEOREM 2. The function

$$u(x, y) = \frac{1}{i} \ln \left[\frac{\theta\left(\frac{ix}{2}U + \frac{iy}{2}V - \frac{1}{4}M + \frac{1}{2}\lambda + z\right)}{\theta\left(\frac{ix}{2}U + \frac{iy}{2}V + \frac{1}{4}M + \frac{1}{2}\lambda + z\right)} \right]^2 \tag{3.10}$$

is a smooth real solution of the sine-Gordon equation $u_{xy} = \sin u$ under the conditions listed above on the Riemann surface Γ, its branch points P, Q, and the local parameters p, q, while the vector z must lie on some imaginary component of the Jacobian $J(\Gamma)$. If the Riemann surface Γ has only one pair of real branch points, then there is another component of smooth real solutions of the equation $u_{xy} = \sin u$

$$u(x, y) = \frac{1}{i}\ln\left[\frac{\theta\left(\frac{x}{2}U - \frac{y}{2}V + \left(\frac{1}{2}, 0\right) + z\right)}{\theta\left(\frac{x}{2}U - \frac{y}{2}V + z\right)}\right]^2, \tag{3.11}$$

where the vector z is purely real. The formulas listed exhaust all smooth real finite-zone solutions of the equation $u_{xy} = \sin u$.

Proof for Genus g = 2. The smoothness and reality of the solutions (3.10) follows directly from Lemma 3. The smoothness and reality of solutions of the form (3.11) is verified just as in Lemma 3. We give the proof of necessity of the reality conditions listed in the theorem. The necessity of the reality of the curve Γ and its branch points P, Q follows from the theory of L–A-pairs and is actually proved in [1]. The necessity of the condition $\zeta + \bar{\zeta} \equiv \Delta + \lambda$ follows from Lemma 3. This equation has on J(Γ) nontrivial solutions if and only if the vector Δ is real. This means that the points P and Q lie on one oval of the anti-involution τ. This uniquely fixes the anti-involution on the Riemann surface for types I and II; for type III another involution τ' also serves, where $\tau'(z, w) = (\bar{z}, -\bar{w})$. This leads precisely to the formulas listed above and only to them. The theorem is proved.

The proof of this theorem for the case of higher genus is carried out practically unchanged.

This theorem allows one easily to calculate the collection of components of smooth real solutions of the sine-Gordon equation constructed from a real hyperelliptic surface Γ of genus g with a fixed pair of real branch points, lying on one oval, where Γ has n real ovals ($1 \le n \le g + 1$). It is only necessary to exclude "trivial" components, for which the solutions differ only in sign. We get 2^{n-2} components of real solutions for $n \ge 2$ and two components for n = 1. A complete list of smooth real solutions for genus g = 2 is given in the following section. The reason for the appearance of two components for curves with one oval is clear from the proof of Theorem 2: actually these two components correspond to two different real curves $w^2 = P_5(z)$ and $w^2 = -P_5(z)$, isomorphic as complex Riemann surfaces.

Before moving to the effectivization of the formulas in genus 2, we mention the problem of the density of the finite-zone solutions of the sine-Gordon equation constructed in the space of all periodic (modulo 2π) solutions of this equation. As Novikov indicated to the authors, up to now there are no approaches to this problem. This is connected, in particular, with the non-self-adjointness of the corresponding L–A-pair (cf. [3]).

4. Effectivization of the Formulas Obtained for Two-Zone Solutions of the Sine-Gordon Equation

In this section we shall show that from the construction of the two-zone solutions one can completely discard the Riemann surface, and we get formulas in closed form for real smooth solutions in terms of theta-functions of two variables. The basic possibility of realizing such a program is clear in advance, since any Riemann 2 × 2-matrix $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$ (i.e., symmetric matrix with positive-definite imaginary part) in general position is the period matrix of some Riemann surface (see, e.g., [5, Chap. 4]).

