PERIODIC PROBLEMS FOR THE KORTEWEG – DE VRIES EQUATION IN THE CLASS OF FINITE BAND POTENTIALS

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Introduction

It was recently shown [1, 2] that the Korteweg–de Vries (KdV) equation $\dot{u} = 6uu' - u'''$, well known from the theory of nonlinear waves, is closely related to the spectral theory of the Sturm–Liouville operator $L = -(d^2/dx^2) + u$. In case of quickly decreasing initial conditions $u(x, 0)$ this allows us to solve the Cauchy problem for the KdV equation, using the well known apparatus of the inverse-scattering problem [3-5]. At the same time, all potentials with vanishing reflection coefficients form a set of finite-dimensional invariant manifolds for the KdV equation. It was shown in [6] that the corresponding solutions of the KdV equation describe interactions of a finite number of solutions of simple wave type (solitons); therefore, the invariant manifolds mentioned are called N-soliton solution manifolds.

In the case of periodic problems for the KdV equation it was shown by Novikov [7] that the analog of N-soliton solutions is the manifold of functions $u(x)$, such that the operator $-(d^2/dx^2) + u(x)$ has exactly N gaps in the spectrum (such potentials are henceforth called finite-band or N-band. It was shown in [7] that any stationary solution of the N-th analog KdV equation (see Theorem 2.2 below) is an N-band potential. In the present paper we prove Novikov's hypothesis on obtaining all finite-band potentials. Besides, all finite-band potentials are explicitly described in the language of the theory of Abelian functions allowing complete description of the dynamics of the KdV equation and its analogs on manifolds of N-band potentials (see [8]). It should be noted that a description of finite-band potentials by the theory of Abelian functions, similar to that given here, was independently obtained (by somewhat different methods) by Its and Matveev [9].

We further mention the approach suggested by Marchenko [10] for solving the periodic KdV problem, based on approximating the matrix elements of the translation matrix by polynomial expressions in the energy. This approximating process is terminated for periodic finite-band potentials; possibly, the methods of paper [10] would be useful in solving the problem of approximating an arbitrary potential by finite-band ones. The studies of Marchenko are based on the differential equations for the time evolution of the translation matrix, obtained by him independently of [7].

The first examples of finite-band potentials can be extracted from Ince's work [11]; the potentials of the Lamé equation $u(x) = N(N+1)\wp(x)$ (here $\wp(x)$ is the elliptic Weierstrass function) are N-band functions. Methods of constructing other examples of finite-band potentials were suggested by Akhiezer in the continual generalization of the theory of orthogonal polynomials on a system of intervals [12]. The idea of the method of [12] is essentially used in the present paper. Finally, the problem of describing single-band potentials was solved completely by Hochstadt [13].

We formulate the basic result of this paper. Let $\{\Gamma_N\}$ be the set of hyperelliptic Riemann surfaces of order N, on which a branch point is marked (let it be infinity ∞). There exists over the space $\{\Gamma_N\}$ a single subdivision $\{J(\Gamma_N)\}$, whose layer is the Jacobi manifold $J(\Gamma_N)$ of the surface Γ_N, while in each layer a point is marked, corresponding to the divisor (∞) (it is easily seen that this point on the Jacobi manifold is a second-order point). This manifold $\{J(\Gamma_N)\}$ is called the full manifold of moduli of hyperelliptic Jacobians (with a distinguished second-order point). The set of all N-band potentials coincides with the manifold $\{J(\Gamma_N)\}$. At the same time the subdivision $\{J(\Gamma_N)\}$ remains invariant with respect to the action

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 9, No. 3, pp. 41-51, July-September, 1975. Original article submitted September 17, 1974.

of dynamic systems, determined by the KdV equation and its higher analogs, and the action of these dynamic systems of the $J(\Gamma_N)$ torus is given by rectangular sheaths.

It should be noted that many results of this paper [especially the differential equations (2.12) and (3.9)] can be generalized without difficulty to the case of an infinite number of bands, but at the same time the effectiveness of conducting potentials is lost to a large extent.

