Invariant manifolds for a singular ordinary differential equation

Stefano Bianchini
SISSA, via Bonomea 265, 34136, Trieste, Italy
email: bianchin@sissa.it
and
Laura V. Spinolo *
Ennio De Giorgi Center, Scuola Normale Superiore, Pisa, Italy and Institute of Mathemathics, University of Zurich, Switzerland
email: laura.spinolo@math.uzh.ch

Abstract

We study the singular ordinary differential equation

$$\frac{dU}{dt} = \frac{1}{\zeta(U)}\phi_s(U) + \phi_{ns}(U), \tag{0.1}$$

where $U \in \mathbb{R}^N$, the functions $\phi_s \in \mathbb{R}^N$ and $\phi_{ns} \in \mathbb{R}^N$ are of class $\mathcal{C}^2$ and ζ is a real valued $\mathcal{C}^2$ function. The equation is singular because $\zeta(U)$ can attain the value 0. We focus on the solutions of (0.1) that belong to a small neighbourhood of a point $\bar U$ such that $\phi_s(\bar U) = \phi_{ns}(\bar U) = \vec 0$ and $\zeta(\bar U) = 0$. We investigate the existence of manifolds that are locally invariant for (0.1) and that contain orbits with a prescribed asymptotic behaviour. Under suitable hypotheses on the set $\{U : \zeta(U) = 0\}$, we extend to the case of the singular ODE (0.1) the definitions of center manifold, center-stable manifold and of uniformly stable manifold. We prove that the solutions of (0.1) lying on each of these manifolds are regular: this is not trivial since we provide examples showing that, in general, a solution of (0.1) is not continuously differentiable. Finally, we show a decomposition result for a center-stable manifold and for the uniformly stable manifold.

An application of our analysis concerns the study of the viscous profiles with small total variation for a class of mixed hyperbolic-parabolic systems in one space variable. Such a class includes the compressible Navier Stokes equation.

Key words: singular ordinary differential equation, stable manifold, center manifold, invariant manifold.

1 Introduction

In this work we study the singular ordinary differential equation

$$\frac{dU}{dt} = \frac{1}{\zeta(U)}\phi_s(U) + \phi_{ns}(U). \tag{1.1}$$

In the previous expression, $U \in \mathbb{R}^N$ and the functions ϕ_s and ϕ_{ns} are $\mathcal{C}^2$ (continuously differentiable with continuously differentiable derivatives) and take values in $\mathbb{R}^N$. The function ζ is as well regular and it takes real values. We say that the equation is singular because $\zeta(U)$ can attain the value 0.

Equation (1.1) is related to a class of problems studied in singular perturbation theory. Consider system

$$\begin{cases} \varepsilon d\mathbf{x}/dt = f(\mathbf{x}, \mathbf{y}, \varepsilon) \\ d\mathbf{y}/dt = g(\mathbf{x}, \mathbf{y}, \varepsilon), \end{cases} \tag{1.2}$$

*Corresponding author. Phone number: +41-044-6355865. Fax number: +41-(0)44-63 55706

where $\mathbf{x}$ and $\mathbf{y}$ are vector valued functions and ε is a parameter. In singular perturbation theory one is typically concerned with the limit $\varepsilon \to 0$ and with the corresponding behaviour of the solution $(\mathbf{x}, \mathbf{y})$. Note that (1.1) can be viewed as as an extension of (1.2) because (1.2) can be written in the form (1.1): in this case, the singularity $\zeta(U)$ in (1.1) is identically equal to ε and hence $d\zeta/dt = 0$.

Being the literature concerning (1.2) extremely wide, it would be difficult to give an overview here. Consequently, we just refer to the notes by Jones [11] and to the rich bibliography contained therein. In particular, [11] provides a nice overview of Fenichel's papers [8, 9, 10]. These works provide several ideas and techniques used in the present paper: in particular, in the following we introduce the notions of *slow dynamic* and *fast dynamic*, which can be viewed as extensions of the notions of *fast* and *slow time scale* discussed in Fenichel's works. We refer to Remark 1.1 at the end of Section 1.1 for further comments on the analogies between the present analysis and Fenichel's.

The main novelty of the present work is that we consider the case when ζ is a nontrivial function of the unknown U. In particular, this means that in general $d\zeta/dt \neq 0$ and hence that we have to face the possibility that $\zeta\big(U(0)\big) \neq 0$, but $\zeta\big(U(t)\big) = 0$ for a finite value of t. This is exactly what happens in the examples (2.12) and (2.17) discussed in Section 2 here. Other examples are provided in a previous work by the same authors [6], Section 2. Note that, in all these cases, there is a loss of regularity at the time t_0 at which $\zeta\big((U(t)\big)$ reaches the value 0, $t_0 = \min\big\{t \in [0, +\infty[: \ \zeta\big(U(t)\big) = 0\big\}$. More precisely, the first derivative dU/dt either has a discontinuity or blows up at $t = t_0$.

Our goal here is to study the solutions of (1.1) that lie in a neighborhood of a point $\bar U$ such that

$$\phi_s(\bar U) = \phi_{ns}(\bar U) = \vec 0, \quad \zeta(\bar U) = 0. \tag{1.3}$$

We are concerned with the existence of invariant manifolds. More precisely, the problem is the following.

Consider first the non singular ODE

$$\frac{dU}{dt} = f(U) \tag{1.4}$$

and assume that the point $\bar U$ is an equilibrium, namely $f(\bar U) = \vec 0$. In a neighbourhood of $\bar U$ one can define a *center* and a *center-stable manifold*, which are both locally invariant for (1.4). We recall that, loosely speaking, a center-stable manifold contains the orbits of (1.4) that either do not blow up or blow up more slowly than $e^{\eta t}$ when $t \to +\infty$. Here η is a small enough constant depending on the system. More precisely, the orbits that lie on a center-stable manifold are those having the asymptotic behaviour described before and solving a suitable system which, in a small neighbourhood of $\bar U$, coincides with (1.4).

We are also interested in the *uniformly stable* manifold relative to E, which is defined as follows. Assume there exists a manifold E containing $\bar U$ and entirely constituted by equilibria of (1.4). By uniformly stable manifold we mean the slaving manifold that contains all the orbits that decay with exponential speed to some point in E when $t \to +\infty$. Note that the uniformly stable manifold does not coincide, in general, with the classical *stable* manifold. Indeed, the stable manifold contains the orbits that decay exponentially fast to the given equilibrium $\bar U$, while on the uniformly stable manifold we only require that the limit belongs to E. The existence of a center-stable and of the uniformly stable manifold can be obtained as consequence of the Hadamard-Perron Theorem discussed in the book by Katok and Hasselblatt [13, Chapter 6, page 242].

In the present paper we prove that, under suitable hypotheses, one can extend the definitions of center, center-stable and of uniformly stable manifold to the case of the singular ODE (1.1). The manifolds we define are all locally invariant for (1.1) and satisfy the following property:

(P) If U is an orbit lying on the manifold and $\zeta\big(U(0)\big) \neq 0$, then $\zeta\big(U(t)\big) \neq 0$ for every t.

This, in particular, rules out the losses of regularity (blow up or discontinuity in the first derivative) mentioned before.

We proceed as follows. First, we consider the non singular ODE

$$\frac{dU}{d\tau} = \phi_s(U) + \zeta(U)\phi_{ns}(U). \tag{1.5}$$

Due to (1.3), the value $\bar{U}$ is an equilibrium for (1.5). Also, equation (1.5) is formally obtained from (1.1) *via* the change of variable $\tau = \tau(t)$, defined as the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\tau}{dt} = \dfrac{1}{\zeta[U(t)]} \\ \\ \tau(0) = 0. \end{cases} \tag{1.6}$$

However, the function $\tau(t)$ is well defined only if $\zeta[U(t)] \neq 0$ for every t. In the present work we always refer to the formulation (1.5) and we prove the existence of locally invariant manifolds satisfying property **(P)**. We then show that *a posteriori* the change of variable (1.6) is well defined and that system (1.5) is equivalent to (1.1) on these manifolds.

We assume that

1. the set $\{U : \zeta(U) = 0\}$ is an hypersurface in $\mathbb{R}^N$ and the intersection between $\{U : \zeta(U) = 0\}$ and $\mathcal{M}^c$ contains only equilibria. Here $\mathcal{M}^c$ is any center manifold of the equilibrium $\bar{U}$ of the system (1.5).

We then define a *manifold of slow dynamics* as a center manifold of the equilibrium point $\bar{U}$ of (1.5) (any center manifold works). To simplify the exposition, in the following we *fix* a manifold of slow dynamics. To define the *manifold of fast dynamics* we assume

2. there exists a one-dimensional manifold which is transversal to the hypersurface $\{U : \zeta(U) = 0\}$ and is entirely constituted by equilibria of (1.5). In the following, we denote by E this manifold: note that, by construction, $E \subseteq \mathcal{M}^c$.

As a remark, we point out that we are *not* assuming that E contains *all* the equilibrium points of (4.1). For example, in the case we discuss in Section 2.1.1 the set of equilibria is a three-dimensional manifold: assumption 2 is anyhow satisfied because such a three-dimensional manifold contains a one-dimensional submanifold transversal to $\{U : \zeta(U) = 0\}$.

The *manifold of fast dynamics* is then defined as the uniformly stable manifold of (1.1) relative to the manifold E. Namely, all the fast dynamics converge exponentially fast to some equilibrium in E.

We also assume that

3. the singular hypersurface $\{U : \zeta(U) = 0\}$ is invariant for (1.5).

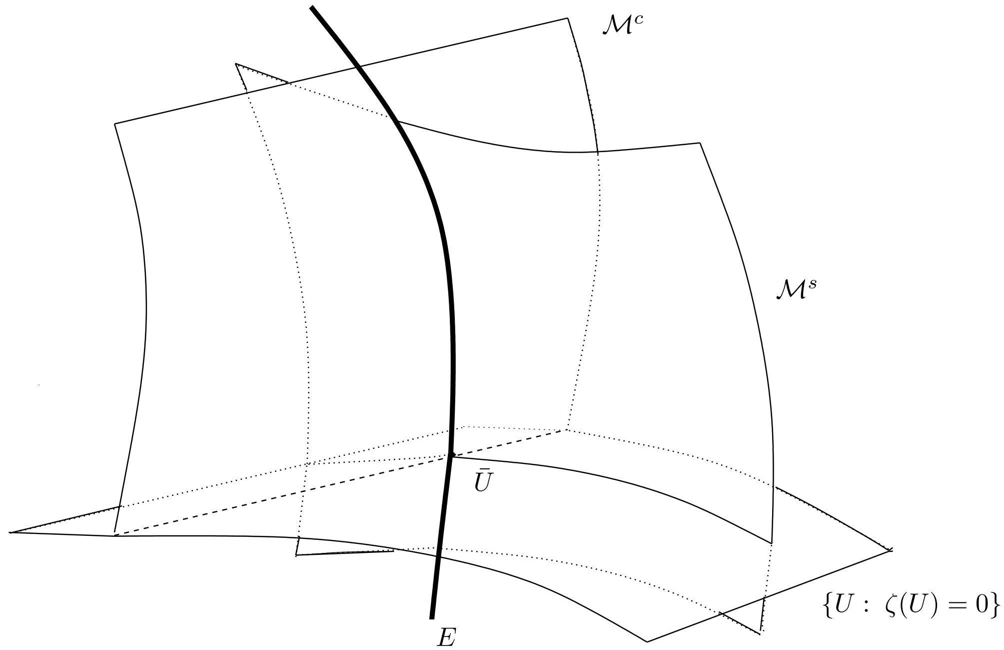
As a consequence of the above assumptions, it turns out that equation (1.1) restricted on the manifold of the slow dynamics is nonsingular and hence it can be extended to the hypersurface $\{U : \zeta(U) = 0\}$. We then assume that

4. the singular hypersurface $\{U : \zeta(U) = 0\}$ is invariant for the solutions of (1.1) that lie on the manifold of the slow dynamics.

We can now define a center manifold of (1.1) as a center manifold of the equilibrium $\bar{U}$ of the system reduced on the manifold of slow dynamics, see Theorem 4.1 (it turns out that $\bar{U}$ is indeed an equilibrium point for that system). One can show that property **(P)** is satisfied on any center manifold and that the losses of regularity are ruled out. Here by *loss of regularity* we mean the blows up or discontinuities in the first derivative that were mentioned before and that may be exhibited by the solutions of (1.1) as shown by examples (2.12) and (2.17) in Section 2.

To extend to the case of the singular ODE (1.1) the definition of center-stable and uniformly stable manifold we need some more work. As mentioned before, due to assumption 2, there exists a manifold of equilibria transversal to the singular hypersurface: we denoted this manifold by E. To define the uniformly stable manifold of (1.1) relative to E we need to study the solutions of (1.1) which converges to a point in E with exponential speed. Note that this speed can be either bounded or unbounded as $\zeta \to 0$, so we are looking for a composition of both fast and slow dynamics. Roughly speaking, to define a center-stable manifold we have to study orbits that are local solutions of (1.1) and that do not blow too fast when $t \to +\infty$. Therefore, we have to deal again with a composition of slow and fast dynamics.

Figure 1: An illustration of Assumptions 1,. . . , 4 and of the statement of Theorem 1.1: the bold line represent the manifold E, which is transversal to the singular hypersurface $\{U : \zeta(U) = 0\}$ and is entirely made by equilibria: note that $E \subseteq \mathcal{M}^c$. The dashed line represents the intersection between the center manifold $\mathcal{M}^c$ and the singular hypersurface. Such an intersection is entirely made by equilibria by Assumption 1. Finally, $\mathcal{M}^s$ is the uniformly stable manifold, containing orbits that converge exponentially fast to a point in E.