We introduce the notation needed. We consider the four linearly independent functions

$$\hat{\theta}[n](z) = \theta\left[\frac{n}{2};\, 0\right](z \mid 2B), \tag{4.1}$$

where $n \in (Z_2)^2$, i.e., the coordinates of the vector $n = (n_1, n_2)$ are equal to 0 or 1. One has the following identity ("addition theorem" for theta-functions, cf. [5]):

$$\theta(z + u)\theta(z - u) = \sum_{n \in (Z_2)^2} \hat{\theta}[n](2z)\hat{\theta}[n](2u). \tag{4.2}$$

The values of the functions $\hat{\theta}[n](z)$ and their derivatives at zero are called the theta-constants. The theta-constants are functions of the Riemann matrix B. Let us agree to omit the argument zero from the theta-constants: $\hat{\theta}[n] \equiv \hat{\theta}[n](0)$, $\hat{\theta}_{ij}[n] \equiv \hat{\theta}_{ij}[n](0)$. We impose on the Riemann 2 × 2-matrix B the following nondegeneracy condition:

$$D = \det\begin{vmatrix} \hat{\theta}_{11}[0, 0] & \hat{\theta}_{12}[0, 0] & \hat{\theta}_{22}[0, 0] & \hat{\theta}[0, 0] \\ \hat{\theta}_{11}[1, 0] & \hat{\theta}_{12}[1, 0] & \hat{\theta}_{22}[1, 0] & \hat{\theta}[1, 0] \\ \hat{\theta}_{11}[0, 1] & \hat{\theta}_{12}[0, 1] & \hat{\theta}_{22}[0, 1] & \hat{\theta}[0, 1] \\ \hat{\theta}_{11}[1, 1] & \hat{\theta}_{12}[1, 1] & \hat{\theta}_{22}[1, 1] & \hat{\theta}[1, 1] \end{vmatrix} \neq 0 \tag{4.3}$$

(this condition eliminates Riemann matrices, reducing an integral-valued symplectic transformation to diagonal form).

THEOREM 3. Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$ be an arbitrary Riemann matrix with the nondegeneracy condition (4.3), satisfying one of the three reality conditions:

$$\text{Type I:} \quad \bar{B} = -B, \tag{4.4}$$

$$\text{Type II:} \quad \bar{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - B, \tag{4.5}$$

$$\text{Type III:} \quad \bar{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - B. \tag{4.6}$$

Then the function

$$u(x, y) = \frac{1}{i} \ln \left[\frac{\theta\,[0, 0; p_1, p_2] \left(\frac{i(x - x_0)}{2} U + \frac{i(y - y_0)}{2} V \right)}{\theta\,[0, 0; q_1, q_2] \left(\frac{i(x - x_0)}{2} U + \frac{i(y - y_0)}{2} V \right)} \right]^2 \tag{4.7}$$

is a smooth real solution of the equation $u_{xy} = \sin u$, where the characteristics p_1, p_2, q_1, q_2 for the types I-III have the form

	Type I		Type II	Type III
p_1	$^3/_4$	$^3/_4$	0	0
p_2	0	$^1/_2$	$^1/_4$	0
q_1	$^1/_4$	$^1/_4$	$^1/_2$	$^1/_2$
q_2	0	$^1/_2$	$^1/_4$	0

, (4.8)

the vectors $U = (U_1, U_2)$, $V = (V_1, V_2)$ have the form

$$U_1 = 1, \quad U_2 = \frac{q_{12} + \sqrt{q_{12}^2 - 4q_{11}q_{22}}}{2q_{11}}, \quad V_1 = q_{11}, \quad V_2 = \frac{q_{12} - \sqrt{q_{12}^2 - 4q_{11}q_{22}}}{2}, \tag{4.9}$$

and the quantities $q_{ij} = q_{ij}(B)$ are calculated according to the following formulas [the determinant D is defined by (4.3)]:

$$q_{11} = D^{-1} \det \begin{vmatrix} \hat\theta[0,0] & \hat\theta_{12}[0,0] & \hat\theta_{22}[0,0] & 0 \\ 0 & \hat\theta_{12}[1,0] & \hat\theta_{22}[1,0] & \hat\theta[1,0] \\ \hat\theta[0,1] & \hat\theta_{12}[0,1] & \hat\theta_{22}[0,1] & 0 \\ 0 & \hat\theta_{12}[1,1] & \hat\theta_{22}[1,1] & \hat\theta[1,1] \end{vmatrix}, \tag{4.10}$$