§1. Background from the Theory of Second-Order Differential Operators with Periodic Coefficients

Consider the operator $L = -(d^2/dx^2) + u(x)$, where $u(x)$ is a smooth real function, periodic with period T. In the solution space of the equation

$$Ly = Ey \tag{1.1}$$

we introduce basis functions $c(x, x_0, E)$ and $s(x, x_0, E)$ with the following initial conditions at point x_0:

$$c(x_0) = s'(x_0) = 1, \quad c'(x_0) = s(x_0) = 0. \tag{1.2}$$

The functions c and s are integral functions of the spectral parameter E. The linear translation operator $\hat{T}$ is determined on the solutions of Eq. (1.1),

$$(\hat{T}y)(x) = y(x+T). \tag{1.3}$$

Let $\alpha_{ij} = \alpha_{ij}(x_0, E)$ be the matrix of the operator $\hat{T}$ in the basis (1.2) (i, j = 1, 2). The matrix elements α_{ij} are, obviously, integral functions of E. Besides, $\det(\alpha_{ij}) = 1$. Consequently, the characteristic polynomial of the matrix (α_{ij}) is of the form $\lambda^2 - 2r\lambda + 1$, where $r = \frac{1}{2}\,\mathrm{Sp}(\alpha_{ij})$, since the eigenvalue of the operator $\hat{T}$ is independent of x_0 and r is a function of E only. The spectral bands are determined by the condition $|r(E)| \le 1$; the eigenvalues of the periodic and antiperiodic problems for the operator L are found from the equation $1 - r^2(E) = 0$. It is well known (see [14]) that the integral function $1 - r^2(E)$ has only real zeros of order not higher than two. The presence of doubly degenerate zeros E_n of the function $1 - r^2(E)$ corresponds to E_n being degenerate levels of the spectrum of the periodic (or antiperiodic) problems for the operator L. Therefore, the matrix of the operator $\hat{T}(E_n)$ is ±1 in any basis. Consequently, in this case

$$\alpha_{12}(x_0, E_n) = \alpha_{21}(x_0, E_n) \equiv 0. \tag{1.4}$$

Conversely, if E_n is a simple root of the function $1 - r^2(E)$, the matrix of the operator $\hat{T}(E_n)$ is not reduced to diagonal form, i.e., the operator $\hat{T}(E_n)$ has only one eigenvector. Boundaries of spectral bands are, obviously, responsible only for simple roots of $1 - r^2(E)$ (see [14] and [7]).

Let us stay within one of the spectral bands. The eigenvalues of the operator $\hat{T}(E)$ are then complex conjugate and are of the form $\exp(\pm ip(E))$, where $p(E)$ is real. Therefore, in this case the operator $\hat{T}$ has two eigenfunctions $\psi_\pm$, with $\psi_- = \overline{\psi}_+$. We normalize the functions $\psi_\pm$ by the condition

$$\psi_\pm(x_0) = 1. \tag{1.5}$$

Such functions are henceforth denoted by $\psi_\pm(x, x_0, E)$ (the indices ± are often omitted). Let $\chi = -i\psi'/\psi$.

LEMMA 1.1. The function $\chi = \chi(x, E)$: a) is independent of the choice of the point x_0, b) is periodic in x with period T, c) satisfies the equation $-i\chi' + \chi^2 + u - E = 0$, d) its imaginary part χ_I is determined by its real part χ_R, $\chi_I = \frac{1}{2} \cdot \chi_R'/\chi_R$, and e) for $E \to \infty$ we have the asymptotic expansion

$$\chi(x, E) \sim k + \sum_{n=1}^{\infty} \frac{\chi_n(x)}{(2k)^n} \quad (k^2 = E). \tag{1.6}$$

Proof. Part a follows from the fact that by changing x_0 the function ψ changes only by a constant factor. Part b follows from $\psi(x+T) = e^{ip}\psi(x)$. The asymptotic expansion (1.6) is well known (see [14]).