In both cases (uniformly stable and center-stable manifold) the analysis can be seen as an extension of the exponential splitting methods for non singular ODEs like (1.4). However, as mentioned before what *a priori* can go wrong is that in the change of time scale defined by the Cauchy problem (1.6) some regularity is missing. The main result of this paper is the following (a more precise statement is given in Theorem 4.2):

Theorem 1.1. *There is a sufficiently small constant $\delta > 0$ such that the following holds. In the ball of center $\bar{U}$ and of radius δ in $\mathbb{R}^N$ one can define two continuously differentiable manifolds $\mathcal{M}^s$ and $\mathcal{M}^{cs}$ which are both locally invariant for* (1.1) *and enjoy the following properties.*

- *$\mathcal{M}^s$, is the uniformly stable manifold of* (1.1) *relative to E, while $\mathcal{M}^{cs}$ is a center-stable manifold for* (1.1). *In particular, $\mathcal{M}^s \subseteq \mathcal{M}^{cs}$.*
- *If U is a solution satisfying $\zeta\big(U(0)\big) \neq 0$ and lying on either $\mathcal{M}^s$ or $\mathcal{M}^{cs}$, the Cauchy problem* (1.6) *defines a diffeomorphism $\tau : [0, +\infty[\to [0, +\infty[$. In other words, if we restrict to either $\mathcal{M}^s$ or $\mathcal{M}^{cs}$, then the formulations* (1.1) *and* (1.5) *are equivalent, provided that $\zeta\big(U(0)\big) \neq 0$. In particular, property* **(P)** *is satisfied on both $\mathcal{M}^s$ and $\mathcal{M}^{cs}$.*
- *If $U(\tau)$ is a solution lying on either $\mathcal{M}^s$ or $\mathcal{M}^{cs}$, then it can be decomposed as*

$$U(\tau) = U_f(\tau) + U_{sl}(\tau) + U_p(\tau), \tag{1.7}$$

where U_{sl} lies on the manifold of slow dynamics and $U_f(\tau)$ lies on a suitable invariant manifold of exponentially decreasing orbits. The perturbation term $U_p(\tau)$ is small in the sense that

$$|U^p(\tau)| \leq k_p \big|\zeta\big(U(0)\big)\big|\, |U_f(0)| e^{-c\tau/4}$$

for suitable positive constants $c,\ k_p > 0$.

From the technical point of view, the key points in the proof of Theorem 1.1 are the following two. First, we introduce a change of variables which allows us to write system (1.5) in a more convenient form. The precise statement is given in Proposition 4.1.

The second main point in the proof of Theorem 1.1 is the analysis of a family of slaving manifolds for system (1.5). This analysis relies on the presence of a splitting based on exponential decay estimates and it is in the spirit of the above-mentioned Hadamard Perron Theorem. The main results here are Theorem 3.1 and Proposition 3.1. Loosely speaking, Proposition 3.1 tells us the following. Fix a manifold $\mathcal{S}$, locally invariant for (1.5) and entirely made by slow dynamics. Then there exists a slaving manifold containing orbits that decay to an orbit in $\mathcal{S}$ exponentially fast, with respect to the τ variable. Also, Proposition 3.1 ensures that any solution U lying on the slaving manifold admits a decomposition like (1.7), namely

$$U(\tau) = U_f(\tau) + U_{sl}(\tau) + U_p(\tau)$$

where U_{sl} lies on $\mathcal{S}$ and $U_f(\tau)$ is exponentially decreasing to $\vec{0}$. The perturbation term $U_p(\tau)$ is small and vanishes when $\zeta(U) = 0$, so on the singular hypersurface $\{U : \zeta(U) = 0\}$ there is no interaction, but a complete decoupling. Notwithstanding that in the following we need only the case when $\zeta(U) = 0$, in the statement of Proposition 3.1 we actually consider slightly more general conditions ensuring that the interaction term vanishes. From the technical point of view, the most complicated point in the analysis is proof of the $\mathcal{C}^1$ regularity of the slaving manifold, since it involves studying the Frechét differentiability of suitable maps between Banach spaces.

As a final remark, we point out that an application of our analysis concerns the study of the viscous profiles with small total variation for a class of mixed hyperbolic-parabolic systems in one space dimension. The connection between these viscous profiles and the singular ordinary differential equation (1.1) is discussed in [5], where we also explain what we mean by viscous profiles and by mixed hyperbolic-parabolic systems in this context. In [5] we also discuss a remark due to Fréderic Rousset [15] about the Lagrangian and the Eulerian formulation of the Navier Stokes equation. Loosely speaking, the connection between viscous profiles and singular ODEs like (1.1) is that the equation satisfied by the viscous profiles may be singular when the system does not satisfy a condition of block linear degeneracy defined in [6]. In particular, this happens in the case of the Navier Stokes equation written in Eulerian coordinates. As we see in Section 2.1.1, the analysis developed in the present paper applies to the study of the viscous profiles of the Navier Stokes.

We recall that viscous profiles provide useful information when studying the parabolic approximation of an hyperbolic system of conservation laws and that in several interesting situations one is concerned with viscous profiles lying on a center, a center-stable or on the uniformly stable manifold. The literature concerning these topics is extremely wide and here we just refer to the books by Dafermos [7] and by Serre [17] and to the rich bibliography contained therein. For the applications of the viscous profiles to the study of the parabolic approximation of an hyperbolic system, see for example Bianchini and Bressan [4] and Ancona and Bianchini [2]. Concerning the analysis of viscous profiles, we only refer to Benzoni-Gavage, Rousset, Serre and Zumbrun [3], to Liu [14], to Zumbrun [19], and to the references therein. For an alternative approach to the analysis of the viscous profiles of the compressible Navier Stokes equation, see Wagner [18] and the bibliography in there.

The exposition is organized as follows. In Section 1.1 we discuss a linear system: this allows us to introduce in a simplified context the main ideas of the analysis done in Section 4. In Section 1.1 we also outline the main steps of the extension of the analysis to the general nonlinear case by relying on the analogy with the linear case.

In Section 2 we define our hypotheses and in Section 2.1.1 we show that they are satisfied by the viscous profiles of the compressible Navier Stokes equation. Also, in Section 2.2 we discuss two examples showing that, if our hypotheses are not satisfied, then the first derivative dU/dt of a solution U of (1.1) may blow up in finite time.

In Section 3 we introduce preliminary results that are used in Section 4. In particular, in Section 3 we focus on the analysis of the nonsingular ODE (1.4) and, by relying on suitable assumptions, we define a class of invariant manifolds (Theorem 3.1 and Proposition 3.1).

In Section 4 we get back to the singular ODE (1.1). In particular, in Section 4.2 we define the notions of slow and fast dynamic and we extend the definition of center manifold to the case of the singular ODE (1.1). In Section 4.3 we extend the notions of uniformly stable and center-stable manifold by applying the analysis in Section 3: the main result here is Theorem 4.2. Finally, Section 4.4 is devoted to the proof of Proposition 4.1, a technical result which reduces our system to a more convenient form.

1.1 The linear system

In the first part of this section we discuss the linear system

$$\begin{cases} dV/dt = A_s V/\zeta + A_{ns} V & V \in \mathbb{R}^d \\ d\zeta/dt = 0. \end{cases} \tag{1.8}$$

We are interested in the behavior for $\zeta \to 0^+$ (the analysis of the limit $\zeta \to 0^-$ does not involve additional difficulties). Note that, from the second line in (1.8) we deduce that ζ is a parameter and hence the problem can be tackled by relying on by-now standard techniques in singular perturbation theory. Our goal here is introducing in a simplified context the main steps of the analysis.

In the second part of this section we outline the extension to the general nonlinear case and we refer to Section 4 for the detailed exposition. In view of this extension, we split the analysis of the linear case in four main steps.

1. We introduce the change of variable $\tau = t\,\zeta$ which transforms system (1.8) in

$$\begin{cases} dV/d\tau = A_s V + \zeta A_{ns} V \\ d\zeta/d\tau = 0 \end{cases} \tag{1.9}$$

   In the following, we denote by n_- the number of eigenvalues of A_s having strictly negative real part (each of them counted according to its multiplicity) and by n_+ the number of eigenvalues with strictly positive real part. We denote by n_0 the multiplicity of the eigenvalue 0 and, relying on Assumption 1 in the introduction, we assume that there are no purely imaginary eigenvalues. Also, if we write the Jordan form of A_s, then in the block corresponding to the eigenvalue 0 all the entries are 0. Finally, we denote by E the line

$$E := \big\{ (V, \zeta) : \; V = \vec{0} \big\} \subseteq \mathbb{R}^{d+1}, \tag{1.10}$$

   which satisfies Assumption 2.

2. We now let $\zeta \to 0^+$: we are concerned with the behavior of the eigenvalues of the matrix $A_s + \zeta A_{ns}$. Due to results concerning the perturbation of finite-dimensional linear operators (see for example the book by Kato [12], page 64 and followings), these eigenvalues can be classified as follows:

   (a) n_- eigenvalues converge to the eigenvalues of A_s with strictly negative real part. We denote by $M^-(\zeta)$ the eigenspace of $A_s + \zeta A_{ns}$ associated to these eigenvalues.

   (b) n_+ eigenvalues converge to the eigenvalues of A_s with strictly positive real part. We denote by $M^+(\zeta)$ the eigenspace of $A_s + \zeta A_{ns}$ associated to these eigenvalues.

   (c) the remaining n_0 eigenvalues converge to 0 as $\zeta \to 0^+$. We denote by $M^0(\zeta)$ the eigenspace of $A_s + \zeta A_{ns}$ associated to these eigenvalues.

   When $\zeta \to 0^+$, the subspace $M^-(\zeta)$ converges to $M^-(0)$, which is the eigenspace of A_s associated to eigenvalues with strictly negative real part. The convergence occurs in the following sense: $M^-(\zeta)$ is the range of a linear application $P^-(\zeta) \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$. As $\zeta \to 0^+$, $P^-(\zeta)$ converges to $P^-(0)$ and the range of $P^-(0)$ is exactly $M^-(0)$. Similarly, when $\zeta \to 0^+$, the subspaces $M^+(\zeta)$ and $M^0(\zeta)$ converge respectively to $M^+(0)$ and $M^0(0)$, the eigenspaces of A_s associated to the eigenvalues with strictly positive and zero real part. We refer again to Kato [12] for a complete discussion.

If $V \in M^0(\zeta)$, then (1.9) is equivalent to

$$\begin{cases} dV^0/d\tau = \zeta\Big[L_0 A_{ns} R_0 + \zeta\mathcal{O}(1)\Big]V^0 \\ d\zeta/d\tau = 0, \end{cases}$$

where R_0 and L_0 are two matrices that do not depend on ζ. The matrix R_0 has dimension $N \times n_0$ and its columns constitute a basis of $M^0(0)$. The matrix L_0 is $n_0 \times N$-dimensional and satisfies $L_0 R_0 = I_{n_0}$. Also, $V^0 = L^0 V$ and $\mathcal{O}(1)$ denotes an $n_0 \times n_0$-dimensional matrix, which possibly depends on ζ but remains bounded as $\zeta \to 0^+$ (its exact expression is not relevant here). Going back to the original variable t, one gets

$$\begin{cases} dV^0/dt = \Big[L_0 A_{ns} R_0 + \zeta\mathcal{O}(1)\Big]V^0 \\ d\zeta/dt = 0, \end{cases} \tag{1.11}$$

and hence V^0 can be regarded as a *slow dynamic*, because it satisfies the non singular ODE (1.11). In view of the future extension to the general nonlinear case, we point out that the set

$$\mathcal{M}^{sl} := \big\{(V, \zeta) : \; V \in M^0(\zeta)\big\} \subseteq \mathbb{R}^{d+1} \tag{1.12}$$

is a center manifold of system (1.9).

3. We now consider the case when V belongs to $M^-(\zeta)$. We then have that (1.9) is equivalent to

$$\begin{cases} dV^-/d\tau = \Big[A_s^- + \zeta\,\mathcal{O}(1)\Big]V^- \\ d\zeta/d\tau = 0 \end{cases}$$

where $V^- \in \mathbb{R}^{n_-}$ and A_s^- is a $n_- \times n_-$-dimensional matrix which does not depend on ζ and whose eigenvalues have all strictly negative real part. In the previous equation, the entries of the vector V^- are the coordinates of V with respect to a basis of $M^-(\zeta)$ and $\mathcal{O}(1)$ denotes a $n_- \times n_-$-dimensional matrix which possibly depends on ζ but remains bounded as $\zeta \to 0^+$ (its exact expression is not important here). If ζ is sufficiently small, then all the eigenvalues of the matrix $\big[A_s + \zeta\mathcal{O}(1)\big]$ have strictly negative real part and hence the solution $V^-(\tau)$ converges exponentially fast to $\vec{0}$. More precisely, one has

$$|V^-(\tau)| \leq e^{-c\tau/2}|V^-(0)|,$$

where $c > 0$ satisfies $-c > \lambda$ for every λ eigenvalue of A_s. Going back to the original variable t, the function $V^-(t)$ satisfies

$$|V^-(t)| \leq e^{-ct/2\zeta}|V^-(0)|.$$

and hence the speed of exponential decay gets faster and faster as $\zeta \to 0^+$. In this sense, we can regard V^- as a *fast dynamic*.

In view of the future extension to the general nonlinear case, we point out that the set

$$\mathcal{M}^f := \big\{(V, \zeta) : \; V \in M^-(\zeta)\big\} \subseteq \mathbb{R}^{d+1} \tag{1.13}$$

is the uniformly stable manifold of system (1.9) relative to the manifold E defined as in (1.10). In the previous expression, ζ varies in a sufficiently small neighborhood of 0.

4. By applying the above-mentioned techniques from [12], one gets that the eigenvalues of $L_0 A_{ns} R_0 + \zeta\mathcal{O}(1)$ can be divided into 3 groups:

   (a) eigenvalues that converge to the eigenvalues of $L_0 A_{ns} R_0$ with strictly negative real part. We denote by $M^{0-}(\zeta)$ the corresponding eigenspace.