$$q_{12} = D^{-1} \det \begin{vmatrix} \hat\theta_{11}[0,0] & \hat\theta[0,0] & \hat\theta_{22}[0,0] & 0 \\ \hat\theta_{11}[1,0] & 0 & \hat\theta_{22}[1,0] & \hat\theta[1,0] \\ \hat\theta_{11}[0,1] & \hat\theta[0,1] & \hat\theta_{22}[0,1] & 0 \\ \hat\theta_{11}[1,1] & 0 & \hat\theta_{22}[1,1] & \hat\theta[1,1] \end{vmatrix}, \tag{4.11}$$

$$q_{22} = D^{-1} \det \begin{vmatrix} \hat\theta_{11}[0,0] & \hat\theta_{12}[0,0] & \hat\theta[0,0] & 0 \\ \hat\theta_{11}[1,0] & \hat\theta_{12}[1,0] & 0 & \hat\theta[1,0] \\ \hat\theta_{11}[0,1] & \hat\theta_{12}[0,1] & \hat\theta[0,1] & 0 \\ \hat\theta_{11}[1,1] & \hat\theta_{12}[1,1] & 0 & \hat\theta[1,1] \end{vmatrix}. \tag{4.12}$$

For Riemann matrices of type III there is another family of real solutions of the form

$$u(x, y) = \frac{1}{i} \ln \left[\frac{\theta \left[0, 0; \frac{1}{2}, 0 \right] \left(\frac{x - x_0}{2} U - \frac{y - y_0}{2} V \right)}{\theta \left(\frac{x - x_0}{2} U - \frac{y - y_0}{2} V \right)} \right]^2. \tag{4.13}$$

Proof. First we choose an arbitrary Riemann matrix (without reality conditions), we construct from it the corresponding function $\theta(z) = \theta(z \mid B)$ and we shall seek a solution of (1.18) in the form (1.17), where ξ is an arbitrary vector, Δ has the form

$$\Delta = \frac{1}{2} M + \frac{1}{2} BN, \quad M, N \in Z^g, \tag{4.14}$$

and the vectors U, V are as yet unknown. To find these vectors, we use (1.10). Under the condition $\varepsilon^2 = 1$ we have

$$\theta(z+\Delta)\theta(z-\Delta) = \alpha\theta^2(z) + \partial_U\partial_V\theta(z)\cdot\theta(z) - \partial_U\theta(z)\partial_V\theta(z). \tag{4.15}$$

We transform this equation, using the addition theorem (4.2) (cf. [5]). We get a system of 2^g relations

$$2\sum_{i,j} U_i V_j \hat{\theta}_{ij}[n] + \alpha\hat{\theta}[n] = \hat{\theta}[n](2\Delta), \qquad n \in (\mathbf{Z}_2)^g. \tag{4.16}$$

For $g > 2$ the compatibility conditions for this system give nontrivial relations on the Riemann matrix B, necessary (and apparently sufficient; cf. [5]) for the matrix B to be the period matrix of a hyperelliptic Riemann surface. But for $g = 2$ these equations are easy to solve. We choose as Δ the vector $\Delta = (1/2, 0)$. Then the system (4.16) can be rewritten in the form

$$2U_1V_1\hat{\theta}_{11}[n] + 2(U_1V_2 + U_2V_1)\hat{\theta}_{12}[n] + 2U_2V_2\hat{\theta}_{22}[n] + \alpha\hat{\theta}[n] = (-1)^{n_1}\hat{\theta}[n], \tag{4.17}$$

where $n = (n_1, n_2)$ assumes the values (0, 0), (1, 0), (0, 1), (1, 1). Solving it by Cramer's rule, we get

$$U_1V_1 = q_{11}, \quad U_1V_2 + U_2V_1 = q_{12}, \quad U_2V_2 = q_{22}, \tag{4.18}$$

where the quantities q_{ij} have the form (4.10)-(4.12). Whence, obviously (4.9) follows. Thus, we have verified that the formulas listed in the theorem give solutions of the sine-Gordon equation. We verify that these solutions are smooth and real under the conditions (4.4)-(4.6) on the matrix B. For this it suffices to prove the reality of the vectors U and V, defined by (4.9). We divide the proof of reality into a series of lemmas.

LEMMA 4. For Riemann matrices B, satisfying one of the reality conditions (4.4)-(4.6), the quantities $q_{ij} = q_{ij}(B)$, defined by (4.10)-(4.12), are real.