We note that it follows from parts d and e that the function $\chi_R(x, E)$ has the following asymptotic expansion for $E \to \infty$:

$$\chi_R(x, E) \sim k + \sum_{n=0}^{\infty} \chi_{2n+1}(x). \tag{1.7}$$

COROLLARY. The following identities hold:

$$\psi(x, x_0, E) = \sqrt{\frac{\chi_R(x_0, E)}{\chi_R(x, E)}} \exp\left\{i \int_{x_0}^{x} \chi_R(x, E)\right\}, \tag{1.8}$$

$$\psi_+\psi_- = |\psi|^2 = \frac{\chi_R(x_0, E)}{\chi_R(x, E)}, \tag{1.9}$$

$$p(E) = \int_{x_0}^{x_0+T} \chi_R(x, E)\,dx + 2\pi n. \tag{1.10}$$

LEMMA 1.2. The variational derivative of p(E) equals

$$\frac{\delta p(E)}{\delta u(x)} = -\frac{1}{2\chi_R(x, E)}. \tag{1.11}$$

Proof. If L_1 and L_2 are two operators with potentials u_1 and u_2, respectively, and $L_i y_i = E y_i$ ($i = 1, 2$), the following identity holds:

$$\frac{d}{dx}\{y_1, y_2\} = (u_1 - u_2)\, y_1 y_2. \tag{1.12}$$

Here $\{y_1, y_2\} = y_1' y_2 - y_1 y_2'$ is the Wronskian. Assuming $y_1 = \psi_{1+}$, $y_2 = \psi_{2-}$ in (1.12) and integrating over the period, we obtain

$$i\left(e^{i(p_1-p_2)} - 1\right)\left(\chi_1(x_0) + \bar{\chi}_2(x_0)\right) = \int_{x_0}^{x_0+T} (u_1 - u_2)\, \psi_{1+}\psi_{2-}\, dx. \tag{1.13}$$

Transforming in (1.13) from differences to variation, we obtain (1.11).

LEMMA 1.3.

$$\chi(x, E) = \frac{\sqrt{1 - r^2(E)}}{\alpha_{21}(x, E)} + \frac{i}{2}\,\frac{\alpha_{11}(x, E) - \alpha_{22}(x, E)}{\alpha_{21}(x, E)}. \tag{1.14}$$

Proof. Since $\psi(x, x_0, E)$ is the eigenvector of the matrix $\alpha_{ij}(x_0, E)$, normalized by condition (1.5),

$$\psi(x, x_0, E) = c(x, x_0, E) + i\xi(x_0, E)\, s(x, x_0, E), \tag{1.15}$$

where $\xi(x_0, E) = \frac{\sqrt{1 - r^2(E)}}{\alpha_{21}(x_0, E)} + \frac{i}{2}\,\frac{\alpha_{11}(x_0, E) - \alpha_{22}(x_0, E)}{\alpha_{21}(x_0, E)}$. On the other hand, it follows from the definition of χ that the Wronskian is

$$\{\psi(x, x_0, E), c(x, x_0, E)\} = i\chi(x_0, E). \tag{1.16}$$

Comparing (1.16) and (1.15) and taking into account part a of Lemma 1.1, we obtain (1.14).

§2. Finite Band Potentials

Now let the potential u(x) have only a finite number of spectral bands. By §1 this is equivalent to the case where the function $1 - r^2(E)$ has only a finite number of simple roots (their number is, obviously, odd). Let these roots be $E_1, \ldots, E_{2N+1}$ (i.e., the operator L has exactly N gaps). We point out that $\sqrt{[1 - r^2(E)]/R(E)}$ is then, obviously, continued to an integral analytic function; all roots of this function are simple and coincide with the degenerate roots of the function $1 - r^2(E)$. By (1.4), therefore, $\alpha_{21}(x, E)$ and $\alpha_{12}(x, E)$ are divided by this radicand, i.e.,

$$\alpha_{21}(x, E) = \tilde{\alpha}_{21}(x, E) \cdot \sqrt{\frac{1 - r^2(E)}{R(E)}}. \tag{2.1}$$

Substituting (2.1) in (1.14), we obtain $\chi_R(x, E) = \frac{\sqrt{R(E)}}{\tilde{\alpha}_{21}(x, E)}$.