   (b) eigenvalues that converge to the eigenvalues of $L_0 A_{ns} R_0$ with strictly positive real part. We denote by $M^{0+}(\zeta)$ the corresponding eigenspace.

(c) eigenvalues that converge to the eigenvalues of $L_0 A_{ns} R_0$ with zero part. We denote by $M^{00}(\zeta)$ the corresponding eigenspace.

If $V(t) \in M^{0-}(\zeta)$, then $V(t)$ converges exponentially fast to the equilibrium $\vec{0}$ when $t \to +\infty$, but the speed of exponential decay does not blow up as $\zeta \to 0^+$.

In view of the future extension to the general nonlinear case we point out that the set

$$\mathcal{M}^s := \big\{ (V, \zeta) : \; V \in M^-(\zeta) \oplus M^{0-}(\zeta) \big\} \tag{1.14}$$

can be regarded as an *uniformly stable space* of (1.8) since every orbit lying on M^s decays exponentially fast to an equilibrium $(\vec{0}, \zeta)$. Also, the speed of exponential decay is uniformly bounded from below by a constant which does not depend on ζ.

Conversely, the set

$$\mathcal{M}^{00} := \big\{ (V, \zeta) : \; V \in M^{00}(\zeta) \big\} \tag{1.15}$$

can be regarded as a *center manifold* of the original equation (1.8). Note that, by construction, the manifold $\mathcal{M}^{00}$ is entirely contained in the manifold of the slow dynamics $\mathcal{M}^{sl}$, which is defined as in (1.12). As a consequence, if the function $(V(t), \zeta(t))$ lies on $\mathcal{M}^{00}$, then it satisfies the nonsingular equation (1.11).

In Section 4 we extend the previous considerations from the linear system (1.8) to the general nonlinear case (1.1). We proceed in four steps, which can be respectively viewed as the extensions of steps 1,. . . ,4 above.

1. We introduce the change of variables $\tau(t)$ defined by the Cauchy problem (1.6), which transforms system (1.1) into (1.5). However, in the nonlinear case the change of variable is not *a priori* well defined since the function $\zeta[U(t)]$ could in principle attain the value 0. We proceed as follows: as in the linear case we carry on the analysis by referring to system (1.5) and then we show that *a posteriori* the change of variables (1.6) is well defined.

2. In Section 4.2 we define a manifold of the slow dynamics as a center manifold of (1.5), thus extending definition (1.12). By relying on Assumption 2 we can show that, as in the linear case, system (1.1) restricted to the manifold of the slow dynamics is actually nonsingular. Moreover, by using Assumption 4 one can prove that on the manifold of the slow dynamics the change of variable (1.6) is well defined and hence that (1.1) and (1.5) are equivalent.

3. In Section 4.2 we also extend the definition of fast dynamic given by (1.13). Let E be the one-dimensional manifold transversal to the hypersurface $\{U : \; \zeta(U) = 0\}$ and containing only equilibria. The existence of E is provided by Assumption 1. The manifold of the fast dynamic is then the uniformly stable manifold of the system (1.5) relative to the manifold E.

4. As pointed out in Step 2 above, system (1.6) restricted to the manifold of the slow dynamics is actually nonsingular and hence we can define a center manifold $\mathcal{M}^{00}$ of the equilibrium point $\bar{U}$. In this way we obtain an extension of the definition (1.15).

   The extension of the definition of uniformly stable manifold (1.14), done in Section 4.3, is more technical and requires additional work. In particular, this is when the analysis in Section 3 comes into play. The main result here is Theorem 4.2, which provides the existence of a uniformly stable manifold $\mathcal{M}^s$. One of the most delicate point in the analysis is showing that on $\mathcal{M}^s$ the change of variables (1.6) is well defined and hence that system (1.5) is equivalent to system (1.1). The manifold $\mathcal{M}^s$ contains orbits that decay exponentially fast (in the t variable) to an equilibrium point in E and hence can be viewed as an extension of the object defined in (1.14). In the linear case (1.14), any orbit lying on $\mathcal{M}^s$ can be decomposed as

$$(V, \zeta) = (V_f, 0) + (V_s, \zeta),$$

   where $V_f \in M^-(\zeta)$ and hence V_f is exponentially decreasing in both the τ and the t variable. Conversely, (V_s, ζ) lies on the manifold of the slow dynamics (1.12) and V_s, which belongs to $\in M^{0-}(\zeta)$, is exponentially decreasing in the t variable only. By relying on the analysis in Section 3 we get that this decomposition result can be extended to the general nonlinear case, provided that we add a perturbation term taking into account possible interactions due to the nonlinearity: this is property 5 in the statement of Theorem 4.2. In particular, we show that the perturbation term is small with respect to the other two, in the sense specified by equation (4.12).

Also, Theorem 4.2 describe how the notion of center-stable manifold can be extended to the case of the singular equation (1.5) by defining a manifold $\mathcal{M}^{cs}$ which contains orbit that do not experience a fast blow up as $t \to 0^+$. We show that on $\mathcal{M}^{cs}$ system (1.5) is equivalent to system (1.1) and that a decomposition result similar to the previous one holds.

Remark 1.1. We are now able to provide more details concerning the main analogies between our analysis and Fenichel's (see [8, 9, 10] and the lecture notes by Jones [11]). Indeed, despite some technical differences, we can single out three main ideas due to Fenichel and used in the present work.

1. As previously mentioned, the notions of *slow* and *fast dynamic* can be viewed as extensions of the notions of *slow* and *fast time* scale respectively. In particular, Step 2 above should be related to Fenichel's First Invariant Manifold Theorem (see Jones [11, page 49]).
2. A key point in Fenichel's analysis is the study of the interaction of slow and fast time scales via the construction of suitable slaving manifolds (see Jones [11, Chapters 2 and 3]). As the above outline shows, this idea is used several times in the present paper, in particular the manifold of the fast dynamics itself is defined as a slaving manifold.
3. Last, Proposition 4.1 can be viewed as an extension of Fenichel's Normal Form (see Jones [11, page 82]). Indeed, in both cases the idea is introducing a local change of variables that "straightens" the manifolds under consideration and allows to write the equations in a more convenient form.

As we mentioned at the beginning of Section 1, the main difference between our analysis and Fenichel's is that we are concerned with the case when $d\zeta/dt \neq 0$. Hence, a priori we may have that $\zeta(U) \neq 0$ at $t = 0$, but ζ reaches the singular hypersurface $\{U : \ \zeta(U) = 0\}$ in finite time. Part of the analysis in Section 4 is devoted to show that, under the assumptions discussed in Section 2, this behavior does not occur if we restrict to solutions U lying on suitable invariant manifolds.

2 Hypotheses and examples

In this section we define the hypotheses we assume in the work and we discuss some examples.

More precisely, in Section 2.1 we state our assumptions, which can be divided into two groups: Hypotheses 1, 2, 3, allow to avoid some technical complications, but could be actually omitted at the price of much heavier notations. On the other side, Hypotheses 4, 5, 6, 7, 8 are much more important and they are used throughout Section 4. Note, however, that in Section 3 we are not directly concerned with the singular ODE (1.1) and that we do not use Hypotheses 4, 5, 6, 7, 8.

Moreover, in Section 2.2 we discuss three counterexamples. They show that, if our hypotheses are violated, then the results discussed in the following sections do not hold. In particular, there might be solutions of (1.1) that are not continuously differentiable.

Finally, in Section 2.1.1 we verify that the conditions introduced in Section 2.1 are satisfied by the viscous profiles of the compressible Navier Stokes equation written in Eulerian coordinates.

2.1 Hypotheses

Set

$$F(U) = \phi_s(U) + \zeta(U)\phi_{ns}(U), \tag{2.1}$$

where ϕ_s, ϕ_{ns} and ζ are the same as in (1.1). Then (1.5) can be written as

$$\frac{dU}{d\tau} = F(U) \tag{2.2}$$

To simplify the exposition, we assume the following:

Hypothesis 1. The initial datum $U(0)$ of (2.2) satisfies $\zeta\big(U(0)\big) > 0$.

The case $\zeta\big(U(0)\big) < 0$ does not involve additional difficulties. The main difference is that, if $\zeta\big(U(0)\big) < 0$, then the change of variable defined by (1.6) has negative derivative. As a consequence, when $t \to +\infty$ the function $\tau(t) \to -\infty$. Loosely speaking, the statements given in the present paper can be extended to the case $\zeta\big(U(0)\big) < 0$ in the following way. All the statements concerning the fast dynamics and referring to the *stable* space or to *stable*-like manifolds have to be replaced by analogous statements concerning the *unstable* space or *unstable*-like manifolds. However, we will not consider the case $\zeta\big(U(0)\big) < 0$ explicitly.

Before stating the other hypotheses, we recall that we want to study (1.1) and (2.2) in the neighbourhood of an equilibrium point $\bar{U}$ such that $F(\bar{U}) = \vec{0}$ and $\zeta(\bar{U}) = 0$. It is not restrictive to take $\bar{U} = \vec{0}$. Namely, in the following we assume

$$F(\vec{0}) = \vec{0} \qquad \zeta(\vec{0}) = 0. \tag{2.3}$$

Also, we can assume the following. Fix a positive constant $\delta > 0$ and consider a smooth cut-off function $\rho(U)$ satisfying

$$\rho(U) = \begin{cases} 1 & |U| \le \delta \\ 0 & |U| \ge 2\delta. \end{cases}$$

In the following, instead of studying system (2.2) we focus on

$$\frac{dU}{d\tau} = \rho(U)F(U).$$

However, to simplify the notations instead of writing each time $\rho(U)F(U)$ we assume that Hypothesis 2 holds.

Hypothesis 2. The function F satisfies the following condition: if $|U| \ge 2\delta$ then $F(U) = \vec{0}$.

The exact size of the constant δ will be discussed in the following.

Note that Hypothesis 2 is not restrictive if the goal is to study the solutions of (2.2) that remain confined in a neighbourhood of the origin of size δ. Loosely speaking, the analysis developed in Sections 3 and 4 can be extended to the orbits of systems that violate Hypothesis 2 as far as these orbits remain in a neighbourhood of the origin with size δ. In particular, the manifold described in Sections 3 and 4 are no more *invariant* if Hypothesis 2 is violated: they are just *locally invariant*.

Also, we can assume with no loss of generality that all the eigenvalues of $DF(\vec{0})$ have non positive real part. Indeed, this condition is satisfied provided that we restrict to orbits lying on a *center-stable* manifold for (2.2). As mentioned in the introduction, the existence of a center-stable manifold can be obtained as a consequence of the Hadamard Perron Theorem, which is discussed in the book by Katok and Hasselblatt [13, Chapter 6, page 242]. Also, note that if $\zeta\big(U(0)\big) < 0$ then it is not restrictive to assume that all the eigenvalues of $DF(\vec{0})$ have *non negative* real part: this can be obtained considering the solutions that lie on a *center unstable* manifold.

Hypothesis 3. The Jacobian $DF(\vec{0})$ admits only eigenvalues with non positive real part.

Also, we assume the following non degeneracy condition:

Hypothesis 4. The gradient $\nabla\zeta$ satisfies $\nabla\zeta(\vec{0}) \neq \vec{0}$.

Let $\mathcal{S}$ be the singular set

$$\mathcal{S} := \big\{ U : \ \zeta(U) = 0 \big\}. \tag{2.4}$$

Hypothesis 4 implies, via the Implicit Function Theorem, that in a small enough neighbourhood of $\vec{0}$ the set $\mathcal{S}$ is actually an hypersurface

Hypothesis 5. Let $\mathcal{M}^c$ be any center manifold of the equilibrium point $\vec{0}$ for system (2.2). If $|U| \le \delta$ and U belongs to the intersection $\mathcal{M}^c \cap \mathcal{S}$, then U is an equilibrium for (2.2), namely $F(U) = \vec{0}$.

Concerning equilibria, we also assume the following

Hypothesis 6. There exists a manifold $\mathcal{M}^{eq}$ containing $\vec{0}$, transversal to $\mathcal{S}$ and entirely made by equilibria of (2.2).

Let n_{eq} be the dimension of $\mathcal{M}^{eq}$. We recall that the hypersurface $\mathcal{S}$ and $\mathcal{M}^{eq}$ are transversal if the intersection $\mathcal{S} \cap \mathcal{M}^{eq}$ is locally a manifold having dimension $n_{eq} - 1$. Note that, by construction, $\mathcal{M}^{eq} \subseteq \mathcal{M}^{c}$. Also, we point out that we are not assuming that $\mathcal{M}^{eq}$ is *the* manifold of equilibria, namely there may be equilibria that do not belong to $\mathcal{M}^{eq}$. We are just assuming that any point in $\mathcal{M}^{eq}$ is an equilibrium for (2.2).

Hypothesis 7. For every $U \in \mathcal{S}$,

$$\nabla \zeta(U) \cdot F(U) = 0. \tag{2.5}$$

Because of Hypothesis 7 and of the regularity of the functions ζ and F, the function

$$G(U) = \frac{\nabla \zeta(U) \cdot F(U)}{\zeta(U)}$$

can be extended and defined by continuity on the hypersurface $\mathcal{S}$.