Proof. For the theta-constants from (4.4)-(4.6) we get these relations:

$$\overline{\hat{\theta}[n]} = \exp(-2\pi i\langle n, \lambda\rangle^2)\hat{\theta}[n], \quad \overline{\hat{\theta}_{kj}[n]} = \exp(-2\pi i\langle n, \lambda\rangle^2)\hat{\theta}_{kj}[n], \tag{4.19}$$

where $\lambda = 0$, $(1/2, 1/2)$ or $(1/2, 0)$ for types I, II, III, respectively. For type I we get thus that all theta-constants are real. For types II, III the theta-constants $\hat{\theta}[0, 0]$, $\hat{\theta}[1, 0]$, $\hat{\theta}[0, 1]$, $\hat{\theta}[1, 1]$ under complex conjugation are multiplied, respectively, by $1, -i, -i, 1$, or by $1, -i, 1, -i$ (the same thing holds for the constants $\hat{\theta}_{jk}[n]$). In all three cases the reality of the quantities q_{ij} follows from this. The lemma is proved.

By virtue of the lemma proved it suffices to verify the positivity of the discriminant

$$\delta = q_{12}^2 - 4q_{11}q_{22}. \tag{4.20}$$

LEMMA 5. The discriminant $\delta = \delta(B)$ does not vanish for any Riemann matrix B, satisfying the nondegeneracy condition (4.3).

Proof. We note first of all that the vectors U, V are determined from the system (4.17) uniquely up to transposition and to transformations $U \to kU$, $V \to k^{-1}V$. Further, any Riemann 2×2-matrix with condition (4.3) determines a Riemann surface Γ of genus 2. For this Riemann surface Γ and a suitable pair of its branch points, the vectors U and V hence necessarily have the form (1.12). It is well known that these vectors are different (cf., e.g., [5]). The lemma is proved.

Now we consider the determinants $\tilde{q}_{11}$, $\tilde{q}_{12}$, $\tilde{q}_{22}$ standing in the numerators of (4.10)-(4.12) (i.e., $q_{ij} = D^{-1}\tilde{q}_{ij}$). They are now defined for all Riemann matrices B [without the restriction (4.3)]. We shall show that even if the determinant D vanishes, the quantity $\tilde{\delta} = \tilde{q}_{12}^2 - 4\tilde{q}_{11}\tilde{q}_{22}$ can become zero only on certain curves in the three-dimensional space of Riemann matrices. In fact, if the determinant D is zero, this means that the Riemann matrix B has, in some basis of the lattice, diagonal form $B = \begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$. The corresponding theta-functions of two variables then split into the product of two one-dimensional theta-functions. Hence $\tilde{q}_{11} = \tilde{q}_{22} \equiv 0$, $\tilde{q}_{12} \neq 0$ for almost all τ_1, τ_2. It is easy to verify that under a change of basis of the period lattice given by an integral symplectic transformation, the quantity $\tilde{\delta}$ can only be multiplied by a nonzero factor. This means that for almost all Riemann matrices lying on the level $D = 0$, the quantity $\tilde{\delta} \neq 0$.

Further, if B is the period matrix of a real Riemann surface of types I-III, then B satisfies the reality conditions (4.4)-(4.6), respectively. For such a Riemann matrix the discriminant δ is positive. From the preceding arguments it follows that δ cannot change sign. This completes the proof of the theorem.

5. Concluding Remarks

1. The two-zone solutions of the sine-Gordon equation constructed above from the Riemann matrix B are such that the function $\exp iu(x, y)$ is almost periodic in x and y with the pair of periods T_1^x, T_2^x and T_1^y, T_2^y, respectively. These periods are determined up to a Lorentz transformation

$$T_i^x \leftrightarrow T_i^y, \quad T_i^x \mapsto kT_i^x, \quad T_i^y \mapsto k^{-1}T_i^y \tag{5.1}$$

and up to transformations of the form

$$(T_i^{x,y})^{-1} \mapsto \sum_{j=1}^{2} m_{ij}(T_j^{x,y})^{-1}, \tag{5.2}$$

where (m_{ij}) is an integral unimodular matrix, generated by a change of basis of the period lattice. The explicit form of these periods is

