Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg–de Vries equation

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A method of connecting the Korteweg–de Vries (KdV) equation, known from the theory of nonlinear waves, with the Schrödinger equation was discovered in 1967.[1] This method is applied in the present paper to a study of a periodic problem. We find exact analytical formulae for a class of solutions $u(x,t)$ such that at any moment in time t the potential $u(x,t)$ of the Schrödinger operator has only a finite number of forbidden bands in the Bloch spectrum. We find in this connection all potentials with a finite number of bands. This class of solutions contains as a degenerate limiting case the well known N-soliton solutions of the KdV equation, which decrease rapidly as $|x|\to\infty$.

INTRODUCTION

It is well known that the nonlinear Korteweg—de Vries (KdV) wave equation

$$u_t=6uu_x-u_{xxx}$$

reduces to the inverse problem of scattering theory for the Schrödinger (Sturm-Liouville) operator

$$L=\frac{-d^2}{dx^2}+u(x),\quad u(x)=u(x,t)\,|_{t=\text{const}},$$

if the solution u(x, t) decreases rapidly as $|x|\to\infty$ (see [1,2]). The most effective study has in this case been made of the so-called "multisoliton" solutions which describe the interaction of a finite number of solitons—solutions of the kind u(x − ct). They have the form u(x, t) where at any time t the potential u is nonreflective (the reflection coefficient vanishes identically). Although the algebraic mechanism connecting the KdV equation with the Schrödinger operator continues to function also in the case of periodic boundary conditions, nobody had succeeded in applying it seriously to an effective study of the KdV equation until the recent work by the present authors [3,4] and by Its and Matveev. [5]

The basis of this procedure is the fact, noted by one of us, [3] that a strictly periodic (and conditionally periodic) analog of the many-soliton solutions consists in those u(x, t) for which at any time t the potential u(x, t) has only a finite number of forbidden bands in the Bloch spectrum. Such a class of potentials, which we shall call in what follows finite-band potentials, contain as a degenerate limiting case all non-reflective potentials which decrease fast as $|x|\to\infty$; all finite-band potentials and the corresponding solutions of the KdV equation can be found in terms of exact, albeit complicated, formulae. The solutions of the form u(x − ct) are in the periodic case potentials with a single forbidden band. This is a Weierstrass elliptic function $2\wp(x)$ + constant. Even a consideration of their simplest perturbations leads to a two-band (i.e., two-forbidden-band) conditionally periodic potential u(x, t) with two, generally speaking, noncommensurate periods, where $u(x+\Delta_1, t+\Delta_2)=u(x,t)$ (i.e., after a period $T=\Delta_2$, the picture is re-established with a shift $x\to x-\Delta_1$).

In this paper we describe a class of periodic and conditionally periodic finite-band potentials and the corresponding family of solutions of the KdV equation. Although many facts can easily be generalized also to the case of an infinite number of bands, to a large extent the results lose their effectivity. We must note here that finite-band periodic potentials turn out to be relatively numerous among the periodic functions in contrast to the non-reflective Bargmann potentials: apparently one can approximate any smooth periodic potential by a finite-band one, although we have not proved this. We note that the procedure developed in the present paper is applicable also to other nonlinear equations which are "fully integrable" by the scattering theory method and which occur in a study of a periodic problem; it is now already known that their number is large [6-8] (Zakharov and Shabat [9] have developed a regular method to find them).

1. FINITE-BAND POTENTIALS AND INTEGRALS OF THE KdV EQUATION

Lax [10], using the procedure of [1], has noted that the basis for the connection between the KdV equation and the Schrödinger operator is the representation of the right-hand side $6uu'-u'''$ as a commutator

$$6uu'-u'''=[A,L],$$
$$L=-\frac{d^2}{dx^2}+u,\quad A=-4\frac{d^3}{dx^3}+3\left(u\frac{d}{dx}+\frac{d}{dx}u\right),\tag{1.1}$$

whence it follows that the equations

$$\dot u=6uu'-u'''\text{ and }\dot L=[A,L]\tag{1.2}$$

are equivalent.

If φ is an eigenfunction, $L\varphi=E\varphi$, we easily get from (1.1) the relation

$$(L-E)\dot\varphi=(L-E)A\varphi.\tag{1.3}$$

We fix two eigenfunction bases

$$\varphi(x,x_0,E),\quad \bar\varphi(x,x_0,E),$$
$$x=x_0,\quad \varphi=1,\quad \varphi'=ik,\quad k^2=E;\tag{1.4}$$
$$c(x,x_0,E),\quad s(x,x_0,E),$$
$$x=x_0,\quad c'=0,\quad c=1,\quad s'=1,\quad s=0.\tag{1.4'}$$

For a periodic potential u(x) with period T the translation operator produces when acting upon the eigenfunctions a shift over the period T:

$$(\hat T\varphi)(x)=\varphi(x+T).\tag{1.5}$$

We obtain in both cases (1.4) and (1.4′) a second rank matrix with a determinant equal to unity:

$$\hat T(x_0,k)=\begin{pmatrix} a & b\\ \bar b & \bar a\end{pmatrix},\quad |a|^2-|b|^2=1\tag{1.6}$$

in the base (1.4), or

$$\hat T(x_0,k)=\begin{pmatrix} \alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix},\quad \alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=1,\tag{1.6'}$$

where the α_{ij} are real, $k^2=E$, in the base (1.4′).