Hypothesis 8. Let $U \in \mathcal{S}$ be an equilibrium for (2.2), namely $\zeta(U) = 0$ and $F(U) = \vec{0}$. Then

$$G(U) = 0. \tag{2.6}$$

In Section 1.1 we introduce the notion of *slow* and *fast* dynamics. Hypotheses 7 and 8 and can be then reformulated by saying that the set $\mathcal{S}$ is invariant for the manifold of the slow and of the fast dynamics respectively. The reason why we impose this condition is because we want that requirement **(P)** in the introduction is satisfied and that the Cauchy problem (1.6) defines a smooth change of variables $\tau(t)$, $\tau : [0, +\infty[\to [0, +\infty[$. In Section 2.2 we discuss Examples (2.12) and (2.17) showing that, if either Hypothesis 7 or Hypothesis 8 is violated, then there may be functions U solving (2.2) such that $\zeta(U)$ is nonzero at $t = 0$ but $\zeta\big(U(t)\big) = 0$ for a finite value of t. Also, in both Examples (2.12) and (2.17) the solution U is not smooth and the Cauchy problem (1.6) does not define a regular change of variables

Remark 2.1. Consider system (1.1) and assume that $f(U)$ is a regular, real valued function such that $f(\vec{0}) > 0$. Clearly, (1.1) is equivalent to

$$\frac{dU}{dt} = \frac{1}{\zeta(U) f(U)} \phi_s(U) f(U) + \phi_{ns}(U) \tag{2.7}$$

and $\zeta(U) f(U) \to 0^+$ if and only if $\zeta(U) \to 0^+$, at least in a sufficiently small neighbourhood of $U = \vec{0}$. By direct check, one can verify that Hypotheses 1 ... 8 are verified by the pair (ζ, F) if and only if they are verified by the pair $(\zeta f, Ff)$.

Remark 2.2. As we will see in Section 4, Hypothesis 5 can be reformulated by saying that the slow dynamics intersecting the singular manifold $\{U : \ \zeta(U) = 0\}$ are equilibria for system (2.2). Heuristically, this means that we require that the limit as $\zeta\big(U(0)\big) \to 0^+$ of a solution of (1.1) is a solution of the limit system. In other words, we want to rule out the possibility of a relaxation effect.

2.1.1 The case of the compressible Navier Stokes in Eulerian coordinates

In this section we show that Hypotheses 1, 3, ..., 8 are satisfied by the ODE for the viscous profiles of the compressible Navier Stokes equation written in Eulerian coordinates. Also, Hypothesis 2 is not restrictive if the goal is to study the viscous profiles entirely contained in a small neighbourhood of an equilibrium point.

The case of the Navier Stokes written in Lagrangian coordinates was already discussed in several paper, see for example Rousset [16]. When the equation is formulated by using Lagrangian coordinates, the ODE satisfied by the viscous profiles is not singular.

The compressible Navier Stokes written in Eulerian coordinates is

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \Big(\rho v^2 + p\Big)_x = \Big(\nu v_x\Big)_x \\ \Big(\rho e + \rho \dfrac{v^2}{2}\Big)_t + \Big(v\Big[\dfrac{1}{2}\rho v^2 + \rho e + p\Big]\Big)_x = \Big(k\theta_x + \nu v v_x\Big)_x. \end{cases} \tag{2.8}$$

Here, the unknowns are $\rho(t, x)$, $v(t, x)$ and $\theta(t, x)$. The function ρ represents the density of the fluid, v is the velocity of the particles in the fluid and θ is the absolute temperature. The function $p = p(\rho, \theta) > 0$ is the pressure and satisfies $p_\rho > 0$, while e represent the internal energy. In the case of a polytropic gas, the following relation holds: $\theta = e(\gamma - 1)/R$, R being the universal gas constant and γ a constant specific of the gas. Finally, $\nu(\rho) > 0$ and $k(\rho) > 0$ represent the viscosity and the heat conduction coefficients respectively.

After some manipulations (see [5] for details), one gets that the equation satisfied by the steady solutions of the compressible Navier Stokes can be written in the form

$$\frac{dU}{dx} = \frac{1}{\zeta(U)} F(U)$$

provided that $U = \big(\rho, v, \theta, \vec{z}\big)^T$, $\zeta(U) = v$ and

$$F(U) = \begin{pmatrix} A_{21}^t \vec{z}/a_{11} \\ v\,\vec{z} \\ b^{-1}\Big[A_{22}v - A_{21}A_{21}^T/a_{11}\Big]\vec{z} \end{pmatrix} \tag{2.9}$$

The equation satisfied by the traveling waves of the compressible Navier Stokes equation in one space variable is similar, the only difference being that the singular value is $v = \sigma$, where σ is the speed of the traveling wave.

In (2.9),

$$A_{21}(\rho,\, v,\, \theta,\, \vec{z}) = \frac{1}{\theta}\begin{pmatrix} p_\rho \\ 0 \end{pmatrix} \tag{2.10}$$

and A_{21}^t denotes its transpose, while $\vec{z} = \big(v_x,\, \theta_x\big)^t$. The function a_{11} is real valued and strictly positive if ρ is bounded away from 0, which we always assume in the following. The matrix b has dimension 2×2 and all its eigenvalues have strictly positive real part (the exact expression is not important here). Finally,

$$A_{22}(\rho,\, v,\, \theta,\, \vec{z}) = \frac{1}{\theta}\begin{pmatrix} \rho v - \nu' \rho_x & p_\theta \\ p_\theta - \nu v_x/\theta & \rho v e_\theta/\theta - k'\rho_x/\theta \end{pmatrix}. \tag{2.11}$$

Note that any point $\bar{U} = (\bar{\rho}, v = 0, \bar{\theta}, \vec{z} = \vec{0})$ satisfies $F(\bar{U}) = \vec{0}$, $\zeta(\bar{U}) = 0$. Also, the matrix A_{22} depends on ρ_x but, plugging $\rho_x = -A_{21}^T \vec{z}/(a_{11}v)$ into (2.11), one gets that $A_{22}v$ evaluated at a point $(\rho,\, v = 0,\, \theta,\, \vec{z} = \vec{0})$ is the null matrix. Thus, the Jacobian DF satisfies

$$DF(\bar{\rho}, 0, \bar{\theta}, \vec{0}) = \begin{pmatrix} 0 & -A_{21}^T/a_{11} \\ \vec{0} & \mathbf{0}_2 \\ \vec{0} & -b^{-1}A_{21}A_{21}^T/a_{11} \end{pmatrix},$$

where $\mathbf{0}_2$ denotes the 2×2 null matrix. Since $A_{21}A_{21}^T/a_{11}$ admits only eigenvalues with non negative real part, then DF admits only eigenvalues with non positive real part and hence Hypothesis 3 is satisfied.

Hypothesis 4 is satisfied since $\zeta(U) = v$, while to verify that Hypotheses 5, ..., 8 are satisfied we first point out that the center space of DF is

$$\Big\{(\rho, v, \theta, \vec{z}) :\; A_{21}^T(\bar{\rho}, 0, \bar{\theta}, \vec{0})\,\vec{z} = 0\Big\}$$

Hence, any center manifold $\mathcal{M}^c$ has dimension 4 since A_{21}^T is given by (2.10) and $p_\rho > 0$. Also, $\mathcal{M}^c$ is transversal to the singular hyperplane $\mathcal{S} = \{v = 0\}$ and hence the intersection $\mathcal{S} \cap \mathcal{M}^c$ has dimension 3. Note that, by the Implicit Function Theorem, the set

$$\big\{(\rho, v, \theta, \vec{z}) : \; v = 0, \; A_{21}^T(\rho, 0, \theta, \vec{z})\, \vec{z} = 0\big\}$$

is locally a three-dimensional manifold included in $\mathcal{S} \cap \mathcal{M}^c$. Since this manifold and $\mathcal{S} \cap \mathcal{M}^c$ have the same dimension, they must locally coincide and hence Hypothesis 5 is satisfied.

We can verify Hypothesis 6 by defining

$$\mathcal{M}^{eq} = \big\{(\rho, v, \theta, \vec{z}) : \; \rho = \bar{\rho}, \; \theta = \bar{\theta}, \; \vec{z} = \vec{0}\big\}.$$

Since $\nabla\zeta(U) \cdot F(U) = vz_1$, z_1 being the first component of $\vec{z}$, then Hypothesis 7 is satisfied. To verify Hypothesis 8, we observe that by relying on (2.10) one deduces that when $v = 0$ the equilibria of F must satisfy $p_\rho^2 z_1 = 0$, which implies $z_1 = 0$.

In conclusion, we have that the analysis developed in the present paper applies to the study of the viscous profiles with small total variation of the Navier Stokes equation written in Eulerian coordinates.

2.2 Examples

2.2.1 Example (2.12)

Example (2.12) deals with a system which satisfies Hypotheses 1, 3 ... 6, but does not satisfy Hypothesis 7. We exhibit a solution of this system which has a blow up in the first derivative and hence it is not continuously differentiable. The loss of regularity experienced in Example (2.12) regards a solution U such that $\zeta[U(0)] \neq 0$, but $\zeta(U)$ reaches the value 0 for a finite value of t.

Consider the system

$$\begin{cases} du_1/dt = -u_2/u_1 \\ du_2/dt = -u_2, \end{cases} \tag{2.12}$$

which can be written in the form (1.1) provided that $U = \big(u_1, \, u_2\big)^T$, $\zeta(U) = u_1$ and

$$\phi_s(U) = \begin{pmatrix} -u_2 \\ 0 \end{pmatrix} \qquad \phi_{ns}(U) = \begin{pmatrix} 0 \\ -u_2 \end{pmatrix}.$$

In this case, the function $F(U)$ defined by (2.1) is

$$F(U) = \begin{pmatrix} -u_2 \\ -u_2 u_1 \end{pmatrix}.$$

By direct check, one can verify that Hypotheses 1 ... 6 and Hypothesis 8 are satisfied by (2.12). On the other side, Hypothesis 7 is not verified in this case. Indeed, the singular hypersurface $\mathcal{S}$ defined by (2.4) is in this case the line $\{u_1 = 0\}$ and

$$\nabla\zeta \cdot F = -u_2$$

is in general different from 0 on $\mathcal{S}$.

The solution of (2.12) can be explicitly computed and it is given by

$$\begin{cases} u_1(t) = \sqrt{u_1(0) + u_2(0)\big(e^{-t} - 1\big)} \\ \\ u_2(t) = u_2(0)e^{-t} \end{cases} \tag{2.13}$$

Choosing $u_2(0) > u_1(0) > 0$, one has that the solution $u_1(t)$ can reach the value 0 for a finite t. Note that at that point t the first derivative du_1/dt blows up: thus, the solution (2.13) of (2.12) is not $\mathcal{C}^1$.

2.2.2 **Example** (2.14)

Example (2.2.2) deals with system (2.14), which is apparently very similar to (2.12). However, in the case of (2.14) Hypotheses 1, 3 ... 8 are all verified. We show the solutions of (2.14) are regular. Also, if $\zeta[U(0)] \neq 0$ then $\zeta[U(t)] \neq 0$ for all values of t.

Consider system

$$\begin{cases} du_1/dt = -u_2 \\ du_2/dt = -u_2/u_1, \end{cases} \tag{2.14}$$

which can be written in the form (1.1) provided $U = \big(u_1,\, u_2\big)^T$, $\zeta(U) = u_1$ and

$$\phi_s(U) = \begin{pmatrix} 0 \\ -u_2 \end{pmatrix} \qquad \phi_{ns}(U) = \begin{pmatrix} -u_2 \\ 0 \end{pmatrix}.$$

Then the function $F(U)$ defined by (2.1) is

$$F(U) = \begin{pmatrix} -u_2 u_1 \\ -u_2 \end{pmatrix}.$$

By direct check, one can verify that Hypotheses 1 ... 8 are all verified in this case.

To study system (2.14) we can proceed as follows. From (2.14) we have

$$\frac{du_1/dt}{u_1} = -\frac{u_2}{u_1} = du_2/dt$$

and hence

$$\ln\left[\frac{u_1(t)}{u_1(0)}\right] = u_2(t) - u_2(0).$$

Eventually, we obtain

$$u_1(t) = u_1(0)e^{u_2(t)-u_2(0)}. \tag{2.15}$$

Choose $u_1(0) > 0$. To prove that $u_1(t) \neq 0$ for all t it is enough to show that $u_2(t)$ is well defined (and in particular finite) for every $t > 0$. In the following we also prove that $u_2(t)$ is also $\mathcal{C}^\infty$ for every $t \geq 0$. This guarantees that no loss of regularity occurs.

Plugging (2.15) into the second line of (2.14) we get

$$du_2/dt = -\frac{u_2}{u_1(0)}e^{u_2(0)-u_2(t)}. \tag{2.16}$$

Note that $u_2 = 0$ is an equilibrium for (2.16). Also, if $u_2(0) < 0$ then $du_2/dt \geq 0$ and hence $u_2(0) \leq u_2(t) < 0$ for every t. Conversely, if $u_2(0) > 0$ then $du_2/dt \leq 0$ and hence $0 \leq u_2(t) < u_2(0)$ for every t. In both cases, we get that $u_2(t)$ is well defined and regular for every $t \geq 0$.

2.2.3 **Example** (2.17)

With Example (2.2.3) we discuss a system which satisfies Hypotheses 1, 3 ... 7, but does not satisfy Hypothesis 8. As in Example (2.12), we exhibit a solution for which $\zeta[U(0)] \neq 0$, but $\zeta(U)$ reaches the value 0 for a finite value of t. When this happens, a loss of regularity occurs.

We consider system

$$\begin{cases} du_1/dt = -u_3 \\ du_2/dt = -u_2/u_1 \\ du_3/dt = -u_3, \end{cases} \tag{2.17}$$

which takes the form (1.1) provided that we set $U = \Big(u_1,\, u_2,\, u_3\Big)^T$, $\zeta(U) = u_1$ and

$$\phi_s(U) = \begin{pmatrix} 0 \\ -u_2 \\ 0 \end{pmatrix} \qquad \phi_{ns}(U) = \begin{pmatrix} -u_3 \\ 0 \\ -u_3 \end{pmatrix}.$$

The function $F(U)$ defined by (2.1) is then

$$F(U) = \begin{pmatrix} -u_3 u_1 \\ -u_2 \\ -u_3 u_1 \end{pmatrix}.$$

By direct check, one can verify that Hypotheses 1 ... 7 are verified by (2.17). On the other side, Hypothesis 8 is not satisfied in this case. Indeed, the hypersurface $\mathcal{S} = \{U : \zeta(U) = 0\}$ is the plane $\{u_1 = 0\}$. Thus, the set of points such that $\zeta(U) = 0$ and $F(U) = \vec{0}$ is $\{u_1 = u_2 = 0\}$ and

$$\nabla\zeta \cdot DF \cdot \Big(\nabla\zeta\Big)^T = -u_3$$

is in general different from zero on this line.