$$\begin{pmatrix} T_1^{x^{-1}} \\ T_2^{x^{-1}} \end{pmatrix} = i\tilde{B}^{-1}U, \quad \begin{pmatrix} T_1^{y^{-1}} \\ T_2^{y^{-1}} \end{pmatrix} = i\tilde{B}^{-1}V, \tag{5.3}$$

where the matrix $\tilde{B}$ has for types I-III the following form:

$$\text{Type I:} \quad \tilde{B} = B; \tag{5.4}$$

$$\text{Type II:} \quad \tilde{B} = \begin{pmatrix} B_{11} - B_{12} & -1 + B_{11} + B_{12} \\ B_{12} - B_{22} & -1 + B_{12} + B_{22} \end{pmatrix}; \tag{5.5}$$

$$\text{Type III:} \quad \tilde{B} = \begin{pmatrix} -1 + 2B_{11} & -1 + 2B_{12} \\ -1 + 2B_{12} & -2 + 2B_{22} \end{pmatrix}. \tag{5.6}$$

The solutions periodic in x (or in y) are singled out by the condition of commensurability

$$n_1T_1^x + n_2T_2^x = 0 \quad \text{or} \quad n_1T_1^y + n_2T_2^y = 0, \tag{5.7}$$

where n_1, n_2 are integers. If all periods in x and in t are equal to infinity, then the two-zone solutions constructed become two-soliton solutions of the sine-Gordon equation (cf. [3]).

2. In conclusion we give a family of two-zone solutions, expressed in terms of elliptic functions. We take the Riemann matrix:

$$B = \frac{1}{4}\begin{pmatrix} \tau + \tilde{\tau} & \tau - \tilde{\tau} \\ \tau - \tilde{\tau} & \tau + \tilde{\tau} \end{pmatrix}, \quad \operatorname{Im}\tau > 0, \quad \operatorname{Im}\tilde{\tau} > 0, \tag{5.8}$$

where type I: $\operatorname{Re}\tau = \operatorname{Re}\tilde{\tau} = 0$, type II: $\operatorname{Re}\tau = 2$, $\operatorname{Re}\tilde{\tau} = 0$, type III: $\operatorname{Re}\tau = \operatorname{Re}\tilde{\tau} = 1$ [the period matrices of Riemann surfaces of the form $w^2 = P_3(z^2)$*]. Making the substitution $t = x + y$, $\xi = x - y$, we rewrite the sine-Gordon equation in the form

$$u_{tt} - u_{\xi\xi} = \sin u. \tag{5.9}$$

We set

$$\theta_2(z) = \theta\left[\frac{1}{2};\, 0\right](z \mid \tau), \quad \theta_3(z) = \theta(z \mid \tau) \tag{5.10}$$

(for the standard notation for theta-functions of Jacobi, cf. [11]); the notation $\tilde{\theta}_2(z)$ and $\tilde{\theta}_3(z)$ (where $\tau \to \tilde{\tau}$) has analogous meaning. Then solutions of (5.9), elliptic in t and in ξ (it is clear that $\exp iu$ will be an elliptic function), have the form

$$u(t, \xi) = \frac{1}{i}\ln\left[\frac{\theta_3(z)\tilde{\theta}_3(w) - \theta_2(z)\tilde{\theta}_2(w)}{\theta_3(z)\tilde{\theta}_3(w) + \theta_2(z)\tilde{\theta}_2(w)}\right]^2, \tag{5.11}$$

where

$$z = i\omega(t - t_0) + \frac{n_1 + n_2}{2}, \quad w = i\varkappa(\xi - \xi_0) + \frac{n_1 - n_2}{2}, \tag{5.12}$$

*It would be interesting to study properties of solutions of other nonlinear equations, integrable by the method of the inverse scattering problem, corresponding to Riemann surfaces with rich symmetry. In connection with the classification of such surfaces, cf. [7].