If $k^2 = E$, at infinity $\sqrt{R(E)}$ has the asymptotic $k \cdot E^N$. By (1.7), therefore, the integral function $\tilde{\alpha}_{21}(x, E)$ is bound to have an asymptotic E^N at infinity, i.e., it is an N-th order polynomial in E. We denote this polynomial by $\tilde{\alpha}_{21}(x, E) = P(E, x) = \prod_{i=1} (E - \gamma_i(x))$.

We, thus, have the following result.

THEOREM 2.1. For a finite-band potential with band boundaries $E_1, \ldots, E_{2N+1}$ the function $\chi(x, E)$, is of the form

$$\chi(x, E) = \left(\sqrt{R(E)} - \frac{i}{2}\frac{dP(E, x)}{dx}\right) \Big/ P(E, x). \quad (2.2)$$

The roots $\gamma_i(x)$ of the polynomial P(E, x) are real and are located in gaps or on their boundaries.

Proof. It remains to prove only the assertion on the location of the roots $\gamma_i(x)$. It follows from the definition of P that $\gamma_i(x_0)$ are roots of the function $\alpha_{21}(x_0, E)$. Consequently, $E = \gamma_i(x_0)$ is an eigenvalue of the operator L at the segment $[x_0, x_0 + T]$ with vanishing boundary conditions. Hence, there follows the reality of the roots $\gamma_i(x_0)$. It follows directly from the unimodularity of the matrix (α_{ij}) that the equality $\alpha_{21}(x_0, E) = 0$ can be satisfied only if E is in a gap or on a boundary. That $\gamma_i(x_0)$ lies exactly in one gap is obvious from alternate considerations.

From Theorem 2.1 we derive a statement, inverse to the basic theorem of [7], which we recall here. We define a set of functionals $I_n\{u\}$, putting

$$I_n\{u\} = \int_T \chi_{2n+3}(x)\, dx. \quad (2.3)$$

Here $\chi_{2n+1}(x)$ are the expansion coefficients (1.7) of the function $\chi_R(x, E)$ for $E \to \infty$. All $\chi_{2n+1}(x)$ are polynomials in u and their derivatives. The equation

$$\dot{u} = -\frac{1}{2}\frac{\partial}{\partial x}\frac{\delta}{\delta u}\sum_{n=0}^{N} c_n I_n \quad (2.4)$$

is called the N-th analog of the KdV equation.

In particular, an ordinary differential equation of order 2N,

$$\frac{\delta}{\delta u}\sum_{n=0}^{N} c_n I_n = d \quad (2.5)$$

is obtained to determine the stationary solutions of Eq. (2.4). It was shown in [7] that any solution of Eq. (2.5) is an N-band potential. We show the inverse theorem.

THEOREM 2.2. Let u(x) be an N-band potential. Then u(x) satisfies some differential equation of form (2.5)

Proof. From Eqs. (1.10), (1.11), and (2.2) we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2k)^{2n+1}}\frac{\delta I_{n-1}}{\delta u(x)} = -\frac{1}{2}\frac{P(E, x)}{\sqrt{R(E)}} \quad (k^2 = E). \quad (2.6)$$

From the explicit form of Eq. (2.6) we see that if $\frac{1}{2}\frac{P(E, x)}{\sqrt{R(E)}} = \sum_{n=0}^{\infty}\frac{\beta_{n-1}(x)}{(2k)^{2n+1}}$ is the Taylor expansion at an infinitely remote point, then the quantities $\beta_N(x), \beta_{N+1}(x), \ldots$ are linearly expressed in terms of $\beta_{-1}(x) = -1, \beta_0(x), \ldots, \beta_{N-1}(x)$ with constant coefficients. Therefore, this statement is also valid for the series on the left-hand side of (2.6). We obtain

$$\frac{\delta I_N}{\delta u(x)} + \sum_{n=-1}^{N-1} c_n \frac{\delta I_n}{\delta u(x)} = 0. \quad (2.7)$$

Since $\delta I_{-1}/\delta u(x) = -1$, putting $c_{-1} = d$ we obtain an equation of type (2.5).