An explicit solution of (2.17) can be obtained as follows. From the third and the first equation we get respectively

$$u_3(t) = u_3(0)e^{-t}$$
$$u_1(t) = u_1(0) - u_3(0) + u_3(0)e^{-t}.$$

Assume that $u_3(0) = Au_1(0)$ for some constant A whose exact value is determined in the following. The equation satisfied by u_2 becomes

$$\frac{du_2}{dt} = -\frac{u_2}{Au_1(0)e^{-t} + u_1(0)(1-A)}.$$

Thus, we obtain

$$\frac{d}{dt}\Big[\ln\Big(u_2(t)\Big)\Big] = \frac{1}{u_1(0)(A-1)}\frac{d}{dt}\Big[\ln\Big(u_1(0)(1-A)e^t + Au_1(0)\Big)\Big]$$

and hence

$$u_2(t) = B\Big[(1-A)e^t + A\Big]^{\frac{1}{(A-1)u_1(0)}}$$

for a suitable constant B. If $(A-1)u_1(0) > 1$, then the first derivative du_2/dt blows up at $t = \ln(A/A-1)$. Note that this is exactly the value of t at which $u_1(t)$ attains 0.

In general, for every $u_1(0) > 0$ if $1/(A-1)u_1(0)$ is not a natural number, then the solution is not in $\mathcal{C}^m$ for $m = [1/(A-1)u_1(0)] + 1$. Here $[1/(A-1)u_1(0)]$ denotes the entire part. Thus, we have a loss of regularity in higher derivatives.

3 Uniformly stable manifolds

In the present section we investigate the existence and the structure of suitably invariant manifolds for system (2.2). The precise statement is given in Proposition 3.1.

This is the most technical section of the paper and its goal is furnishing the tools that are then applied in Section 4 to the analysis of the singular ODE (1.1). From the technical point of view, the main result in here is Theorem 3.1, which allows to prove Proposition 3.1. By relying on Proposition 3.1, in Section 4 we extend to the general nonlinear case the results discussed in Section 1.1 in the linear case.

In this section we rely on Hypotheses 2 and 3, but we do not use Hypotheses 1, 4,. . . , 8. Conversely, in Section 4 we use all the hypotheses introduced in Section 2.1.

3.1 Notations and preliminary results

3.1.1 Fréchet differentiability of the fixed point of a family of maps

In the following we rely on the Implicit Function Theorem to study the regularity of the fixed points of a family of maps depending on a parameter. For the convenience of the reader we now discuss the abstract framework we use in Section 3.3 and we state in Lemma 3.1 the precise regularity result we need. For the definition of Fréchet differential and for a discussion about differential calculus in infinite dimensional Banach spaces, we refer to the book by Ambrosetti and Prodi [1].

In Section 3.3 we are in the following situation: let X be a closed subset with non empty interior in a Banach space $\tilde{X}$ and let Y be an open subset of another Banach space $\tilde{Y}$. We are concerned with a given map $T : X \times Y \to \tilde{X}$ and we prove that, for every $y \in Y$, $T(\cdot,\, y)$ takes values in X and is a strict contraction, namely there exists some constant $k < 1$ such that

$$\|T(x_1,\, y) - T(x_2,\, y)\|_{\tilde{X}} \leq k \|x_1 - x_2\|_{\tilde{X}} \quad \forall\, x_1,\ x_2 \in X.$$

By relying on the Contraction Mapping Theorem, we define a function $x : Y \to X$ which maps y into the fixed point of $T(\cdot,\, y)$.

Lemma 3.1 deals with the regularity of the function $x(y)$. Before stating it, we introduce some notations: we assume that, for any point $(\bar{x},\, \bar{y})$ in the interior of $X \times Y$, the function $T(\cdot,\, \bar{y})$ is Fréchet differentiable at $\bar{x}$ and we denote by $T_x(\bar{x},\, \bar{y}) \in \mathcal{L}(\tilde{X},\, \tilde{X})$ its differential. We also assume that the map $T(\bar{x},\, \cdot)$ is Fréchet differentiable at $\bar{y}$ and we denote by $T_y(\bar{x},\, \bar{y}) \in \mathcal{L}(\tilde{Y},\, \tilde{X})$ its differential.

We can now state the regularity result we need in the following.

Lemma 3.1. *Assume that the map $x(y)$ is Lipschitz continuous and that the point $(\bar{x},\, \bar{y})$ is in the interior of $X \times Y$ and satisfies $\bar{x} = x(\bar{y})$. Also, assume that $T(\bar{x}, \cdot)$ is Fréchet differentiable at $\bar{y}$, that the map $T(\cdot,\, y)$ is Fréchet differentiable at x for (x, y) in a neighbourhood of $(\bar{x},\, \bar{y})$ and that the differential $T_x(x, y)$ is continuous in there. Then the function x is Fréchet differentiable at $\bar{y}$ and the differential is*

$$\Big[I - T_x\big(\bar{x},\, \bar{y}\big)\Big]^{-1} \circ T_y\big(\bar{x},\, \bar{y}\big), \tag{3.1}$$

where I denotes the identity.

Note that the map $\Big[I - T_x\big[x(\bar{y}),\, \bar{y}\big]\Big]$ is invertible because $T(\cdot,\, \bar{y})$ is a strict contraction on X.

Remark 3.1. As a matter of fact, in the following we get the Lipschitz continuity of the map x as a consequence of this condition: for every $y_1,\ y_2 \in Y$,

$$\|T\big(x(y_1),\, y_1\big) - T\big(x(y_1),\, y_2\big)\|_X \leq L \|y_1 - y_2\|_Y. \tag{3.2}$$

This is enough to conclude because

$$\begin{aligned}
\|x(y_1) - x(y_2)\|_X &= \|T\big(x(y_1),\, y_1\big) - T\big(x(y_2),\, y_2\big)\|_X \\
&\leq \|T\big(x(y_1),\, y_1\big) - T\big(x(y_1),\, y_2\big)\|_X + \|T\big(x(y_1),\, y_2\big) - T\big(x(y_2),\, y_2\big)\|_X \\
&\leq L\|y_1 - y_2\|_Y + k\|x(y_1) - x(y_2)\|_X.
\end{aligned}$$

Since $k < 1$, we get that $x(y)$ is Lipschitz continuous.

3.1.2 First change of variables

Consider system (2.2). Let V^- be the eigenspace of the Jacobian $DF(\vec{0})$ associated to eigenvalues with strictly negative real part. Also, let V^0 be the eigenspace associated to the eigenvalues with 0 real part. Also, fix $\mathcal{V}^0$, a center manifold of (2.2) around the equilibrium $\vec{0}$. Finally, let $\mathcal{V}^-$ be the stable manifold. The manifolds $\mathcal{V}^0$ and $\mathcal{V}^-$ are tangent at the origin to V^0 and V^- respectively. Note that $\mathbb{R}^N = V^0 \oplus V^-$ because $DF(\vec{0})$ admits only eigenvalues with non positive real part. Due to the Local Invertibility Theorem, in a sufficiently small neighbourhood of the origin we can define a local diffeomorphism Υ in such that the following conditions are satisfied. Let $\tilde{U} = \Upsilon(U)$, then $\tilde{U} = (\tilde{X}^-, \tilde{X}^0)$, where $\tilde{X}^0$ has the same dimension as V^0 and $\tilde{X}^-$ has the same dimension as V^-. The stable manifold of (3.3) is the subspace $\{\tilde{X}^0 \equiv \vec{0}\}$, while the subspace $\{\tilde{X}^- \equiv \vec{0}\}$ is a center manifold. By construction,

$$\frac{d\tilde{U}}{d\tau} = \tilde{f}(\bar{U}), \tag{3.3}$$

where $\tilde{f}(\tilde{U}) = D\Upsilon\big(\Upsilon^{-1}(\bar{U})\big)F\big(\Upsilon^{-1}(\bar{U})\big)$. In the following, we assume that the constant δ in (2.2) is small enough to have that the local diffeomorphism Υ is defined in the ball of radius 2δ and center at the origin. Also, to simplify the notations we do not write $\tilde{U}$, $\tilde{X}^-$ and $\bar{X}^0$, but just U, X^- and X^0.

3.1.3 A priori estimates

We rewrite system (3.3) as

$$\begin{cases} dX^-/d\tau = f^-(X^-,\, X^0) \\ dX^0/d\tau = f^0(X^-,\, X^0) \end{cases} \tag{3.4}$$

The subspaces $\{X^- = \vec{0}\}$ and $\{X^0 = \vec{0}\}$ are locally invariant for (3.4) since they represent respectively a center and the stable manifold. Thus, $f^-(\vec{0},\, X^0) \equiv \vec{0}$ for every X^0 and $f^0(X^-,\, \vec{0}) \equiv 0$ for every X^- and

$$f^-(X^-,\, X^0) = A^-(X^-,\, X^0)X^- \qquad f^0(X^-,\, X^0) = \hat{A}^0(X^-,\, X^0)X^0 \tag{3.5}$$

for suitable matrices A^- and A^0. By construction, $A^-(\vec{0},\, \vec{0})$ admits only eigenvalues with strictly negative real part and $\hat{A}^0(\vec{0},\, \vec{0})$ has only eigenvalues with zero real part. As a consequence, the following holds. Let n_- denote the dimension of X^- and fix a constant $c > 0$ satisfying $Re\lambda < -c$ for every λ eigenvalue of $A^-(\vec{0},\, \vec{0})$. Then there exists a constant $C_- > 0$ such that

$$\forall\, \underline{X}^- \in \mathbb{R}^{n_-}, \qquad |e^{A^-(\vec{0},\, \vec{0})t}\underline{X}^-| \leq C_- e^{-ct}|\underline{X}^-|. \tag{3.6}$$

Also, if δ is small enough and $|X^-(0)| < \delta$, then the solution of the Cauchy problem

$$\begin{cases} dX^-/d\tau = f^-(X^-,\, \vec{0}) \\ X^-(\tau = 0) = X^-(0) \end{cases}$$

satisfies

$$|X^-(\tau)| \leq C_- e^{-c\tau/2}|X^-(0)|,$$

where $c > 0$ is as before a constant such that $Re\lambda < -c$ for every λ eigenvalue of $A^-(\vec{0},\, \vec{0})$.

Plugging (3.5) in (3.4) we get

$$\begin{cases} dX^-/d\tau = A^-(X^-,\, X^0)X^- \\ dX^0/d\tau = \hat{A}^0(X^-,\, X^0)X^0. \end{cases} \tag{3.7}$$

In view of the applications discussed in Section 4 it is convenient to take into account the following situation. Assume that there exists a continuously differentiable manifold $\mathcal{Z}_0$ containing the stable manifold $\{X^0 = \vec{0}\}$ and satisfying

$$f^0(X^-,\, X^0) = \vec{0} \qquad \forall\, (X^-,\, X^0) \in \mathcal{Z}_0. \tag{3.8}$$

Actually, this assumption is not restrictive, in the sense explained in Remark 3.2 at the end of Section 3.1.3.

Applying, if needed, a local diffeomorphism, we can assume that $X^0 = (\zeta,\, u_0)$ and that $\mathcal{Z}_0 = \{\zeta = \vec{0}\}$. Since the stable manifold is entirely contained in $\mathcal{Z}_0$, such a diffeomorphism does not produce any change on X^-, but only on X^0. In the following we assume that the constant δ in Hypothesis 2 is small enough to have that the local diffeomorphism is defined in the ball of radius 2δ and center at the origin.

Consider the system restricted on the center manifold $\{X^- = \vec{0}\}$: since the subspace $\{\zeta = \vec{0}\}$ is entirely made by equilibria, then we get that the equation

$$dX^0/d\tau = \hat{A}^0(\vec{0},\, X^0)X^0$$

becomes

$$\begin{cases} d\zeta/d\tau = \hat{B}(\vec{0},\, \zeta,\, u_0)\zeta \\ du_0/d\tau = \hat{C}(\vec{0},\, \zeta,\, u_0)\zeta, \end{cases} \tag{3.9}$$

where $\hat{B}$ and $\hat{C}$ are suitable matrices. Note that, by construction, $\hat{B}(\vec{0},\, \vec{0},\, \vec{0})$ admits only eigenvalues with zero real part. Fix a constant ε such that $Re\lambda < -\varepsilon < 0$ for any λ eigenvalue of $A^-(\vec{0},\, \vec{0})$: also, we impose $\varepsilon < c$, where c is the same as in (3.6). Assuming that the constant δ in Hypothesis 2 is sufficiently small we can assume that every solution ζ of (3.9) satisfies

$$|\zeta(\tau)| \leq \mathcal{O}(1) e^{\varepsilon|\tau|} |\zeta(0)| \tag{3.10}$$

for some suitable constant $\mathcal{O}(1)$. Since in (3.9) the matrix $\hat{C}$ is uniformly bounded, we get that

$$|u_0(\tau) - u_0(0)| \leq \mathcal{O}(1) e^{\varepsilon|\tau|} |\zeta(0)| \tag{3.11}$$

for a constant $\mathcal{O}(1)$ (possibly different from the one in (3.10)). We introduce the following notation: given a point $\underline{X}^0 = (\zeta,\, u_0)$ on the center manifold we call $\underline{Y}^0$ the point

$$\underline{Y}^0 = (\vec{0},\, u_0). \tag{3.12}$$

Clearly $\underline{Y}^0$ depends on $\underline{X}^0$, but to simplify the notations we won't express this dependence explicitly. Combining (3.10) and (3.11) we then obtain

$$|X^0(\tau) - \underline{Y}^0(0)| \leq k_0 e^{\varepsilon|\tau|} |\zeta(0)| \tag{3.13}$$

for a suitable constant k_0.

