A PERIODICITY PROBLEM FOR THE KORTEWEG–DE VRIES AND STURM–LIOUVILLE EQUATIONS. THEIR CONNECTION WITH ALGEBRAIC GEOMETRY

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1. As was shown in the remarkable communication [4] the Cauchy problem for the Korteweg–de Vries (KdV) equation $u_t = 6uu_x - u_{xxx}$, familiar in theory of nonlinear waves, is closely linked with a study of the spectral properties of the Sturm–Liouville operator $L\psi = E\psi$, where $L = -d^2/dx^2 + u(x)$. For rapidly decreasing initial conditions $u(x, 0)$, where $\int_{-\infty}^{\infty} u(x, 0)(1 + |x|)dx < \infty$, the KdV equation can in a sense be completely solved by going over to scattering data for the opreator L; these data determine the potential u, as was shown in [3], [8], [10]. It was later pointed out in [7] that this procedure actually provides the basis for representing the operator of multiplication by the right side of the KdV equation in terms of the commutator $[A, L] = (6uu_x - u_{xxx})$, where $A = -4d^3/dx^3 + 3(u\,d/dx + (d/dx)u)$. It follows from this that the KdV equation is equivalent to the equation $\dot{L} = [A, L]$.

In the case of functions $u(x, t)$ periodic in x, and even more in the case of conditionally periodic functions, it has not proved possible to make any serious use of the connection between the operator L and the KdV equation. Recently, in [2], [9], the present authors made substantial progress in this problem, and discovered deep links with algebraic geometry. It should be mentioned that a substantial part of Dubrovin's results [2] was obtained simultaneously and independently by Matveev and Its [6], Both the articles [2], [6] employed an idea of N. I. Ahiezer [l], the full significance of which has only recently been appreciated.

2. Finite-zone potnetials and higher analogs of the KdV equation (see [9]). For a periodic potential $u(x)$, the eigenfunctions of the operator L are determined as usual on the entire axis by the conditions:

(1) $L\psi(x, x_0, E) = E\psi(x, x_0, E)$;
(2) $\psi(x, x_0, E) = 1$, $x = x_0$;
(3) $\psi_\pm(x + T, x_0, E) = e^{\pm ip(E)}\psi(x, x_0, E)$,

where $\psi_- = \bar{\psi}_+$, T is the period, and $p(E)$ is a real function which is not defined for all E. The domains of defintion of $p(E)$ are called allowed zones, and their complements forbidden zones, or lacunae, of which there are as a rule infinitely many (their lengths decrease as $E \to \infty$).

a) We call u a *finite-zone potential* if there are only a finite number of lacunae in the corresponding spectrum of the operator L (e.g., $u = \text{const}$).

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b) A *higher order KdV equation* is defined as an equation $\dot u = Q(u, u', \dots, u^{(N)})$ for which there exists an operator $A = d^N/dx^N + \sum_{i=1}^{N} P_i d^{N-i}/dx^{N-i}$ such that the commutator $[A, L]$ is the operator of multiplication by the right side $Q(u, u', \dots, u^{(N)})$. It is assumed here that Q and all the P_i are polynomials in the function u and its derivatives with respect to x, with constant (real) coefficients.

The higher order KdV equations are described as follows in [7]: if $\chi(x, E) = -id\ln\psi/dx$, then as $E \to \infty$ we have the asymptotic expansion

$$\chi \sim k + \sum_{n\geq 1} \frac{\chi_n(x)}{(2k)^n}, \quad -i\chi' + \chi^2 + u = E, \quad k^2 = E;$$

all the $\chi_n(x)$ are polynomials in $u, u', u'', \dots$, while the integrals

$$I_m = \int_{x_0}^{x_0+T} \chi_{2m+1}(x)\, dx$$

are such that $\dot I_m = 0$ by virtue of the initial and all the higher order KdV equations. All the higher order KdV equations have the form

$$\dot u = \frac{d}{dx} \frac{\delta}{\delta u(x)} \left(\sum_{q=1}^{m+2} c_q I_q \right).$$