For rapidly decreasing potentials one usually chooses $x_0 = \pm\infty$, $\hat{T} = \hat{T}(k)$ in the base from two exponents. In the case of a finite period the choice of the point x_0 is arbitrary and when we change x_0 (in the base (1.4)) we have the equation

$$d\hat{T}/dx_0=[Q,\hat{T}],\qquad Q=ik\begin{pmatrix}1&0\\0&-1\end{pmatrix}-\frac{iu}{2k}\begin{pmatrix}1&-1\\1&-1\end{pmatrix}. \tag{1.7}$$

The Bloch eigenufnctions $\psi_\pm(x, x_0, E)$ are determined by the conditions

$$L\psi_\pm=E\psi_\pm,\ \hat{T}\psi_\pm=\exp(\pm ip(E))\psi_\pm,\ \psi_\pm|_{x=x_0}=1, \tag{1.8}$$

where the dispersion law p(E) is determined in the allowed bands. The trace of the matrix $\hat{T}$ is of the form

$$\mathrm{Sp}\hat{T}=a+\bar{a}=2a_R \tag{1.9}$$

in the base (1.4) and is independent of x_0. The allowed bands are determined by the condition

$$\tfrac{1}{2}|\mathrm{Sp}\hat{T}|=|a_R|\leqslant 1. \tag{1.10}$$

We note that the eigenvalues of the matrix $\hat{T}$ have the form $a_R \pm (a_R^2-1)^{1/2}$, or $a_R = \cos p(E)$. The points of the discrete spectrum E_n of the periodic and the antiperiodic problems: $\psi(x+T) = \pm\psi(x)$, are determined by the conditions $a_R(E_n) = \pm 1$. They are the edges of the forbidden bands only when these levels are nondegenerate (or the matrix $\hat{T}$ is a Jordan matrix for $E = E_n$). If $a_R = \pm 1$, but the matrix $\hat{T}$ is diagonal (and equal to ± 1) the forbidden band is collapsed to nothing. This is characterized by a condition similar to the non-reflectivity condition:

$$b(x_0, k_n)\equiv 0,\quad k_n^2=E_n. \tag{1.11}$$

The finite-band character of the potential means that all higher periodic and antiperiodic levels E_n are twofold degenerate.

We find, clearly, from (1.6) that in the points of the spectrum E_n of the periodic and antiperiodic problems: $a_R = \pm 1$, we have the equations

$$|a_I|=|b|,\quad E=E_n, \tag{1.12}$$

where $a = a_R + ia_I$, $|b| \neq 0$ in the non-degenerate points of the spectrum for all x_0.

If $\chi(x, E) = -i d(\ln\psi)/dx$, then χ will be independent of the point x_0 and will satisfy the Riccati equation which expresses, in particular, its imaginary part in terms of its real part:

$$-i\chi'+\chi^2+u=E,\quad \chi_I=\tfrac{1}{2}(\ln\chi_R)', \tag{1.13}$$

and allows an asymptotic expansion as $E \to \infty$

$$\chi(x,k^2)\sim k+\sum_{n\geqslant 1}\frac{\chi_n(x)}{(2k)^n}. \tag{1.14}$$

By virtue of (1.13) all functions $\chi_n(x)$ are polynomials in u(x) and its derivatives with respect to x, while the $\chi_{2m}(x)$ are total derivatives.

It is well known that all integrals

$$I(k)=\int\chi(x,k^2)\,dx,\quad I_{m-1}=\int\chi_{2m+1}(x)\,dx,\quad m\geqslant 0, \tag{1.15}$$

are conserved by virtue of the KdV equation. Moreover (Lax and Gardner [10,11]) all "higher KdV equations"

$$\dot{u}=\frac{\partial}{\partial x}\frac{\delta I_m}{\delta u(x)} \tag{1.16}$$

admit of a representation in the form (1.1):

$$\dot{L}=[A_m,L],\quad A_m=\frac{d^{2m+1}}{dx^{2m+1}}+\sum_{i=0}^{2m}P_i\frac{d^i}{dx^i}, \tag{1.17}$$

where all P_i are polynomials in u and its derivatives with respect to x. The KdV equation itself is obtained for m = 1, and the operators A_0, A_1, and A_2, and the integrals I_{-1}, I_0, I_1, and I_2 take the following form:

$$\begin{gathered}I_{-1}=\int u\,dx,\quad I_0=\int u^2\,dx,\quad A_0=2\frac{d}{dx},\\ I_1=\int\left[u^3+\frac{1}{2}(u')^2\right]dx,\\ A_1=-4\frac{d^3}{dx^3}+3\left(u\frac{d}{dx}+\frac{d}{dx}u\right),\\ I_2=\int\left[\frac{1}{2}(u'')^2-\frac{5}{2}u^2u''+\frac{5}{2}u^4\right]dx,\\ A_2=16\frac{d^5}{dx^5}-20\left(u\frac{d^3}{dx^3}+\frac{d^3}{dx^3}u\right)+30u\frac{d}{dx}u+5\left(u''\frac{d}{dx}+\frac{d}{dx}u''\right).\end{gathered} \tag{1.18}$$