Finally, note that, since both A^- and $\hat{A}$ are zero when $|(X^-,\, X^0)| \geq 2\delta$, then any non constant solution of (3.7) satisfies

$$|X^0(\tau)| \leq 2\delta \qquad |X^-(\tau)| \leq 2\delta \qquad \forall\, \tau. \tag{3.14}$$

Remark 3.2. The hypothesis that there exists a manifold of zeroes $\mathcal{Z}_0$ is not restrictive. Indeed, assume that the set of the zeroes of f^0 coincides with the stable manifold $\{X^0 = \vec{0}\}$. In this case, we can set $\zeta = X^0$, there is no component u_0 and, given $\underline{X}^0$, the element $\underline{Y}^0$ is just $\underline{X}^0$ itself. This notation ensures that the estimate (3.13) still holds. As it will be clear in the the following, the only fact about $\mathcal{Z}_0$ we use in the proof of Proposition 3.1 is estimate (3.13). As a consequence, Proposition 3.1 can be extended to the case that $\mathcal{Z}_0$ is just the stable manifold.

In other words, the presence of a manifold of zeroes wider then the stable manifold is not strictly necessary for the existence of the uniformly stable manifold introduced in Theorem 3.1. However, it allows to get a sharper estimate in (3.22).

3.1.4 Linear change of variables

In the statement of the following lemma we denote by n_c the dimension of X^0, and hence we have that $N = n_c + n_-$. The proof is standard, so we omit it.

Lemma 3.2. *For every $M > 0$, there exists a linear change of variables $\mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ such that the function X^0 written by using the new coordinates satisfies*

$$dX^0/d\tau = \hat{A}(X^-,\, X^0)X^0, \tag{3.15}$$

for a suitable matrix such that

$$\hat{A}^0(\vec{0},\, \vec{0}) = \bar{A}^0 + N^0, \tag{3.16}$$

where $\bar{A}^0$ and N^0 enjoy the following properties:

$$|e^{\bar{A}^0 t} X^0| \leq |X^0| \quad \forall t > 0,\ X^0 \in \mathbb{R}^{n_c} \tag{3.17}$$

and

$$|N^0 X^0| \leq \frac{1}{M} |X^0|,\ X^0 \in \mathbb{R}^{n_c}. \tag{3.18}$$

We specify in the following how we chose the constant M.

Remark 3.3. If we apply the linear change of coordinates introduced in Lemma 3.2, then it is no more true that $X^0 = (\zeta,\, u_0)$ where $\{\zeta = \vec{0}\}$ is the manifold $\mathcal{Z}_0$ of equlibria for f^0. However, estimate (3.13) still holds, provided that we change if needed the value of the constant k_0 and we take, instead of $X^0(\tau)$, $\underline{Y}_0$ and $\zeta(0)$, their images trough the linear change of variables.

3.2 Uniformly stable manifold of an orbit

We are now ready to state the main result of this section.

Theorem 3.1. *Let Hypotheses 2 and 3 hold. If the constant δ in Hypothesis 2 is sufficiently small, then the following holds.*

Fix an orbit $Y^0(\tau) = \big(\vec{0},\, X^0(\tau)\big)$ of

$$\begin{cases} dX^-/d\tau = A^-(X^-,\, X^0)X^- \\ dX^0/d\tau = \hat{A}^0(X^-,\, X^0)X^0 \end{cases} \tag{3.19}$$

that lies on the center manifold and satisfies $|X^0(0)| < \delta$. Then we can define a uniformly stable manifold relative to $Y^0(\tau)$. This manifold is defined in the ball of radius δ and center at the origin, is parameterized by $\{X^0 = \vec{0}\}$ and is tangent to this subspace at the origin. Also, it is locally invariant for (3.19)*, meaning that if the initial datum lies on the manifold, then $\big(X^-(\tau),\, X^0(\tau)\big)$ belongs to the uniformly stable manifold for $|\tau|$ sufficiently small. Every orbit lying on the uniformly stable manifold relative to $Y^0(\tau)$ can be decomposed as*

$$X(\tau) = Y^0(\tau) + Y^-(\tau) + U^p(\tau), \tag{3.20}$$

where the components $Y^- = \big(X^-(\tau), \vec{0}\big)$ and $U^p(\tau)$ satisfy respectively

$$|X^-(\tau)| \leq k_- |X^-(0)| e^{-c\tau/2} \tag{3.21}$$

and

$$|U^p(\tau)| \leq k_p |\zeta(0)|\, |X^-(0)| e^{-c\tau/4}. \tag{3.22}$$

In (3.21) *and* (3.22)*, c, k_- and k_p are suitable constants. In particular, c is the same as in* (3.6)*.*

In (3.22), ζ is the component of $X^0 = (\zeta,\, u_0)$ according to the decomposition introduced in Section 3.1.2.

3.3 Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1, which is divided in several steps: in Section 3.3.1 we introduce the spaces of functions we use in the proof. In Section 3.3.2 we are concerned with the component $Y^-(\tau)$ in (3.20). In Section 3.3.3 we deal with the "perturbation" term $U^p(\tau)$ in (3.20). Both the components $Y^-(\tau)$ and $U^p(\tau)$ are obtained as fixed points of suitable contractions: in Section 3.3.4 we study their regularity by relying on Lemma 3.1. Finally, in Section 3.3.5 we conclude the proof of Theorem 3.1 by putting together all the considerations carried on in Sections 3.3.1. 3.3.2, 3.3.3 and 3.3.4.

We have to introduce some notations. Let $(\vec{0},\, X^0(\tau))$ be a given orbit as in the statement of Theorem 3.1, then we denote by $\underline{X}^0 = X^0(0)$ and by $\underline{Y}^0$ the corresponding projection, defined as in (3.12). Also, if we write $\underline{X}^0 = (\zeta(0),\, u_0(0))$ then we set

$$\underline{\zeta} = \zeta(0). \tag{3.23}$$

By definition, $X^0(\tau)$ is a solution of the Cauchy problem

$$\begin{cases} dX^0/d\tau = \hat{A}^0(\vec{0},\, X^0) \\ X^0(0) = \underline{X}^0 \end{cases} \tag{3.24}$$

3.3.1 Definition of the functional spaces

Let n_- denote the dimension of X^-. Also, n_c denotes the dimension of X^0, as in the statement of Lemma 3.2.

In the following we use the following Banach spaces of functions:

$$\mathcal{Y}^- = \Big\{ X^- \in \mathcal{C}^0\big([0, +\infty[, \mathbb{R}^{n_-}\big) : \|X^-\|_- < +\infty \Big\} \tag{3.25}$$

and

$$\mathcal{Y}^0 = \Big\{ X^0 \in \mathcal{C}^0\big([0, +\infty[, \mathbb{R}^{n_c}\big) : \|X^0\|_0 < +\infty \Big\}, \tag{3.26}$$

where the norms $\|\cdot\|_-$ and $\|\cdot\|_0$ are defined as follows:

$$\|X^-\|_- = \sup_\tau \big\{ e^{c\tau/2}|X^-(\tau)| \big\} \qquad \|X^0\|_0 = \sup_\tau \big\{ e^{-\varepsilon|\tau|}|X^0(\tau)| \big\} \tag{3.27}$$

The constants c and ε are the same as in (3.6) and (3.13) respectively. Also, we consider the following closed subsets of Y^- and Y^0:

$$\mathcal{Y}^-_\delta = \Big\{ X^- \in \mathcal{C}^0\big([0, +\infty[, \mathbb{R}^{n_-}\big) : \|X^-\|_- \leq k_-\delta. \Big\} \qquad \mathcal{Y}^0_\delta = \Big\{ X^0 \in \mathcal{C}^0\big([0, +\infty[, \mathbb{R}^{n_c}\big) : \|X^0\|_0 \leq k_0\delta. \Big\} \tag{3.28}$$

We specify in the following how to determine the exact value of the constant k_-, while the constant k_0 is the same as in (3.13). Also, note that the spaces $\mathcal{Y}^-_\delta$ and $\mathcal{Y}^0_\delta$ are equipped with the same norms $\|\cdot\|_-$ and $\|\cdot\|_0$ as $\mathcal{Y}^-$ and $\mathcal{Y}^0$ respectively.

We will also need the space of functions defined as follows. Let c be as (3.6) and let $a \in [0, c[$. Consider the space

$$\mathcal{Y}^p_a = \Big\{ (U^-, U^0) \in \mathcal{C}^0\big([0, +\infty[, \mathbb{R}^{n_c+n_-}\big) : \|X^-\|_{pert} < +\infty \Big\} \tag{3.29}$$

which depends on a because it is equipped with the norm

$$\|(U^-, U^0)\|_{pert} = \sup_\tau \Big\{ e^{(c+a)\tau/4}\Big[|(U^-(\tau)| + |U^0(\tau)|)\Big] \Big\}.$$

Also, we will use the closed subset

$$\mathcal{Y}^p_{\delta a} = \Big\{ (U^-, U^0) \in \mathcal{C}^0\big([0, +\infty[, \mathbb{R}^{n_c+n_-}\big) : \|(U^-, U^0)\|_{pert} \leq k_p\delta^2 \Big\}, \tag{3.30}$$

which is equipped with the same norm as $\mathcal{Y}^p$. We specify in the following how to determine the values of the constants k_p and a.

3.3.2 Analysis of the stable component

This step is devoted to the definition of the component $Y^-(\tau) = (X^-(\tau), \vec{0})$ in (3.20). Fix a vector $\underline{X}^- \in \mathbb{R}^{n_-}$ satisfying $|\underline{X}^-| < \delta$.

We define $X^-(\tau)$ as the solution of the Cauchy problem

$$\begin{cases} dX^-/d\tau = A^-(X^-, \underline{Y}^0)X^- \\ X^-(0) = \underline{X}^-, \end{cases} \tag{3.31}$$

where $\underline{Y}^0$ is given by (3.12). It is known that, for any fixed $\underline{Y}^0$ and $\underline{X}^-$, X^- can be obtained as the fixed point of the application

$$T^- : \mathcal{Y}^-_\delta \to \mathcal{Y}^-_\delta$$

defined by

$$T^-(X^-)[\tau] = e^{\bar{A}^-\tau}\underline{X}^- + \int_0^\tau e^{\bar{A}^-(\tau-s)}\Big[A^-\big(X^-(s), \underline{Y}^0\big) - \bar{A}^-\Big]X^-(s)ds \tag{3.32}$$

where $\bar{A}^- = A^-(\vec{0}, \vec{0})$. The space $\mathcal{Y}^-_\delta$ is defined in (3.28). More precisely, if the constant k_- in (3.28) satisfies $k_- \leq C_-$ and the constant δ is (3.28) is sufficiently small, then the map T^- takes values in $\mathcal{Y}^-_\delta$ and is indeed a contraction. Also, the fixed point satisfies

$$|X^-(\tau)| \leq k_- |\underline{X}^-| e^{-c\tau/2} \tag{3.33}$$

We are now interested in the differentiability of the fixed point with respect to $\underline{Y}^0$ and $\underline{X}^-$. To study it, we recall that the set $\mathcal{Z}_0 \in \mathbb{R}^N$ is given by $\mathcal{Z}_0 = \{(\underline{X}^-, \vec{0}, \underline{u}_0)\}$. We then regard T^- as an application

$$T^- : \mathcal{Z}_0 \times \mathcal{Y}^-_\delta \to \mathcal{Y}^-$$

and we verify that the hypotheses of Lemma 3.1 are satisfied. The space $\mathcal{Y}^-$ is defined by (3.25). The Frechét differential of T^- with respect to $(\underline{X}^-, \underline{Y}^0)$ is a linear map $\mathcal{T}^- \in \mathcal{L}(\mathcal{Z}_0, \mathcal{Y}^-)$ which takes the value

$$\mathcal{T}^-(h^-, h^0)[\tau] = e^{\bar{A}^-\tau}\underline{h}^- + \int_0^\tau e^{\bar{A}^-(\tau-s)}\Big[D_{\underline{Y}^0} A^-\big(X^-(s), \underline{Y}^0\big)[\underline{h}^0]\Big] X^-(s) ds$$

if evaluated at the point $(\underline{h}^-, \underline{h}^0) \in \mathcal{Z}_0$. In the previous expression, $D_{\underline{Y}^0} A^-\big(X^-(s), \underline{Y}^0\big)[\underline{h}^0]$ denotes the differential with respect $\underline{Y}^0$ of the function $A^-\big(X^-(s), \underline{Y}^0\big)$, the differential being applied to the vector $\underline{h}^0$. If $\underline{X}^- = \vec{0}$ then, no matter what $\underline{Y}^0$ is, the differential $\mathcal{T}^-$ maps $(\underline{h}^-, \underline{h}^0)$ into the function $e^{\bar{A}^-\tau}\underline{h}^-$.

The Frechét differential of T^- with respect to X^- is a linear map $\mathcal{S}^- \in \mathcal{L}(\mathcal{Y}^-, \mathcal{Y}^-)$ which takes the value

$$\mathcal{S}^-(h^-)[\tau] = \int_0^\tau e^{\bar{A}^-(\tau-s)}\Big\{\Big[A^-\big(X^-(s), \underline{Y}^0\big) - \bar{A}^-\Big]h^-(s) + \Big[D_{\underline{X}^-} A^-\big(X^-(s), \underline{Y}^0\big)[h^-(s)]\Big] X^-(s)\Big\} ds$$

if evaluated at the point $h^- \in \mathcal{Y}^-$. In the previous expression, $D_{X^-} A^-\big(X^-(s), \underline{Y}^0\big)[h^-(s)]$ denotes the differential with respect X^- of the function $A^-\big(X^-(s), \underline{Y}^0\big)$, the differential being applied to the vector $h^-(s)$.