the vector (n_1, n_2) assumes the values (0, 0), (1, 0) for type I and (1, 1) for types II and III. The "dispersion relations" for the wave vector $(\omega, \varkappa)$ have the form

$$\omega^2 [f_0''\tilde{f}_0 + f_2''\tilde{f}_2] - \varkappa^2 [f_0\tilde{f}_0'' + f_2\tilde{f}_2''] = \frac{1-\alpha}{8} (f_0\tilde{f}_0 + f_2\tilde{f}_2),$$

$$\omega^2 f_1''\tilde{f}_1 - \varkappa^2 f_1\tilde{f}_1'' = -\frac{1+\alpha}{8} f_1\tilde{f}_1,$$

$$\omega^2 [f_1''\tilde{f}_0 + f_0''\tilde{f}_1] - \varkappa^2 [f_1\tilde{f}_0'' + f_0\tilde{f}_1''] = \frac{1-\alpha}{8} (f_1\tilde{f}_0 + f_0\tilde{f}_1)$$

(the variable α can be eliminated from the system). Here we have introduced the functions $f_k(z) = \theta[k/4; 0] \times (z \mid 2\tau)$, k = 0, 1, 2; the functions $\tilde{f}_k(z)$ (where $\tau \to \tilde{\tau}$) have analogous meaning. As usual, the absence of an argument from these functions (or from their derivatives) means that this argument is equal to zero. One should note that in the special case where the periods in ξ are infinite, solutions of type (5.11) were found Gribkov [13] with the help of the Bäcklund transformation.

Supplement. On the Zeros of Theta-Functions of Real Nonseparating Curves

Let Γ be a Riemann surface of genus g with anti-involution τ, having n fixed ovals $(0 \leq n \leq g)$, which altogether do not separate Γ. Let, further, $\theta(z)$ be the theta-function of this Riemann surface, constructed with respect to a basis of cycles of the form (2.27). It assumes real values on g-dimensional tori of the form (2.31).

THEOREM. On all tori of the form (2.31) the function $\theta(z)$ has zeros.

Proof. We parametrize the points of the Jacobian $J(\Gamma)$ by divisors D of degree g:

$$z = \int_{gQ}^{D} \omega + K^Q. \tag{S.1}$$

Here $\omega = (\omega_1, \ldots, \omega_g)$ are normalized holomorphic differentials, Q is a fixed point of Γ, K^Q is the corresponding vector of Riemannian constants (cf. [10]).

LEMMA. A vector z of the form (S.1) satisfies (2.31) if and only if one can find a meromorphic differential with zeros at $D + \tau(D)$ and poles at Q and $\tau(Q)$, i.e., $D + \tau(D) \sim K_\Gamma + Q + \tau(Q)$ (K_Γ is the canonical class).

Proof. Let (2.31) hold. Then the differential sought, $\Omega(P)$, has the form

$$\Omega(P) = \frac{\theta\left(\int_Q^P \omega - z\right) \theta\left(\int_{\tau(Q)}^P \omega + z\right)}{\varepsilon(P, Q)\, \varepsilon(P, \tau(Q))} dp \tag{S.2}$$

(p is a local parameter at the point P, the quantity $\varepsilon(P, Q)$ has the form (1.13); cf. [10, Chap. 2]). The zeros of the numerator are situated at points of the divisors D (first factor) and $\tau(D)$ [second factor by virtue of (2.29)]. Further, (S.2) is the general form of differentials with poles at Q and $\tau(Q)$; whence and from (2.29) it follows that $z + \bar{z} \equiv \lambda$. The lemma is proved.

We note that $\theta(z) = 0$, if and only if the differential $\Omega(P)$ of the form (S.2) is holomorphic. The divisor D in this case contains the point Q.

Now we verify the assertion of the theorem for hyperelliptic curves $w^2 + P_{2g+2}(z) = 0$, where $P_{2g+2}(z)$ is a polynomial of degree 2g + 2 with real coefficients. On such a curve there is a pair of anti-involutions τ, τ', where $\tau(z, w) = (\bar{z}, \bar{w})$, $\tau'(z, w) = (\bar{z}, -\bar{w})$, and also the involution $\gamma(z, w) = (z, -w)$. Let the anti-involution τ be nonseparating. We take $Q = \{z = \infty\}$; then $\tau' Q = Q$. The differentials $\Omega(P)$, figuring in the lemma, have the form

$$\Omega(P) = \frac{R_g(z)\, dz}{\sqrt{P_{2g+2}(z)}}, \tag{S.3}$$

where the polynomial $R_g(z)$ of degree g has real coefficients. Its zeros are symmetric with respect to γ. Thus, the points of interest to us of the Jacobian [of the form (2.31)] are parametrized by divisors D such that $D + \tau(D)$ is invariant with respect to γ. The divisors D, satisfying this condition, have the form