Any equation of the form $\dot{u} = Q(u, u', \ldots)$, where the right-hand side is a polynomial and can be written in the form of a commutator $[A, L] = Q$, is of the form

$$A=\sum_{m=0}^{N}c_mA_m,\quad Q=\frac{d}{dx}\left(\sum_{m=0}^{N}c_m\frac{\delta I_m}{\delta u(x)}\right). \tag{1.19}$$

Let some such equation be given. It turns out that all its periodic stationary solutions u(x) are finite-band potentials, and we obtain thus all finite-band potentials. The conditionally periodic solutions of this equation are also finite-band potentials (see [3,4]). We shall indicate below the algorithm for integrating these equations:

$$\sum_{i=0}^{N}c_i\frac{\delta I_i}{\delta u(x)}=\mathrm{const}. \tag{1.20}$$

By virtue of (1.3) we have for the basis (1.4) of the eigenfunctions the equations

$$\dot{\varphi}=A\varphi+\lambda\varphi+\mu\bar{\varphi},\quad \dot{\bar{\varphi}}=A\bar{\varphi}+\bar{\mu}\varphi+\bar{\lambda}\bar{\varphi}, \tag{1.21}$$

where the matrix

$$\Lambda=\Lambda(x_0,k)=\begin{pmatrix}\lambda&\mu\\ \bar{\mu}&\bar{\lambda}\end{pmatrix} \tag{1.22}$$

has a zero trace $\lambda + \bar{\lambda} = 0$ and has a polynomial dependence on k, on u, and on its derivatives with respect to x in the point $x = x_0$. It is determined from the conditions $\dot{\varphi} = 0$ and $\dot{\varphi}' = 0$ when $x = x_0$. It turns out that we have for the matrix $T(x_0, k)$ the equation

$$\partial\hat{T}/\partial t=[\Lambda,\hat{T}]. \tag{1.23}$$

Comparing (1.23) with (1.7), we get from the condition

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x_0}\hat{T}=\frac{\partial}{\partial x_0}\frac{\partial}{\partial t}\hat{T}$$

the equation

$$\frac{\partial\Lambda}{\partial x_0}-\frac{\partial Q}{\partial t}=[\Lambda,Q]. \tag{1.24}$$

Equation (1.24) gives a new useful algebraic representation of the KdV equation and its higher analogs. For instance, for stationary solutions of Eqs. (1.16) we have Eq. (1.20), whence it follows that

$$\begin{gathered}\partial Q/\partial t=\partial\hat{T}/\partial t=0,\\ d\Lambda/dx_0=[\Lambda,Q],\quad[\Lambda,\hat{T}]=0.\end{gathered} \tag{1.25}$$

Since $\mathrm{Tr}\,\Lambda = 0$, the eigenvalues $\alpha_\pm$ are given by $\alpha_\pm(k) = \pm(\det\Lambda)^{1/2}$, where $\det\Lambda$ is a polynomial of $k^2 = E$, the zeroes of which (see below) are the boundaries of the bands, with coefficients depending on u, u′, ..., $u^{(2N)}$. These coefficients are also a complete set of commuting integrals of the Hamiltonian of Eq. (1.20) which also allows, by virtue of (1.25), a commutator representation with second rank matrices with coefficients which depend polynomially on k. Furthermore, for the matrix elements of the second Eq. (1.25) we get

$$[\Lambda, \hat{T}]_{12}=(\lambda-\bar{\lambda})b+(a-\bar{a})\mu=0,$$

or

$$2\lambda b=2ia_I\mu. \tag{1.26}$$

In the non-degenerate points of the spectrum of the periodic and antiperiodic problems we get from (1.26) by virtue of (1.12)

$$|a_I/b|=|\lambda/\mu|=1 \tag{1.27}$$

when $E = E_n$. Hence it follows that

$$|\lambda|^2-|\mu|^2=\det\Lambda=0. \tag{1.28}$$

The roots of the polynomial det Λ in terms of $k^2 = E$ determine therefore the Bloch bands.

For the case of one or two forbidden bands the matrix Λ and the polynomials $R(k^2) = \det\Lambda$ can easily be evaluated and have the form

$$\begin{gathered}\Lambda=\begin{pmatrix}\lambda & \mu\\ \bar{\mu} & \bar{\lambda}\end{pmatrix},\quad u'''=6uu'+cu',\\ \lambda=ik^{-1}(-u''/2+u^2-4ik^4),\\ \mu=u'+ik^{-1}(u''/2-u^2-2k^2u),\\ 64ER(E)=(8E^2+2cE+d)^2-8aE-d^2,\\ 0={}^1/_2(u')^2-(u^3+{}^1/_2cu^2+du+a).\end{gathered} \tag{1.29}$$

If a = 0, we have

$$u(x)=2\wp(x)-c/6,\quad (\wp')^2=4\wp^3-g_2\wp-g_3,$$
$$g_2={}^1/_{12}c^2-d,\quad g_3={}^1/_{12}cd-(c/6)^3.$$

The function $\mathscr{P}$(x) is a Weierstrass elliptic function; Ince[12] was the first to establish in 1940 that the potential $2\mathscr{P}(x)$ leads to a single forbidden band. In the same paper, devoted to the Lamé equation, it was ineffectively shown that $n(n + 1)\mathscr{P}(x)$ is an n-band potential at integer n (already for n = 2 this class does not exhaust by far all the two-band periodic potentials).