The main theorem of [9] states that all the steady state periodic solutions $u(x)$ of any higher order KdV equation are finite-zone potentials. In addition, all the equations

$$\frac{\delta}{\delta u(x)} \left(\sum c_q I_q \right) = \text{const}$$

prove to be completely integrable Hamiltonian systems. The method of [9] gives an algorithm for evaluating the entire set of commuting integrals $J_1, \dots, J_m$, which are polynomials in $u, u', \dots, u^{(2m-1)}$, where m is the number of lacunae. The set of potentials obtained is invariant under all higher order KdV equations and fills an $(m+1)$-dimensional family of Euclidean spaces R^{2m} (or space R^{3m+1}), dependent on the set of constants $\{c_q\}$. In each $R^{2m}_{\{c_q\}}$ a commutative group R^m acts, the orbits of which are distinguished by the set of integrals (polynomials) $J_1, \dots, J_m$. If an orbit is compact (a torus) and the numbers of revolutions are commensurable, winding of this torus yields a periodic m-zone potential. In general, meromorphic periodic potentials $u(x)$ are obtained with a group of (real and imaginary) periods $T_1, \dots, T_m, T_1', \dots, T_m'$. The entire construction admits obvious complexification, and the complex orbits of the group C^m are Abelian varieties.

3. Finite-zone potentials and algebraic geometry (see [2]). It turns out that, for a periodic potential, an eigenfunction $\psi(x, x_0, E)$ can be continuous analytically with respect to E to a meromorphic function on a Riemann surface R (except for $E = \infty$) having the form $y^2 = \prod_{j=1}^{2m+1}(E - E_j)$, where the E_j are the zone boundaries. In the conditionally periodic case, this is a condition imposed on the investigated class of potentials. If $\chi(x, E) = -id\ln\psi/dx$, then $\chi = \chi_R + i\chi_I$, where $\chi_I = +\frac{1}{2} d(\ln\chi_R)'/dx$.

Hence it follows that

$$\psi(x, x_0, E) = \sqrt{\frac{\chi_R(x_0, E)}{\chi_R(x, E)}} \exp\left\{ \int_{x_0}^{x} \chi_R\, dx \right\}.$$

We can prove the important formula

$$\chi_R(x,E) = \sqrt{R(E)} / \prod_{j=1} (E - \gamma_j(x)),$$

where $R(E) = \prod_{j=1}^{2m+1}(E - E_i)$, and the $\gamma_j(x)$ each lie in a forbidden zone or on the boundary of such a zone. Let $C(x, x_0, E)$ and $S(x, x_0, E)$ be a basis of eigenfunctions, normalized for $x = x_0$ by the conditions

$$C = 1, \quad C' = 0, \quad S = 0, \quad S' = 1.$$

It is clear from the general expression $\psi = C + i\chi(x_0, E)S$ that the poles of ψ lie above the points $\gamma_j(x_0)$. However, a pole of ψ lies on just one of the sheets. The condition for its contradiction to another position is given by the equation

$$\left[\prod_{j=1}^{m}(E - \gamma_j(x))\right]'_{E=\gamma_j(x)} = 2i\sqrt{R(\gamma_j)}$$

or

$$\gamma_j' = 2i\sqrt{R(\gamma_j)} \ / \prod_{j\neq k}(\gamma_j - \gamma_k).$$

Comparison with the results of [9] enables us to find the time dependence $\dot\gamma_j$ in the light of the higher order KdV equations. In particular, for the initial KdV equation we obtain [2]

$$\dot\gamma_j = 8i\left(\sum_{k\neq j}\gamma_k - c\right)\sqrt{R(\gamma_j)} / \prod_{j\neq k}(\gamma_j - \gamma_k),$$

where $c = \frac{1}{2}\sum E_i$. Here, $\gamma_1, \ldots, \gamma_m$ is a set of points (divisors) on R, and $u(x) = -2\sum\gamma_j(x) + \sum E_i$.

These equations may be integrated by an Abelian mapping:

$$\xi_k = \sum_{j=1}^{m}\int_Q^{\gamma_j(x)}\omega_k, \quad \text{where } \omega_k = \frac{E^k\,dE}{\sqrt{R(E)}}, \quad k = 0, \ldots, m-1,$$

is a basis of holomorphic differentials on R. By virtue of the higher order KdV equations, all the derivatives $\dot\xi$ are constant. The parameters ξ_k are defined up to the periods of the forms ω_k along the cycles and define a torus $J(R)$, i.e., a "Jacobian", the rectilinear structure on which is specified by the higher order KdV equations. The set of potentials with given zone-boundaries is itself isomorphic (after complexification) to this Jacobi variety (complex torus).