Both $\mathcal{T}^-$ and $\mathcal{S}^-$ are continuous if viewed as maps from $\mathcal{Z}_0 \times \mathcal{Y}_\delta$ to $\mathcal{L}(\mathcal{Z}_0, \mathcal{Y}^-)$ and $\mathcal{L}(\mathcal{Y}^-, \mathcal{Y}^-)$ respectively. Thus, the hypotheses of Lemma 3.1 are verified and hence the application

$$\mathcal{Z}_0 \to \mathcal{Y}^-_\delta$$

which associates to $(\underline{X}^-, \underline{Y}^0)$ the fixed point of (3.32) is continuously differentiable (in the sense of Frechét). When both $\underline{X}^- = \vec{0}$ and $\underline{Y}^0 = \vec{0}$ the Frechét differential is the functional that maps $(\underline{h}^-, \underline{h}^0) \in \mathcal{Z}_0$ to the function $e^{\bar{A}^-\tau}\underline{h}^-$.

3.3.3 Analysis of the component of perturbation

This step is devoted to the definition the component $U^p(\tau)$ in (3.20). First, we apply the change of variables introduced in Lemma 3.2 and we get that the matrix $\hat{A}(X^-, X^0)$ in (2.2) satisfies

$$\hat{A}(\vec{0}, \vec{0}) = \bar{A}^0 + N^0,$$

where $\bar{A}^0$ and N^0 enjoy (3.17) and (3.18) respectively. By relying on Remark 3.3, we can still use the estimate (3.13).

We impose that $X(\tau) = Y^0(\tau) + Y^{st}(\tau) + U^p(\tau)$ is a solution of (3.7). We then write $U^p(\tau) = \big(U^-, U^0\big)^T$ and, subtracting (3.24) and (3.31) from (3.7), we get

$$\begin{cases} dU^-/d\tau = \bar{A}^- U^- + \Big[A^-(X^- + U^-, X^0 + U^0) - A^-(\vec{0}, \vec{0})\Big]U^- \\ \qquad + \Big[A^-(X^- + U^-, X^0 + U^0) - A^-(X^-, \underline{Y}^0)\Big]X^- \\ dU^0/d\tau = \bar{A}^0 U^0 + N^0 U^0 + \Big[\hat{A}^0(X^- + U^-, X^0 + U^0) - \hat{A}^0(\vec{0}, \vec{0})\Big]U^0 \\ \qquad + \Big[\hat{A}^0(X^- + U^-, X^0 + U^0) - \hat{A}^0(\vec{0}, X^0)\Big]X^0. \end{cases} \tag{3.34}$$

Here, $\bar{A}^- = A^-(\vec{0}, \vec{0})$.

Let $\mathcal{Y}^p_{\delta a}$ be the metric space (3.30) and consider the application T_p, defined for $(U^-,\, U^0) \in \mathcal{Y}^p_{\delta a}$ as follows:

$$
\begin{aligned}
T^1_p(U^-,\, U^0)[\tau] &= \int_0^\tau e^{\bar{A}^-(\tau-s)} \Big\{ \big[A^-\big(X^-(s)+U^-(s),\, X^0(s)+U^0(s)\big) - A^-\big(X^-(s),\, \underline{Y}^0\big)\big] X^-(s) \\
&\quad + \big[A^-\big(X^-(s)+U^-(s),\, X^0(s)+U^0(s)\big) - A^-\big(\vec{0},\, \vec{0}\big)\big] U^-(s) \Big\} ds \\
T^2_p(U^-,\, U^0)[\tau] &= \int_{+\infty}^\tau e^{\bar{A}^0(\tau-s)} \Big\{ \Big[N^0 + \hat{A}^0\big(X^-(s)+U^-(s),\, X^0(s)+U^0(s)\big) - \hat{A}^0\big(\vec{0},\, \vec{0}\big)\Big] U^0(s) \\
&\quad + \Big[\hat{A}^0\Big(X^-(s)+U^-(s),\, X^0(s)+U^0(s)\big) - \hat{A}^0(\vec{0},\, X^0(s)\big)\Big] X^0(s) \Big\} ds.
\end{aligned}
\tag{3.35}
$$

In the previous expression, X^- is the solution of (3.31) and X^0 is the solution of (3.24). We want to show that T_p maps $\mathcal{Y}^p_{\delta a}$ into itself, provided that δ is sufficiently small. We have

$$
\begin{aligned}
|T^1_p(U^-,\, U^0)[\tau]| &\le \int_0^\tau C_- e^{-c(\tau-s)} \Big\{ L\Big[|U^-(s)| + |U^0(s)| + |X^0(s) - \underline{Y}^0| \Big] |X^-(s)| \\
&\quad + L\Big[|X^-(s)| + |U^-(s)| + |X^0(s)| + |U^0(s)| \Big] |U^-(s)| \Big\} ds \\
&\le C_- e^{-c\tau} \int_0^\tau e^{cs} L \Big[2k_p \delta^2 e^{-s(c+a)/4} + k_0 |\underline{\zeta}| e^{\varepsilon s} \Big] k_- |\underline{X}^-| e^{-cs/2} \\
&\quad + C_- e^{-c\tau} \int_0^\tau e^{cs} L \Big[k_- |\underline{X}^-| e^{-cs/2} + 2k_p \delta^2 e^{-s(c+a)/4} + 2\delta \Big] k_p \delta^2 e^{-s(c+a)/4} ds \\
&\le \Big[\frac{8}{c-a} C_- L k_p k_- \delta \Big] \delta^2 e^{-\tau(3c+a)/4} + \frac{2}{c+2\varepsilon} L C_- k_0 k_- \delta^2 e^{-\tau(2c-4\varepsilon)/4} + \Big[\frac{4}{c-a} L C_- k_- k_p \delta \Big] \delta^2 e^{-\tau(3c+a)/4} \\
&\quad + \Big[\frac{4}{c-a} L C_- k_p \delta^2 \Big] k_p \delta^2 e^{-\tau(c+a)/2} + \Big[\frac{8}{3c-a} \delta \Big] L C_- k_p \delta^2 e^{-\tau(c+a)/4}.
\end{aligned}
\tag{3.36}
$$

In the previous expression, C_- is the same constant as in (3.6) and L is a Lipschitz constant for $A^-(X^-,\, X^0)$. The estimate (3.36) is obtained by using (3.13), (3.14), (3.33) and the fact that, since it belongs to $\mathcal{Y}^p_{\delta a}$, then $(U^-,\, U^0)$ satisfies

$$
|U^-(\tau)|,\ |U^0(\tau)| \le k_p \delta^2 e^{-\tau(c+a)/4}. \tag{3.37}
$$

Also, the term $\underline{\zeta}$ is the same as in (3.23) and we rely on the fact that $|\underline{X}^-|,\, |\underline{\zeta}| < \delta$.

We then use estimates (3.17), (3.18), (3.33), (3.37) and (3.14) to get

$$
\begin{aligned}
|T^2_p(U^-,\, U^0)[\tau]| &\le \int_{+\infty}^\tau |N^0 U^0(s)| + L\Big[|X^-(s)| + |U^-(s)| + |X^0(s)| + |U^0(s)| \Big] |U^0(s)| ds \\
&\quad + \int_{+\infty}^\tau L\Big[|X^-(s)| + |U^-(s)| + |U^0(s)| \Big] |X^0(s)| ds \\
&\le \int_{+\infty}^\tau \frac{1}{M} |U^0(s)| ds + L \int_{+\infty}^\tau \Big[k_- |\underline{X}^-| e^{-cs/2} + 2k_p \delta^2 e^{-s(c+a)/4} + 2\delta \Big] k_p \delta^2 e^{-s(c+a)/4} ds \\
&\quad + L k_- \int_{+\infty}^\tau |\underline{X}^-| e^{-cs/2} 2\delta ds + L \int_{+\infty}^\tau 2k_p \delta^2 e^{-s(c+a)/4} 2\delta ds \\
&\le \frac{4}{M(c+a)} k_p \delta^2 e^{-\tau(c+a)/4} + \frac{4Lk_-\delta}{3c+a} k_p \delta^2 e^{-\tau(3c+a)/4} + \frac{4Lk_p\delta^2}{c+a} k_p \delta^2 e^{-\tau(c+a)/2} + \frac{8L\delta}{c+a} k_p \delta^2 e^{-\tau(c+a)/4} \\
&\quad + \frac{4}{c} k_- L \delta^2 e^{-c\tau/2} + \frac{16L\delta}{c+a} k_p \delta^2 e^{-\tau(c+a)/4}
\end{aligned}
\tag{3.38}
$$

In the previous expression L denotes a Lipschitz constant of $\hat{A}^0(X^-,\, X^0)$. By combining (3.36) and (3.38) we get the following. Assume that the constant k_p in (3.30) is sufficiently large (namely, $k_p \ge 4Lk_-/c$).

Then for every $a \leq c - 4\varepsilon$ we can choose δ and M in such a way that T_p take values into $\mathcal{Y}^p_{\delta a}$. Also, estimates similar to (3.36) and (3.38) ensure that one can choose the constants in such a way that T_p is a strict contraction. As a remark, we point out that, the bigger is a, the smaller is δ.

We set $a = 12\varepsilon$ and we choose δ in such a way that T^p is a contraction from $\mathcal{Y}^p_{\delta 12\varepsilon}$ to itself. The constant ε is the same as in (3.10). However, in the following we regard T^p as a map $\mathcal{Y}^p_{\delta 0} \to \mathcal{Y}^p_{\delta 0}$, where $\mathcal{Y}^p_{\delta 0}$ is the space (3.30) obtained setting $a = 0$. In this way, we obtain that T^p is a contraction on $\mathcal{Y}^p_{\delta 0}$, but, due to our choice of δ, the fixed point automatically satisfies the sharper estimate

$$|U^-(\tau)|,\ |U^0(\tau)| \leq k_p \delta^2 e^{-\tau(c+12\varepsilon)/4}. \tag{3.39}$$

Also, in the definition of the space $\mathcal{Y}^p_{\delta 0}$ one can take $\delta^2 = |\underline{\zeta}|\,|\underline{X}^-|$ and hence

$$|U^-(\tau)|,\ |U^0(\tau)| \leq k_p e^{-\tau(c+12\varepsilon)/4} |\underline{\zeta}|\,|\underline{X}^-|, \tag{3.40}$$

where $\underline{X}^-$ is defined by (3.31). Also, to simplify the notations in the previous expression we denote by $\underline{\zeta}$ the point obtained applying the change of coordinates introduced in Lemma 3.2 to the vector $(\underline{\zeta},\, \vec{0})$ defined by (3.23).

3.3.4 Frechét differentiability of the component of perturbation

We are now concerned with the Frechét differentiability of the fixed point of the map T_p defined by (3.35). Since $T_p(U^-,\, U^0)$ depends on X^- and X^0, we regard T_p as a map

$$T : \mathcal{Y}^p_{\delta 0} \times \mathcal{Y} \to \mathcal{Y}^p_{\delta 0}. \tag{3.41}$$

In the previous expression, $\mathcal{Y} = \mathcal{Y}^- \times \mathcal{Y}^0$, where $\mathcal{Y}^-$ and $\mathcal{Y}^0$ are defined by (3.25) and (3.26) respectively. Note that by construction they satisfy $X^- \in \mathcal{Y}^-$ and $X^0 \in \mathcal{Y}^0$.

The proof of the differentiability relies on Lemma 3.1 (taking $Y = \mathcal{Y}$ and $X = \mathcal{Y}^p_{\delta 0}$). We thus verify that the hypotheses of Lemma 3.1 are satisfied.

To simplify the exposition, we write (3.35) as

$$\begin{aligned} T^1_p(U^-,\, U^0)[\tau] = & \int_0^\tau e^{\bar{A}^-(\tau - s)} \Big\{ F\Big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\Big) X^-(s) \\ & + G\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) U^-(s) \Big\} ds \\ T^2_p(U^-,\, U^0)[\tau] = & \int_{+\infty}^\tau e^{\bar{A}^0(\tau - s)} \Big\{ \Big[N^0 + H\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big)\Big] U^0(s) \\ & + L\Big(X^-(s) + U^-(s),\, X^0(s),\, U^0(s)\Big) X^0(s) \Big\} ds, \end{aligned} \tag{3.42}$$

where the functions F, G, H and L satisfy

$$F\Big(X^-(s),\, \vec{0},\, X^0(s),\, \vec{0}\Big) \equiv 0 \quad G\Big(\vec{0},\, \vec{0}\Big) \equiv \vec{0} \quad H\Big(\vec{0},\, \vec{0}\Big) \equiv \vec{0} \quad L\Big(\vec{0},\, X^0,\, \vec{0}\Big) \equiv \vec{0}. \tag{3.43}$$

Note that $X^0(s) \equiv \underline{Y}^0$ is an equilibrium for (3.24).

Relying on (3.39), one can show that the condition (3.2) is verified here, so applying Remark 3.1 we get that the fixed point $(U^-,\, U^0)$ is Lipschitz continuous with respect to $(X^0,\, X^-)$.