2) n = 2. Let

$$\sum_{i=1}^{5} E_i=0;$$

Eq. (1.20) takes the form

$$\begin{gathered}\frac{\delta I_2}{\delta u(x)}+c_1\frac{\delta I_0}{\delta u(x)}=d_1,\\ \lambda=ik^{-1}\{{}^1/_2u^{\mathrm{IV}}-(4uu''+3(u')^2-3u^3)-2u^2k^2+16k^6\},\\ \mu=-u'''+6uu'-4u'k^2+ik^{-1}\{{}^1/_2u^{\mathrm{IV}}+4uu''\\ +3(u')^2-3u^3+k^2(2uu''-4u^2)-8uk^4\},\\ R(E)=E^5+{}^1/_4c_1E^3-{}^1/_{16}d_1E^2+({}^1/_{32}J_1+{}^1/_4c_1^2)E\\ +J_2/2^8+c_1d_1/2^7,\\ J_1=H(p,q)=p_1p_2-({}^1/_2q_2^2+{}^5/_2q_1^2q_2+{}^5/_8q_1^4)\\ +c_1q_1^2-d_1q_1,\\ J_2=p_1^2-2q_1p_1p_2+2(q_2-c_1)p_2^2+q_1^5+2c_1q_1^3\\ +d_1q_1^2-4q_1q_2^2+4c_1q_1q_2-2d_1q_2;\end{gathered} \tag{1.30}$$

where $p_1 = q_2'$, $p_2 = u'$, $q_1 = u$, $q_2 = -{}^5/_2u^2 + u''$. In particular, we have for the potential $v(x) = 6\mathscr{P}(x) - c/2$ the band edges:

$$-{}^1/_4c,\quad -{}^5/_8c\pm{}^1/_8(c^2-16d)^{1/2},\quad -{}^1/_2c\pm{}^1/_2(c^2-12d)^{1/2}. \tag{1.31}$$

2. FINITE-BAND POTENTIALS AND RIEMANN SURFACES

For the periodic potential u(x) the Bloch eigenfunction $\psi_\pm(x, x_0, E)$, normalized by the conditions (1.8), can be analytically continued with respect to E from the allowed bands and turns out to be a meromorphic function on the Riemann surface of the root

$$\left[\prod_{i=1}^{2n+1}(E-E_i)\right]^{1/2},$$

which branches at the band edges E_i. Inside the allowed bands the values of the same function ψ on different sheets then correspond to a pair of linearly independent functions $\psi_\pm(x, x_0, E)$. One sees easily that the zeroes and poles of ψ can lie on the Riemann surface R only on the forbidden bands or their edges on the surface R. It is clear that $\psi_\pm \sim \exp(\pm ik(x - x_0))$ as $E \to \infty$, $k^2 = E$.

From (1.13) we get the following representation

$$\psi(x,x_0,E)=\left(\frac{\chi_R(x_0,E)}{\chi_R(x,E)}\right)^{1/2}\exp\left\{i\int_{x_0}^{x}\chi_R(x,E)\,dx\right\}. \tag{2.1}$$

Moreover, there is for ψ a representation in the base (1.4′):

$$\psi=c+i\chi(x_0,E)s. \tag{2.2}$$

We get easily for $\chi(x, E)$ an expression in terms of the matrix $\hat{T}$:

$$\begin{gathered}\chi_R(x,E)=\frac{k(1-a_R^2)^{1/2}}{(a_I+b_I)},\\ \chi_R(x,E)=\frac{(1-{}^1/_4(\alpha_{11}+\alpha_{22})^2)^{1/2}}{\alpha_{21}}\end{gathered} \tag{2.3}$$

in both bases (1.4) and (1.4′).

One can easily show that the entire function

$$\tilde{\alpha}_{21}=\alpha_{21}(1-a_R^2)^{-1/2}\left(\prod_{i=1}^{2n+1}(E-E_i)\right)^{1/2},$$

has zeroes only at the edges of the forbidden bands and behaves asymptotically like E^n as $E \to \infty$. From this it follows that

$$\begin{gathered}\tilde{\alpha}_{21}=\prod_{j=1}^{n}(E-\gamma_j(x))=P_n(x,E);\\ \chi_R(x,E)=\frac{R^{1/2}(E)}{P_n(x,E)},\quad \chi_I(x,E)=-\frac{1}{2}\frac{P_n'(x,E)}{P_n(x,E)}.\end{gathered} \tag{2.4}$$

Here

$$R(E)=\prod_{i=1}^{2n+1}(E-E_i).$$

From (2.1) we get the identity

$$\psi\bar{\psi}=\psi_+\psi_-=P_n(x,E)/P_n(x_0,E). \tag{2.5}$$

Moreover, it follows from (2.5) that $\psi(x, x_0, E)$ has up to one pole $\gamma_j(x_0)$ and one zero $\gamma_j(x)$ in each of the forbidden bands or at their edges; more precisely, the function has on the Riemann surface R a pole on only one of the sheets: $(\gamma_j(x_0), \sigma_j)$, where $\sigma_j = \pm$.