Let Γ_N by the (hyperelliptic) Riemann surface of the function $\sqrt{R(E)}$. By Theorem 2.1 the function $\chi(x, E)$ is a single-valued algebraic function on the surface Γ_N. We show that the function ψ also has a natural continuation on Γ_N.

THEOREM 2.3. The eigenfunction $\psi(x, x_0, E)$ continued to a meromorphic function on $\Gamma_N \setminus \infty$, has there N poles at the points $E = \gamma_i(x_0)$, N roots at the points $E = \gamma_i(x)$, and also an essentially singular point at infinity with an asymptotic of form $\exp[ik(x - x_0)]$.

Proof. We recall that by Lemma 1.3 we have

$$\psi(x, x_0, E) = c(x, x_0, E) + i\chi(x_0, E)\, s(x, x_0, E). \quad (2.9)$$

Since χ is algebraic on Γ_N, and c and s are integral functions of E, ψ is obviously continued to a single-valued function on Γ_N. The poles of the function ψ can occur only where $\chi(x_0, E)$ can have poles, i.e., at the points $E = \gamma_i(x_0)$. We show that ψ indeed has exactly one pole at each point $E = \gamma_i(x_0)$. Indeed, by Eqs. (1.9) and (2.2) we have

$$\text{norm} \quad (\psi) = \psi(x, x_0, E_+)\cdot\psi(x, x_0, E_-) = \frac{P(x, E)}{P(x_0, E)} \tag{2.10}$$

(here E_+, E_- are two points on Γ_N, located on E).

It follows from (2.10) that norm (ψ) has only simple poles for $E = \gamma_i(x_0)$; therefore, ψ cannot have two poles at the point $E = \gamma_i(x_0)$. The statement on the location of roots of ψ obviously follows from Eq. (2.10).

Adopting free notation in what follows, we denote the roots and poles of the function ψ on Γ_N by the same symbols $\gamma_i(x)$ and $\gamma_i(x_0)$.

One more spectral interpretation of the energy levels $E = \gamma_i(x)$ follows from Theorem 2.3: the discrete spectrum eigenvalue of the operator L on one of the rays $[x_0, \pm\infty]$ with vanishing boundary conditions, i.e., the conditional eigenvalue in the terminology of [15]. We obtain a differential equation for the conditional eigenvalues.

It follows directly from Theorem 2.3 that the function $\chi(x, E)$ also has on $\Gamma_N \setminus \infty$ N poles at the points $\gamma_i(x)$; therefore, the numerator of Eq. (2.2) vanishes for $E = \gamma_i(x)$ on one sheet of Γ_N. Consequently, we have the system of equations

$$P'(E, x)|_{E=\gamma_j(x)} = 2i\sqrt{R(\gamma_j)} \qquad (j = 1, \ldots, N) \tag{2.11}$$

[the sign in front of the root is chosen according to the sheet where the poles $\gamma_j(x)$ are located]. The system (2.11) is easily rewritten in form

$$\gamma_j' = -\frac{2i\sqrt{R(\gamma_j)}}{\prod\limits_{j\neq k}(\gamma_j - \gamma_k)} \qquad (j = 1, \ldots, N). \tag{2.12}$$

Equation (2.12) gives the law of motion of the points γ_j over the cycles on Γ_N, located over the gap. We change variables and integrate the system of Eqs. (2.12). The idea of this replacement is based on the method of [12]. We introduce on Γ_N a basis of cycles a_j, b_k (j, k = 1, . . ., N), so that the intersecting indices have the following form:

$$(a_j, b_k) = \delta_{jk}, \quad (a_j, a_k) = (b_j, b_k) = 0.$$

Let $\omega_1, \ldots, \omega_N$ be a basis of holomorphic differentials (first-order differentials) on Γ_N, normalized by the condition

$$\oint_{a_k} \omega_j = 2\pi i\delta_{jk}. \tag{2.13}$$

Let Ω be a second-order differential on the surface Γ_N with double poles at infinity, normalized by the condition

$$\oint_{a_k} \Omega = 0 \qquad (k = 1, \ldots, N). \tag{2.14}$$

Let, further,

$$\oint_{b_k} \Omega = iU_k. \tag{2.15}$$

We fix a mapping

$$A\colon S^N\Gamma_N \to J(\Gamma_N) \tag{2.16}$$

of the N-th symmetric power of Γ_N into its Jacobi manifold (the Abel mapping). In coordinates this mapping is written as

$$[A(P_1, \ldots, P_N)]_n = \sum_{i=1}^{N}\int_{\infty}^{P_i} \omega_n \quad (n = 1, \ldots, N). \tag{2.17}$$