$$D = \sum P_i + \sum (P_i' + \tau' P_i') + \sum (P_i'' + \gamma P_i''),$$

where $\tau' P_i = P_i$. We denote by $A_1', \ldots, A_m'$ all ovals of the involution τ', not containing Q. Here $m = n - 1$ for $n > 0$, $m = 0$ for $n = 0$ and $g = 2p$, $m = 1$ for $n = 0$ and $g = 2p + 1$. The "number" $(\varepsilon_1, \ldots, \varepsilon_m)$ of a component of the form (2.31) is defined thus: ε_k is the collection of points modulo 2 of the divisor D, lying on the oval A_k'. In each such component there are divisors D, containing the point Q [we recall that in this case the differential (S.3) is holomorphic]. Then $\theta(z) = 0$, where z has the form (S.1) (cf. [10]).

Now we proceed to the general case of a Riemann surface Γ with nonseparating anti-involution τ. Each such surface can be obtained by a deformation from a real hyperelliptic one in the class of real curves (cf. [6]). For the hyperelliptic case we have found in each component of the form (2.31) a divisor D such that $D + \tau(D)$ is the divisor of a holomorphic differential $\Omega = \sum_{i=1}^{g} \alpha_i \omega_i$. This differential is symmetric with respect to τ: $\tau^* \Omega = \overline{\Omega}$, the coefficients α_i are real. Under a continuous deformation the basis differentials vary continuously (cf. [12]). Thus the holomorphic symmetric differential Ω is defined on all curves of the deformation. It also determines for us the zero of the theta-function lying on the corresponding component. The theorem is proved.

LITERATURE CITED

1. V. A. Kozel and V. P. Kotlyarov, "Finite-zone solutions of the sine-Gordon equation," Preprint FTINT Akad. Nauk Ukr. SSR, No. 9-77, Kharkov (1977).
2. I. V. Cherednik, "Reality conditions in 'finite-zone integration,'" Dokl. Akad. Nauk SSSR, 252, No. 5, 1104-1108 (1980).
3. S. P. Novikov (ed.), Theory of Solitons [in Russian], Nauka, Moscow (1980).
4. B. A. Dubrovin, "S. P. Novikov's conjecture in the theory of theta-functions and nonlinear equations of Korteweg–de Vries and Kadomtsev–Petviashvili type," Dokl. Akad. Nauk SSSR, 251, No. 3, 541-544 (1980).
5. B. A. Dubrovin, "Theta-functions and nonlinear equations," Usp. Mat. Nauk, 36, No. 2, 11-80 (1981).
6. S. M. Natanzon, "Space of moduli of real curves," Tr. Mosk. Mat. Ob-va, 37, 219-253 (1978).
7. S. M. Natanzon, "Lobachevskiian geometry and automorphisms of complex curves," in: Geometric Methods in Problems of Analysis and Algebra. Interuniversity Collection of Themes [in Russian], No. 1 (1978), pp. 130-151; No. 2 (1980), pp. 156-158, Yaroslavl' State Univ.
8. I. M. Krichever, "Analog of d'Alembert's formula for the equations of a principal chiral field and the sine-Gordon equation," Dokl. Akad. Nauk SSSR, 253, No. 2, 288-292 (1980).
9. G. Springer, Introduction to the Theory of Riemann Surfaces [Russian translation], IL, Moscow (1960).
10. J. Fay, "Theta-functions on Riemann surface," Lect. Notes Math., 352, Springer-Verlag (1973).
11. G. Bateman and A. Erdei, Higher Transcendental Functions, McGraw-Hill.
12. L. V. Ahlfors and L. Bers, Lectures on Quasiconformal Mappings, Van Nos Reinhold (1966).
13. I. V. Gribkov, "Some solutions of the sine-Gordon equation obtained with the help of the Bäcklund transformation," Usp. Mat. Nauk, 33, No. 2, 191-192 (1978).
14. H. P. McKean, "The sine-Gordon and sinh-Gordon equations on the circle," Commun. Pure Appl. Math., 34, No. 2, 197-257 (1981).