From the condition that there be no pole on the other sheet $(\gamma_j(x_0), \sigma_j')$ and from Eqs. (2.2) and (2.4) we find that the quantity

$$\chi(x_0,E)=\left[R^{1/2}(E)-\frac{i}{2}\frac{dP_n(x_0,E)}{dx_0}\right]\Big/P_n(x_0,E)$$

has no pole when $E = \gamma_j(x_0)$ and the sign in front of the radical $R^{1/2}$ is equal to σ_j'. Hence follows the equation

$$\left.\frac{dP_n(x,E)}{dx}\right|_{E=\gamma_j(x)}=2\sigma_j'iR^{1/2}(\gamma_j). \tag{2.6}$$

Solving (2.6) for γ_j' we get

$$\gamma_j'=\pm2iR^{1/2}(\gamma_j)\Big/\prod_{k\neq j}(\gamma_j-\gamma_k). \tag{2.6′}$$

For the two-band case (n = 2) these equations take the form

$$\gamma_1' = \frac{2iR^{1/2}(\gamma_1)}{\gamma_1-\gamma_2}, \quad \gamma_2' = \frac{2iR^{1/2}(\gamma_2)}{\gamma_2-\gamma_1}, \tag{2.6''}$$

and can be integrated by the substitution (α = 1, 2)

$$\begin{gathered}(\gamma_2-\gamma_1)\,d\tau=dx, \quad \tau=\frac{1}{2i}\int_{E_{2\alpha}}^{\bar\gamma_\alpha}\frac{dz}{R^{1/2}(z)},\\ E_1<E_2<E_3<E_4<E_5,\\ x-x_0=\int_0^\tau[\gamma_2(\tau)-\gamma_1(\tau)]\,d\tau,\\ \gamma_1(\tau)=\bar\gamma_1(\tau), \quad \gamma_2(\tau)=\bar\gamma_2(\tau+\tau_0).\end{gathered} \tag{2.7}$$

The parameter x_0 is chosen here such that $\gamma_1(x_0) = E_2$; the functions γ_1 and γ_2 are periodic in τ and possess the properties

$$E_2\leqslant\gamma_1\leqslant E_3, \quad E_4\leqslant\gamma_2\leqslant E_5. \tag{2.8}$$

From the asymptotic behavior of χ_R as $E \to \infty$ (Eq. (1.14)) one can derive relations that express symmetric polynomials in γ_1 and γ_2 in terms of u, u′, u″, ... In particular, we have for n = 2

$$\begin{gathered}u(x)=-2(\gamma_1+\gamma_2)+\sum_{i=1}^{5}E_i,\\ \gamma_1\gamma_2={}^1/_3(3u^2-u'')-A, \quad A=-{}^1/_2\Sigma E_iE_j+{}^3/_8(\Sigma E_i)^2,\\ \gamma_{1,2}=-{}^1/_4[u\pm(-5u^2+2u''+16A)^{1/2}].\end{gathered} \tag{2.8'}$$

Let us explain the geometrical meaning of Eqs. (2.6′) and (2.6″), which are written on the Riemann surface R. The forbidden band number j corresponds to the section $l_j = [E_{2j}, E_{2j+1}]$ in the E-plane. However, on the Riemann surface R this section corresponds to the cycle a_j – a circle consisting of two sections $(l_j, +)$ and $(l_j, -)$, the ends of which are identical (see Fig. 1). The set of points (γ_j, σ_j) lies on the circles a_j and Eq. (2.6′) holds for them. By varying x we get the motion of every point (γ_j, σ_j) along the circle a_j and the signs of σ_j change after passing through the points E_{2j} or E_{2j+1}.

In fact, (2.6′) describes the motion of all "phase points" $(\gamma_1, \sigma_1, \gamma_2, \sigma_2, \ldots, \gamma_n, \sigma_n)$ over an n-dimensional torus. It is convenient for the integration of Eq. (2.6′) for all n ≥ 2 to give a different description of the same torus. We consider differentials on a Riemann surface which have no poles (of first order)

$$\Omega_m=\sum_{k=0}^{n-1}c_{km}\frac{E^k\,dE}{\pm R^{1/2}(E)}; \quad m=1,\ldots,n, \tag{2.9}$$

normalized by the conditions

$$\oint_{a_j}\Omega_m=2\pi i\delta_{jm}. \tag{2.9'}$$

We introduce cycles b_j on the Riemann surface which do not intersect the a_m with m ≠ j, while each b_j intersects a_j in one point, E_{2j} (see Fig. 2). We have the real matrix B_{mj}:

$$B_{mj}=\oint_{b_j}\Omega_m. \tag{2.10}$$

It is known (Riemann) that $B_{mj} = B_{jm}$, that the matrix (B_{mj}) is negative definite, and that it cannot be broken up into blocks (e.g., it cannot be diagonal). At n = 2 this is the complete set of conditions for the matrix B_{mj}.