Akhiezer's theorem states that for poles and roots of the functions with the properties described in Theorem 2.3 the following relation holds in the Jacobi manifold:

$$A(\gamma_1(x), \ldots, \gamma_N(x)) = A(\gamma_1(x_0), \ldots, \gamma_N(x_0)) + \mathbf{U}\cdot(x - x_0). \tag{2.18}$$

Due to the fact that A is a birational isomorphism, Eq. (2.18) can be solved for almost all x, and the roots $\gamma_1(x), \ldots, \gamma_N(x)$ can be found.

To find the potential, however, it is not necessary to explicitly solve the system of Eqs. (2.18) for $\gamma_1(x), \ldots, \gamma_N(x)$. Indeed, for $\chi(x, E)$ expansion (1.7) holds, in which $\chi_1(x) = -u(x)$. On the other hand, from Eq. (2.2) we have that the same coefficient equals $2\sum\gamma_i(x) - \sum E_i$. Therefore, we obtain

$$u(x) = -2\sum\gamma_i(x) + \sum E_i. \tag{2.19}$$

We now compare Eqs. (2.18) and (2.19). For final formulation of the algebraic-geometric description of the manifold of finite-band potentials we define on the Jacobi manifold $J(\Gamma_N)$ the function σ_1,

$$\sigma_1 \circ A((Q_1, \sqrt{R(Q_1)}), \ldots, (Q_N, \sqrt{R(Q_N)})) = Q_1 + \ldots + Q_N; \tag{2.20}$$

σ_1 is obviously an algebraic function on $J(\Gamma_N)$ (in [9] σ_1 was explicitly expressed in terms of Riemann's θ-function). We obtain the following theorem.

THEOREM 2.4. Each potential with band boundaries $E_1, \ldots, E_{2N+1}$ is determined by assigning an initial point on the Jacobi manifold $J(\Gamma_N)$ and is a bounded function $2\sigma_1 + \sum_i E_i$ on the rectilinear sheet of the torus $J(\Gamma)$, protruding from this point with a normal vector U.

COROLLARY. The manifold of N-band potentials coincides with the full manifold of moduli of hyperelliptic Jacobians with second-order distinguished points.

Thus, we see that for given band boundaries a potential is obtained which is, generally speaking, conditionally periodic with N independent periods (this was first pointed out in [7]).

§3. Time Evolution of Finite-Band Potentials due to the KdV Equation and Its High Analogs

Now let u = u(x, t) depend on the parameter t according to an equation of type (2.4). The operator L then also depends on the parameter t. Lax pointed out that a real skew-symmetric operator A of order 2N + 1 with coefficients depending on u, u', ..., can then be found, so that Eq. (2.4) is equivalent to the equation

$$\dot{L} = [A, L] \tag{3.1}$$

([2]; see also [16]). For eigenfunctions (1.1) the equation

$$\frac{\partial y}{\partial t} = Ay + \lambda y + \mu\bar{y} \tag{3.2}$$

holds where λ is independent of x. It was shown in [7] that the eigenvalues of the operator $\hat{T}$, i.e., the functions p(E), are independent of time t. Therefore, if for y one takes the function $\psi(x, x_0, E)$, then $\mu = 0$, $\lambda = \lambda(x_0, E)$. We note that the action of the operator A on the eigenfunction ψ can be represented in the form

$$\begin{aligned} A\psi(x, x_0, E) &= \Lambda(x, E)\psi'(x, x_0, E) + \Xi(x, E)\psi(x, x_0, E) = \\ &= [i\Lambda(x, E)\chi(x, E) + \Xi(x, E)]\psi(x, x_0, E), \end{aligned} \tag{3.3}$$

where Λ and Ξ are real functions, polynomials depending on E and on u, u', ... Taking into account the normalization (1.5), we then obtain

$$-\lambda(x_0, E) = i\Lambda(x_0, E)\chi(x_0, E) + \Xi(x_0, E). \tag{3.4}$$

Differentiating Eq. (3.2) with respect to x, we have

$$\dot{\chi}(x, E) = [\Lambda(x, E)\chi(x, E) - i\Xi(x, E)]' = i\lambda'(x, E). \tag{3.5}$$