It can easily be seen that $p(E) = \int \chi_R\,dx$, where $\psi(x+T) = e^{ip(E)}\psi(x)$. The general equation $\delta p/\delta u(x) = 1/(2\chi_R)$ holds. From the form of the function χ_R and this identity, we obtain the converse of the theorem of §2: any finite-zone potential is a steady state solution of one of the higher order KdV equations. The results of §§2 and 3 make it possible for us to integrate completely a KdV equation with "finite-zone" initial conditions $u(x, 0)$ (whether periodic or conditionally periodic). It was shown in [9] that these solutions represent an analog of the multiple soliton solutions of the KdV equation. While the results of §3 can easily be extended to the case of an infinite number of zones, they cease to be analytically effective as applied to theory of the KdV equation. It seems possible that any smooth periodic potential may be approximated by a finite-zone potential. This question deserves further study.

In the next section it will be shown that a combination of the results of §§2 and 3 enables certain nontrivial facts to be proved in theory of Abelian varieties.

4. The complete variety of moduli of hyperelliptic Jacobians $J(R)$. Over the variety of moduli of hyperelliptic curves V over the complex number field there exists a natural fibering $M \mapsto V$, each fiber of which is the Jacobi variety of the corresponding curve. For genus 2 it is the same as the complete variety of moduli of all (up to isogeny) two-dimensional Abelian varieties (apart from cartesian products of one-dimensional varieties). To see what the variety M is, consider a $(2g+2)$-sheeted covering $\tilde{M}$ over this variety of moduli, connected with the fixing of one of the $2g+2$ branch points, which may be assumed at infinity. This implies the isolation of a second order point on the Jacobian J.

Theorem. *The complete variety $\tilde{M}$ of moduli of hyperelliptic Jacobians $J(R)$ with distinguished point of second order is rational.*

The proof is obtained from a combination of the results of §§2 and 3. In fact, the set of all conditionally periodic finite-zone potentials with given zone-boundaries E_i (assume that $\sum_{i=1}^{2n+1} E_i = 0$) is, after complexification, the Jacobian $J(R)$; see §3. On the other hand, the same varieties, in accordance with §2, fiber the space C^{3n} (if we exclude the points at infinity). In fact, we have an n-dimensional family of fiberings of the spaces $C^{2n}_{\{c_q\}}$ by means of the set of n polynomials $J_1, \ldots, J_n$, which depend parametrically on the remaining coordinates $\{c_q\}$ in C^{3n}. For $n = 2$, these polynomials have the form (see [9])

$$J_1 = p_1 p_2 - \frac{1}{2}\left(q_2^2 + 5q_1^2 q_2 + \frac{5}{4}q_1^4\right) + 8c_2 q_1^2 - c_1 q_1,$$

$$J_2 = p_1^2 - 2q_1 p_1 p_2 + (2q_2 - 16c_2)p_2^2 + q_2^5 + 16c_2 q_1^3 + c_1 q_1^2 + 32c_2 q_1 q_2 - 2c_1 q_2,$$

while the Riemann surface R is given by the equation

$$y^2 = E^5 + 2c_2 E^3 - \frac{c_1}{16}E^2 + \frac{J_1 + 32c_2^2}{32}E + \frac{J_2 + 16c_1 c_2}{16^2},$$

where p_1, p_2, q_1, q_2 are coordinates in C^4, and c_1, c_2 are superfluous coordinates in C^6.

The group C^2 operators (locally) be means of the pair of Hamiltonian systems

$$\dot{p}_j = \frac{\partial J_\alpha}{\partial q_j}, \quad \dot{q}_j = \frac{\partial J_\alpha}{\partial p_j}, \quad \alpha = 1, 2, \ j = 1, 2.$$

Isomorphic curves will be obtained if we multiply the entire set (E_i) be a number which does not destroy rationality. After this, there will remain altogether $2g+2$ isomorphic curves, linked with the isolation of one of the branch points (at infinity).

In conclusion, note that the hyperellipticity is bound up with the order of the operator L. Operators of higher order are now also known (see [5]) to which the same methods are applicable.

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