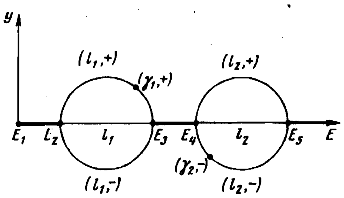

FIG. 1. The functions

$$y=\prod_{i=1}^{2N+1}(E-E_i)^{1/2}$$

where the E_j are the band edges, are on the Riemann surface R determined by the cycles a_j which are situated above the forbidden bands l_j. The poles $\gamma_j,\pm)$ of the Bloch function move along the cycle a_j.

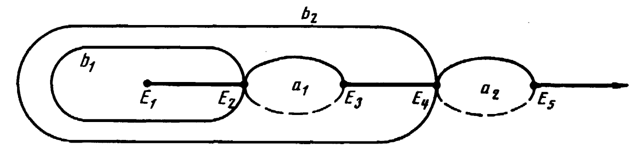

FIG. 2. Basis of closed contours (cycles) on the Riemann surface R (two-band case). The cycles a_j correspond to forbidden bands while the part of the cycle a_j which lies on the lower sheet is shown by a dashed line.

Let $Q_1, \ldots, Q_n$ be a set of points on the Riemann surface R. We consider the complex parameters

$$\eta_m(Q_1,\ldots,Q_n)=\sum_{j=1}^{n}\oint_{E_{2j}}^{Q_j}\Omega_m. \tag{2.11}$$

It is clear that these parameters are not defined uniquely since we have a choice in the path on R connecting the points E_{2j} and Q_j. We can change the path by any integral number of linear combinations of closed contours, the cycles $a_1, \ldots, a_n, b_1, \ldots, b_n$, after which we get

$$\eta_k\sim\eta_k+\sum_{j=1}^{n}m_j\oint_{a_j}\Omega_k+\sum_{j=1}^{n}n_j\oint_{b_j}\Omega_k, \tag{2.12}$$

where m_j and n_j are arbitrary integers.

We have thus a lattice of 2n vectors in the space of the n complex parameters $(\eta_1, \ldots, \eta_n)$ which can be expressed in terms of the basis vectors of n × 2n matrices:

$$(2\pi i\delta_{jk};\quad B_{jk}). \tag{2.13}$$

We arrive thus at a 2n-dimensional torus. The real part of the torus is determined by the matrix B_{jk} and gives us the n-dimensional torus in which we are interested. The lattice (2.13) determines the standard multidimensional Riemann θ-function:

$$\theta(\eta_1,\ldots,\eta_n)=\sum_{m_1,\ldots,m_n}\exp\left\{\frac{1}{2}\sum B_{jk}m_jm_k+\sum m_k\eta_k\right\} \tag{2.13'}$$

(see [13], Vol. I); the substitution (2.11) is reversible and we can write

$$Q_\alpha=Q_\alpha(\eta_1,\ldots,\eta_n); \tag{2.14}$$

we are interested in the substitution (2.11) or (2.14) for the points $Q_j = (\gamma_j, \sigma_j)$ which lie on the cycles a_j on the surface R.

It turns out that the substitution (2.11) or (2.14) integrates Eq. (2.6′) for all n. To be precise, it means that

$$\begin{gathered}\eta_k=\eta_k(\gamma_1(x),\sigma_1,\gamma_2(x),\sigma_2,\ldots),\\ d\eta_k/dx=\text{const}; \quad k=1,\ldots,n.\end{gathered} \tag{2.15}$$

In fact, we can use an idea of Akhiezer [14] to obtain the following expression for η_k':

$$\begin{gathered}d\eta_k/dx=U_k,\\ iU_j=\oint_{b_j}\Omega=-2c_{1m};\end{gathered} \tag{2.16}$$

the (second order) differential $\Omega = (E^n + q_1 E^{n-1} + \ldots + q_n)dE/R^{1/2}(E)$ is here normalized by the conditions

$$\oint_{a_j} \Omega=0, \quad j=1,\ldots n. \tag{2.17}$$

It is well known that one can easily get by using (1.14) the following representation for the potential u(x):

$$u(x)=-2\sum_{j=1} \gamma_j(x)+\sum_{i=1}^{2n\pm1} E_i. \tag{2.18}$$

If $Q_j = (\gamma_j, \sigma_j)$ are points on the Riemann surface, we can write γ_j as a numerical function of the parameters $\eta_1, \ldots, \eta_n$, by virtue of (2.11) and (2.14):

$$\gamma_j=\varkappa_j(\eta_1, \ldots, \eta_n). \tag{2.19}$$

Using (2.18) we get

$$u(x)=-2\sum \varkappa_j+\text{const}=-2\varkappa(\eta_1,\ldots,\eta_n)+\text{const}, \quad \eta_j=xU_j+\eta_j^0 \tag{2.20}$$

by virtue of (2.16).