Using the relation $\chi_I = \frac{1}{2}\cdot\frac{\chi_R'}{\chi_R}$, we obtain

$$\Xi(x, E) = -\frac{1}{2}\Lambda'(x, E). \quad (3.6)$$

Let now the potential u be finite-band. We then obtain from (3.5) an expression for the time derivative of the polynomial P(E, x),

$$\dot{P} = \Lambda P' - \Lambda' P. \quad (3.7)$$

This equality is valid for any E. Substituting $E = \gamma_j$ and taking into account (2.11), we have

$$\dot{P}|_{E=\gamma_j} = 2i\Lambda(\gamma_j)\sqrt{R(\gamma_j)} \qquad (j = 1, \ldots, N) \quad (3.8)$$

[the sign convention is as in (2.11) and (2.12)]. Hence,

$$\dot{\gamma}_j = -\frac{2i\Lambda(\gamma_j)\sqrt{R(\gamma_j)}}{\prod_{k\neq j}(\gamma_j - \gamma_k)} \qquad (j = 1, \ldots, N). \quad (3.9)$$

We show that the system (3.9) reduces to a system with constant coefficients by means of the Abelian mapping. In what follows we work with the KdV equation analog of standard form

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial x}\frac{\delta I_n}{\delta u(x)}. \quad (3.10)$$

For n = 1 we obtain the standard KdV equation $\dot{u} = 6uu' - u'''$. We denote the polynomial $\Lambda(x, E)$ for Eq. (3.10) by $\Lambda_n(x, E)$. We provide an explicit expression for the polynomial $\Lambda_n(x, E)$.

LEMMA 3.1. The following equation holds:

$$\Lambda_n = (4E)^n \frac{\delta}{\delta u}\left(I_{-1} + \frac{I_0}{4E} + \cdots + \frac{I_{n-1}}{(4E)^n}\right). \quad (3.11)$$

To prove the lemma we consider the operator

$$A_z = \frac{1}{8}\left[\frac{1}{\chi_R(x, z)}\frac{d}{dx} - \frac{1}{2}\left(\frac{1}{\chi_R(x, z)}\right)'\right]\frac{1}{L - z} \quad (3.12)$$

(the idea of considering such an operator was suggested by Novikov).

LEMMA 3.2. The commutator of the operators A_z and L is a multiplication operator on the subsequent function

$$[A_z, L] = \frac{1}{4}\frac{d}{dx}\left(\frac{1}{\chi_R(x, z)}\right) = -\frac{1}{2}\frac{d}{dx}\frac{\delta p(x)}{\delta u(x)}. \quad (3.13)$$

Proof. We evaluate the result of this operator acting on the eigenfunctions of the operator L. We have

$$[A_z, L]\psi(x, x_0, E) = (E - L)A_z\psi(x, x_0, E).$$

After the calculation we obtain

$$[A_z, L]\psi(x, x_0, E) = \frac{1}{8(E - z)}\left[-\frac{1}{2}f''' + 2f'(u - E) + fu'\right]\psi(x, x_0, E),$$

where $f = 1/\chi_R(x, z)$. Since up to a constant factor, independent of x, f is simply $|\psi(x, x_0, z)|^2$ [see (1.9)], f satisfies the equation

$$-\frac{1}{2}f''' + 2f'(u - z) + fu' = 0,$$

which also concludes the proof.