It is well known that the function $\kappa(\eta_1, \ldots, \eta_n)$ is a standard algebraic function on the 2n-dimensional torus, given by the lattice (2.13).[4] We can follow Its and Matveev[5] and take for the function κ from the literature a convenient expression for calculations in terms of the Riemann θ-function (see[13]). It then follows for the potential u that

$$u(x)=-2\frac{d^2}{dx^2}\ln\theta(xU_1+\eta_1^0,\ldots,xU_n+\eta_n^0)+\text{const}, \tag{2.21}$$

$$\eta_j^0=-x_0U_j+\sum_{k=1}^{n}\int_{E_{2k}}^{\gamma_k(x_0)}\Omega_j-K_j,$$

$$K_j=\frac{1}{2}\sum_{k=1}^{n}B_{kj}-\pi ij.$$

It follows from Eqs. (2.20) and (2.21) that, generally speaking, the potential u(x) is quasi-periodic with periods $(T_1, \ldots, T_n)$, where

$$T_j^{-1}=\sum_{k=1}^{n} B^{jk}U_k, \tag{2.22}$$

where the matrix B^{jk} is the inverse of the matrix B_{kj} of the periods and, if we continue into the complex region, with periods $(T_1', \ldots, T_n')$ where

$$T_j'=2\pi i/U_j. \tag{2.22'}$$

The $n-1$ relations

$$\sum n_jT_j=0, \tag{2.23}$$

with n_j an integer, are necessary and sufficient for the periodicity of u(x). If, moreover, the $n-1$ relations for the imaginary periods,

$$\sum m_jT_j'=0, \tag{2.23'}$$

are satisfied, we can express the potential in terms of elliptic functions. For the two-band case, n = 2, the compatibility condition for having both Eqs. (2.23) and (2.23′) for five parameters $(E_1, \ldots, E_5)$ gives us an enumerable set of three parameter families. One of them (Ince's case) has already been indicated at the end of Sec. 1 (see (1.30), (1.31), and (2.8′)). To be more precise, we can obtain from the Lamé potential $6\mathscr{P}(x)$ + constant other potentials of this family by changing the time, by virtue of the KdV equation (see Sec. 3), and they will have the same spectrum (correspond to the same Riemann surface R, satisfying conditions (2.23) and (2.23′)).

If we use Eqs. (2.8′) and (1.31) for the potential $u(x) = 6\mathscr{P}(x)$ we get the spectrum explicitly (surface R) and also the form of $\gamma_1(x)$ and $\gamma_2(x)$:

$$R(E)=E^5-\frac{21}{4}g_2E^3-\frac{27}{4}g_3E^2+\frac{27}{4}g_2^2E-\frac{81}{4}g_2g_3, \tag{2.24}$$

$$E_1=3e_1, \quad E_2=-(3g_2)^{1/2}, \quad E_3=3e_2, \quad E_4=(3g_2)^{1/2}, \quad E_5=3e_3,$$

$$4e_i^3-g_2e_i-g_3=0, \quad i=1,2,3;$$

$$\gamma_{1,2}(x)=-\frac{3}{2}[\wp(x)\pm(g_2-3\wp^2(x))^{1/2}].$$

It is, finally, relevant to note the general uniformity for the Bloch dispersion law p(E):

$$p(E)+n\pi=\int_{x_0}^{x_0+T}\chi_R(x,E)\,dx,$$

$$\frac{dp}{dE}=\int_{x_0}^{x_0+T}\frac{dx}{2\chi_R(x,E)}, \quad \frac{\delta p}{\delta u(x)}=-\frac{1}{2\chi_R(x,E)}. \tag{2.25}$$

From the last equation, together with the form of the function χ_R (see (2.3)) we easily get the statement which is the inverse of the result of Sec. 1: any finite-band potential satisfies one of the higher KdV equations (1.16). We note also that in the case of a potential which is periodic with period T it follows from the second Eq. (2.25) that the differential $T^{-1}dp$ is the same as the differential Ω occurring in Eqs. (2.16) and (2.17).

3. TIME-DEPENDENCE OF FINITE-BAND POTENTIALS BY VIRTUE OF THE KdV EQUATION

We consider the "finite-band" solutions u(x, t) of the KdV equation which at any time t give a finite-band potential for the Schrödinger operator. If the finite-band potential $u_T(x)$ is periodic with period T with $T \to \infty$ and if $u_\infty(x)$ decreases rapidly, the potential $u_\infty(x)$ is nonreflective. The family of finite-band solutions of the KdV equation thus contains as a degenerate limiting case the many-soliton solutions. In that case the Riemann surface R of the root

$$\left[\prod_{i=1}^{2n\pm1}(E-E_i)\right]^{1/2}$$

is degenerate as for $T \to \infty$ the band edges converge pairwise to one another, and in the limit the root can be taken. The parameters $(\eta_1, \ldots, \eta_n)$, given by Eq. (2.11) have no meaning at all when $T = \infty$.

We now study the time-dependence of the potential u(x) by virtue of the KdV equation. Firstly, the band edges are integrals of the system. One can show that the derivatives $\dot{\eta}_k$ are constants, by virtue of any of Eqs. (1.20). One can easily evaluate these constants. We denote them by $\dot{\eta}_k = W_k$ for the original KdV equation. We then get from Eqs. (2.20):

$$u(x,t)=-2\varkappa(xU_1+tW_1+\eta_1^0,\ldots,xU_n+tW_n+\eta_n^0)+\text{const}. \tag{3.1}$$

It is, however, convenient to evaluate the time-dependence for the functions γ_j (or the points (γ_j, σ_j) on the cycles a_j). We get from Eq. (1.23) for the matrix $\hat{T}$ in the basis (1.4):

$$a_I+b_I=-2\mu_R(a_I+b_I)+2b_R(\lambda_I+\mu_I). \tag{3.2}$$

Moreover, we find from Eqs. (1.7) and (1.24) the general relations

$$2\mu_R=-\frac{d}{dx_0}\left(\frac{\lambda_I+\mu_I}{k}\right), \quad 2b_R=-\frac{d}{dx_0}\left(\frac{a_I+b_I}{k}\right). \tag{3.3}$$

From (3.3) and (3.2) we get, together with (2.3):

$$\dot{\chi}_R=(\Lambda\chi_R)', \quad \Lambda=(\lambda_I+\mu_I)/k,$$
$$\dot{P}_N=\Lambda' P_N-\Lambda P_N'. \tag{3.4}$$

Moreover, for $E = \gamma_j(x)$ after using (2.6) it follows from (3.4) by analogy with (2.6′) that

$$\dot{\gamma}_j=-4i\sigma_j'\Lambda|_{E=\gamma_j}R^{1/2}(\gamma_j)\Big/\prod_{k\neq j}(\gamma_j-\gamma_k). \tag{3.5}$$

In the case of the KdV equation we have $\Lambda = -2(u + 2E)$, i.e.,

$$\dot{\gamma}_j=\pm 8i\Big(\sum_{k\neq j}\gamma_k-\frac{1}{2}\sum E_k\Big)R^{1/2}(\gamma_j)\Big/\prod_{k\neq j}(\gamma_j-\gamma_k). \tag{3.5'}$$

Through the substitution (2.11) and (2.14) we can integrate Eqs. (3.5) and (3.5′), and the derivatives $\dot{\eta}_k = W_k$ can easily be expressed in terms of the periods of a few differentials on the Riemann surface R with poles at infinity.

For the case of two forbidden bands we get, starting from Eq. (2.7), for the parameters (x_0, τ_0):

$$\dot{\tau}_0=4, \quad \dot{x}_0=4\Big(\bar{\gamma}_1(\tau_0)-\frac{1}{2}\sum E_i\Big),$$
$$u(x, t)=-2(\bar{\gamma}_1(\tau(x-x_0(t)))+\bar{\gamma}_2(\tau(x-x_0(t)+\tau_0(t)))+\text{const}. \tag{3.6}$$

Together with Eq. (2.8) this gives the final form of u(x, t) in the two-band case. In the particular case $u(x, 0) = 6\wp(x)$ we get from Eqs. (2.8′) and (1.30), (1.31):

$$u(x, t)=2\wp(x-\beta_1(t))+2\wp(x-\beta_2(t))+2\wp(x-\beta_3(t)), \tag{3.7}$$
$$\beta_1+\beta_2+\beta_3=0, \quad \int_0^{\beta_1-\beta_2}\frac{dz}{12(g_2-3\wp^2(z))^{1/2}}=t,$$
$$\beta_2-\beta_3=\frac{1}{2}\wp^{-1}[-\wp(\beta_1-\beta_3)+(g_2-3\wp^2(\beta_1-\beta_3))^{1/2}].$$

In conclusion we note that the formulae given here can be improved upon in a number of cases but, in principle, they describe the whole dynamics of the finite-band solutions. The parameters η_k on the torus (determined apart from the lattice periods (2.13)) give "angle variables" which are canonically conjugate to the "action" variables formed from the eigenvalues of the Schrödinger operator by analogy of the work of Zakharov and Faddeev.[15] It is relevant to draw attention to the complexity of the angle variables in the periodic case as compared to the fast decreasing case.

[1] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).

[2] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **60**, 993 (1971) [Sov. Phys.-JETP **33**, 538 (1971)].

[3] S. P. Novikov, Funkts. Anal. **8**, No. 3 (1974) [Funct. Anal. Appl. **8**, No. 3 (1975)].

[4] B. A. Dubrovin, Funkts. Anal. **8**, No. 4 (1974) [Funct. Anal. Appl. **8**, No. 4 (1975)].

[5] A. R. Its and V. B. Matveev, Funkts. Anal. **8**, No. 4 (1974) [Funct. Anal. Appl. **8**, No. 4 (1975)].

[6] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys.-JETP **34**, 62 (1972)].

[7] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **65**, 219 (1973) [Sov. Phys.-JETP **38**, 108 (1974)].

[8] R. Hirota, J. Phys. Soc. Japan **33**, 1456 (1972).

[9] V. E. Zakharov and A. B. Shabat, Funkts. Anal. **8**, No. 3 (1974) [Funct. Anal. Appl. **8**, No. 3 (1975)].

[10] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).

[11] R. M. Miura, C. S. Gardner, and M. D. Kruskal, J. Math. Phys. **9**, 1202 (1968).

[12] E. L. Ince, Proc. Roy. Soc. Edinb. **60**, 83 (1940).

[13] A. Kratzer, Lehrbuch der Thetafunktionen, Teubner, Leipzig, 1903.

[14] N. I. Akhiezer, Dokl. Akad. Nauk SSSR **141**, 263 (1961).

[15] V. E. Zakharov and L. D. Faddeev, Funkts. Anal. **5**, 18 (1971) [Funct. Anal. Appl. **5**, 280 (1971)].

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