Proof of Lemma 3.1. We expand A_z in a power series in $\varkappa^{-1}$, where $\varkappa = \sqrt{z}$,

$$A_z = \sum_n \frac{A_{n-1}}{(2\varkappa)^{2n+1}}. \quad (3.14)$$

It then follows from (3.14) and (1.11) that

$$[A_n, L] = -\frac{1}{2}\frac{d}{dx}\frac{\delta I_n}{\delta u(x)}, \tag{3.15}$$

i.e., the operator A_n provides a Lax commutation representation for Eq. (3.10). We note that for the operator A_z the corresponding function Λ_z is of the form

$$\Lambda_z(x, E) = +\frac{1}{8(E-z)\chi_R(x, z)}. \tag{3.16}$$

Expanding (3.16) in a series in $\varkappa^{-1}$ and again evaluating (1.11), we obtain the assertion of the lemma.

Let Ω_n be a second-order differential on the surface Γ_n with poles of order 2n + 2 at infinity, normalized by the condition $\oint_{a_k} \Omega_n = 0$. Let [see (2.14), (2.15)]

$$iU_k^{(n)} = -\oint_{b_k} \Omega_n. \tag{3.17}$$

THEOREM 3.1. For the Abelian mapping A the system (3.9) transforms into a system with constant coefficients, i.e.,

$$A(\gamma_1(t), \ldots, \gamma_N(t)) = A(\gamma_1(t_0), \ldots, \gamma_N(t_0)) + 2^{2n}U^{(n)}(t-t_0) \tag{3.18}$$

(all γ are at one x).

Proof. From (3.2) and (3.4) we obtain

$$\psi_t(x, x_0, E) = \frac{\varphi_x(t, t_0, E)}{\varphi_{x_0}(t, t_0, E)}\psi_{t_0}(x, x_0, E), \tag{3.19}$$

where the function

$$\varphi_x(t, t_0, E) = \exp\left\{-\int_{t_0}^{t} \lambda_t(x, E)\,dt\right\} \tag{3.20}$$

is, consequently, a single-valued function on Γ_N, meromorphic on $\Gamma_N \setminus \infty$, has N poles at $E = \gamma_i(x, t_0)$ and has N roots at $E = \gamma_i(x, t)$. We evaluate the behavior of $\varphi_x(t, t_0, E)$ at $E \to \infty$. We note that it follows directly from Eqs. (3.11), (3.4), and (1.11) that $\lambda_t(x, E)$ is for $E \to \infty$ of the form

$$\lambda_t(x, E) \sim 2^{2n} ik^{2n+1} + O\left(\frac{1}{k}\right) \qquad (k^2 = E). \tag{3.21}$$

Therefore, the function $\varphi_x(t, t_0, E)$ has at infinity an asymptote of the form $\exp(-2^{2n}ik^{2n+1}(t-t_0))$. Equation (3.18) is now obtained after applying the Akhiezer procedure to the function $\varphi_x(t, t_0, E)$.

Thus, the point coordinates on the Jacobi manifold $J(\Gamma_N)$ are natural angular variables for the Hamiltonian KdV equation (see [17]).

COROLLARY. To identify the full manifold of moduli of hyperelliptic Jacobians of distinguished second-order points with the solution space of equations of type (2.5), obtained by comparing the results of [7] with the results of §2 of the present paper, the tori of $J(\Gamma_N)$ transform into the invariant tori of the fully integrable Hamiltonian systems (2.5), explicitly evaluated in [7].

A discussion of the algebraic-geometric conclusions obtained by such comparison is given in [18].

Theorem 3.1 now allows a complete solution of the Cauchy problem for the KdV equation for finite-band initial conditions. Several specific calculations related to the two-band case are discussed in [19].

Note. As was shown in the Vancouver International Mathematical Congress, simultaneously with Novikov's paper [7] Lax's paper appeared [20], in which it was also shown (though by other methods) that stationary periodic solutions of high analog KdV [see Eq. (2.5)] are finite-band potentials. Unlike [7], Lax's proof is not effective and does not allow one to obtain the totally integrable equations (2.5). The class of almost periodic finite-band potentials is not discussed in Lax's paper. The proof of the hypothesis, formulated by Lax at the end of [20], is contained in Theorem 2 of the author's paper [8] (see also Theorem 2.2 of the present paper).

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