Concerning the Frechét differentiability of T_p with respect to $(X^0,\, X^-)$, we proceed as follows. Fix an element $(U^0,\, U^-,\, X^0,\, X^-) \in \mathcal{Y}^p_{\delta 0} \times \mathcal{Y}$ satisfying the estimates (3.13), (3.14), (3.33) and (3.39). The Frechét differential of T_p with respect to $(X^-,\, X^0)$ computed at the point $(U^0,\, U^-,\, X^0,\, X^-)$ is a linear map $\mathcal{T} \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}^p_0)$. The image of the element $(h^-, h^0) \in \mathcal{Y} = \mathcal{Y}^- \times \mathcal{Y}^0$ is given by

$$
\begin{aligned}
\mathcal{T}^1_p(h^-, h^0)[\tau] = \int_0^\tau & e^{\bar{A}^-(\tau-s)} \Big\{ F\Big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\Big) h^-(s) \\
&+ \Big[D_{X^-} F\Big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\Big) h^-(s) \Big] X^-(s) \\
&+ \Big[D_{X^-} G\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^-(s) \Big] U^-(s) \\
&+ \Big[D_{X^0} F\Big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\Big) h^0(s) \Big] X^-(s) \\
&+ \Big[D_{X^0} G\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^0(s) \Big] U^-(s) \Big\} ds \\
\mathcal{T}^2_p(h^-, h^0)[\tau] = \int_{+\infty}^\tau & e^{\bar{A}^0(\tau-s)} \Big\{ \Big[D_{X^-} H\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^-(s) \Big] U^0(s) \\
&+ \Big[D_{X^-} L\Big(X^-(s) + U^-(s),\, X^0(s),\, U^0(s)\Big) h^-(s) \Big] X^0(s) \\
&+ \Big[D_{X^0} H\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^0(s) \Big] U^0(s) \\
&+ L\Big(X^-(s) + U^-(s),\, X^0(s),\, U^0(s)\Big) h^0(s) \\
&+ \Big[D_{X^0} L\Big(X^-(s) + U^-(s),\, X^0(s),\, U^0(s)\Big) h^0(s) \Big] X^0(s) \Big\} ds.
\end{aligned}
\tag{3.44}
$$

In the previous expression, $\big[D_{X^-} F\big(X^-(s),\ U^-(s),\, X^0(s),\ U^0(s)\big) h^-(s)\big]$ denotes the differential of the matrix valued function F with respect to the variable X^-. The differential is computed at the point $\big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\big)$ and is applied to the vector $h^-(s)$. To prove that indeed

$$\big(\mathcal{T}^1(h^-, h^0),\, \mathcal{T}^2(h^-, h^0)\big) \in \mathcal{Y}^p_0$$

one uses the estimate (3.39) and the identity $L(\vec{0}, X^0, \vec{0}) \equiv \vec{0}$.

We now discuss the Frechét differentiability of T_p with respect to (U^0, U^-). Fix an element $(U^0, U^-, X^0, X^-) \in \mathcal{Y}^p_{\delta 0} \times \mathcal{Y}$. The Frechét differential of T_p with respect to (U^0, U^-), evaluated at the point (U^0, U^-, X^0, X^-), is a linear map $\mathcal{S} \in \mathcal{L}(\mathcal{Y}^p_0, \mathcal{Y}^p_0)$ and the image of the element $(h^-, h^0) \in \mathcal{Y}^p_0$ is given by

$$
\begin{aligned}
\mathcal{S}^1(h^-, h^0)[\tau] = \int_0^\tau & e^{\bar{A}^-(\tau-s)} \Big\{ \Big[D_{U^-} F\Big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\Big) h^-(s) \Big] X^-(s) \\
&+ G\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^-(s) \\
&+ \Big[D_{U^-} G\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^-(s) \Big] U^-(s) \\
&+ \Big[D_{U^0} F\Big(X^-(s),\, U^-(s),\, X^0(s),\, U^0(s)\Big) h^0(s) \Big] X^-(s) \\
&+ \Big[D_{U^0} G\Big(X^-(s),\, U^-(s),\, X^0(s) + U^0(s)\Big) h^0(s) \Big] U^-(s) \Big\} ds \\
\mathcal{S}^2(U^-, U^0)[\tau] = \int_{+\infty}^\tau & e^{\bar{A}^0(\tau-s)} \Big\{ \Big[D_{U^-} H\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^-(s) \Big] U^0(s) \\
&+ \Big[D_{U^-} L\Big(X^-(s) + U^-(s),\, X^0(s),\, U^0(s)\Big) h^-(s) \Big] X^0(s) \\
&+ N^0 h^0(s) + H\Big(X^-(s),\, U^-(s),\, X^0(s) + U^0(s)\Big) h^0(s) \\
&+ \Big[D_{U^0} H\Big(X^-(s) + U^-(s),\, X^0(s) + U^0(s)\Big) h^0(s) \Big] U^0(s) \\
&+ \Big[D_{U^0} L\Big(X^-(s) + U^-(s),\, X^0(s),\, U^0(s)\Big) h^0(s) \Big] X^0(s) \Big\} ds.
\end{aligned}
\tag{3.45}
$$

One can verify that, if $(U^0, U^-, X^0, X^-) \in \mathcal{Y}^p_{\delta 0} \times \mathcal{Y}$, then indeed $\mathcal{S}(h^-, h^0) \in \mathcal{Y}^p_0$. Also, $\mathcal{S}$ is continuous as a map from $X^p \times \mathcal{Y}$ in $\mathcal{L}(\mathcal{Y}^p_0, \mathcal{Y}^p_0)$.

This shows that the hypotheses of Lemma 3.1 are all verified.

3.3.5 Conclusion

Applying Lemma 3.1, we get that the map

$$\mathcal{Y} \to \mathcal{Y}^p_{\delta 0} \tag{3.46}$$

that associates to $(X^-,\ X^0)$ the fixed point of (3.34) is Frechét differentiable and that its differential evaluated at the point $X^-(\tau) \equiv 0$ and $X^0(\tau) \equiv \vec{0}$ is the functional that associates to $(h^-, h^0) \in \mathcal{Y}$ the functions $U^-(\tau) \equiv \vec{0}$, $U^0(\tau) \equiv \vec{0}$. We then perform the linear change of variables which is the inverse of the change of variables introduced in Lemma 3.2. In this way, we go back to the original variables. To simplify the notations, we still denote by $\Big(U^-(\tau),\ U^0(\tau)\Big)$ the functions obtained applying the change of variables.

To define the map that parameterizes the uniformly stable manifold we proceed as follows: the orbit $X^0(\tau)$ is fixed. For every $\underline{X} \in \mathbb{R}^{n-}$, there exists a unique solution of (3.31). Also, in Section 3.3.2 we showed that the map

$$\underline{X} \to X^-(\tau) \tag{3.47}$$

is continuously differentiable in the sense of Frechét. As a consequence, the map obtained composing (3.47) and (3.46) is Frechét differentiable. Note that such a map associates to $\underline{X}^-$ the functions (X^-, U^-, U^0). The function ϕ that parameterizes the uniformly stable manifold is then defined by setting

$$\phi(\underline{X}) = \Big(X^-(0),\ X^0(0) + U^0(0)\Big).$$

Due to the previous considerations, ϕ is continuously differentiable and the manifold is tangent to the stable space $\big\{(X^-, \vec{0}):\ X^- \in \mathbb{R}^-\big\}$ at the origin. Finally, estimate (3.22) is a consequence of (3.40).

This concludes the proof of Theorem 3.1.

3.4 Uniformly stable manifolds

Let $\mathcal{V}^0$ be a given center manifold for (2.2) and assume that Hypotheses 2 and 3 in Section 2 are satisfied. In Theorem 3.1 we consider a fixed orbit lying on $\mathcal{V}^0$ and we construct the uniformly stable manifold relative to that orbit. In this section we discuss what happens if, instead of having a single orbit, we have a whole invariant manifold.

More precisely, let $\mathcal{S}_0$ be an invariant manifold for (3.7) and assume that $\mathcal{S}_0$ is entirely contained in the center manifold $\{X^- = \vec{0}\}$. Also, denote by S^0 the tangent space to $\mathcal{S}_0$ at the origin. Choosing a sufficiently small constant in Hypothesis 2, we can assume that $\mathcal{S}_0$ is parameterized by S^0. By construction, S^0 is contained in $\{X^- = \vec{0}\}$. Also, as in Section 3.1.3 assume that $\mathcal{Z}_0 = \{(X^-, \vec{0}, u_0) : \zeta = \vec{0}\}$ is a manifold of zeroes for the function f_0 in (3.4).

As a consequence of Theorem 3.1, we get the following result:

Proposition 3.1. *Let Hypotheses 2 and 3 hold. Let $\mathcal{S}_0$ be an invariant manifold for* (3.7) *and assume that it is entirely contained in the center manifold $\{X^- = \vec{0}\}$. If the constant δ in Hypothesis 2 is sufficiently small, then the following holds.*

There exists a continuously differentiable manifold $\mathcal{M}^{us}_{\mathcal{S}_0}$ which is defined in the ball of radius δ and center at the origin. Also, $\mathcal{M}^{us}_{\mathcal{S}_0}$ satisfies the following properties:

1. *$\mathcal{M}^{us}_{\mathcal{S}_0}$ is locally invariant for* (3.7)*, meaning that if the initial datum lies on the manifold, then the solution $\big((X^-(\tau),\ X^0(\tau)\big)$ of* (3.7) *lies on $\mathcal{M}^{us}_{\mathcal{S}_0}$ for $|\tau|$ sufficiently small.*

2. *$\mathcal{M}^{us}_{\mathcal{S}_0}$ is parameterized by $\mathrm{S}^0 \times V^-$ and it is tangent to this space at the origin. Here, S^0 is the tangent space to $\mathcal{S}^0$ at the origin and $V^- = \{(X^-, \vec{0}):\ X^0 = \vec{0}\}$.*

3. *Any orbit $Y(\tau)$ lying on $\mathcal{M}^{us}_{\mathcal{S}_0}$ can be decomposed as*

$$Y(\tau) = Y^0(\tau) + Y^-(\tau) + Y^p(\tau), \tag{3.48}$$

where $Y^0(\tau) = \big(\vec{0}, \zeta^0(\tau), u_0(\tau)\big)$ is an orbit lying on $\mathcal{S}_0$. The component $Y^-(\tau) = \big(X^-(\tau), \vec{0}, \vec{0}\big)$ lies on the stable manifold and the perturbation term $Y^p(\tau)$ satisfies

$$|Y^p(\tau)| \leq C|\zeta^0(0)|\,|Y^-(0)|e^{-c\tau/4}, \tag{3.49}$$

for some positive constant C. The constant $c > 0$ in (3.49) *is the same as in* (3.6).

In the following we call $\mathcal{M}^{us}_{\mathcal{S}_0}$ the *uniformly stable manifold relative to* $\mathcal{S}_0$.

Proof. Let $\big(\vec{0},\, X^0(\tau)\big)$ and $\big(X^-(\tau),\, \vec{0}\big)$ be two orbits of (3.7) lying on the center manifold $\{X^- = \vec{0}\}$ and on the stable manifold respectively. We then have $X^0(\tau) \in \mathcal{Y}^0_\delta$, $X^-(\tau) \in \mathcal{Y}^-_\delta$, where the metric spaces $\mathcal{Y}^0_\delta$ and $\mathcal{Y}^-_\delta$ are defined by (3.28). As in Section 3.3, we use the notation $\mathcal{Y} = \mathcal{Y}^-_\delta \times \mathcal{Y}^0_\delta$. Consider the map

$$\Phi : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}^p_{\delta 0}$$

which associates to $X^-(\tau)$ and $X^0(\tau)$ the function $\big(X^-(\tau),\, X^0(\tau),\, U^-(\tau),\, U^0(\tau)\big)$, where $(U^-,\, U^0)$ is the perturbation term constructed in Section 3.3.3. We recall that $\mathcal{Y}^p_{\delta 0}$ is the set obtained setting $a = 0$ in (3.30). As shown in Section 3.3.4, the map Φ is continuously differentiable in the sense of Frechét. Also, let

$$f^- : \{X^0 = \vec{0}\} \times \{X^- = \vec{0}\} \to \mathcal{Y}^-_\delta$$

be the map that associates to $(\underline{X}^-,\, \underline{\zeta},\, \underline{u}^0) \in \mathcal{Z}_0$ the unique solution of the Cauchy problem (3.31) (we recall that $\underline{Y}_0 = (\vec{0},\, \underline{u}_0)$ in (3.49)). The map f^- is continuously differentiable in the sense of Frechét, as it is shown in Section 3.3.2. Also, let

$$f^0 : \{X^- = \vec{0}\} \to \mathcal{Y}^0_\delta$$

be the map that associates to $(\underline{X}^0,\, \vec{0})$ the unique solution of the Cauchy problem (3.24). The map f^0 is also continuously differentiable in the sense of Frechét. Finally, let

$$g^0 : \mathrm{S}^0 \to V^0$$

be a continuously differentiable parameterization of $\mathcal{S}_0$. We define the map

$$\psi : \mathrm{S}^0 \times V^0 \to \mathcal{Y} \times \mathcal{Y}^p_{\delta 0}$$

setting

$$\psi(\underline{X}^0,\, \underline{X}^-) = \Phi\Bigg(f^-\Big(\underline{X}^-,\, g(\underline{X}^0)\Big),\, f^0 \circ g^0(\underline{X}^0) \Bigg). \tag{3.50}$$

The map ψ is then continuously differentiable in the sense of Frechét. By construction, $\psi(\underline{X}^0,\, \underline{X}^-)$ is an element in the form $\big(X^-(\tau),\, X^0(\tau),\, U^-(\tau),\, U^0(\tau)\big)$ and, setting

$$Y(\tau) = \Big(X^-(\tau) + U^-(\tau),\; X^0(\tau) + U^0(\tau)\Big),$$

we get that $Y(\tau)$ can be decomposed as in (3.48). Also, the perturbation term $\big(U^0,\, U^-\big)$ automatically satisfies (3.49). We then define the map

$$\psi_0 : \mathrm{S}^0 \times V^- \to \mathbb{R}^{n_c + n_-}$$

parameterizing $\mathcal{M}^{us}_{\mathcal{S}_0}$ by setting

$$\psi_0(\underline{X}^0,\, \underline{X}^-) = \Big(X^-(0),\; X^0(0) + U^0(0)\Big) = Y(0),$$

where $X^0(\tau)$, $X^-(\tau)$ and $U^0(\tau)$ are given by (3.50).

The map ψ_0 is continuously differentiable, being the composition of maps that are continuously differentiable in the sense of Frechét. Also, by construction the manifold $\mathcal{M}^{us}_{\mathcal{S}_0}$ is invariant for (3.7). To prove that the manifold $\mathcal{M}^{us}_{\mathcal{S}_0}$ is tangent to $\mathrm{S}^0 \times V^-$ at the origin it is enough to observe that the Frechét differential of f^- at $\underline{X}^- = \vec{0}$ is the functional $\underline{h}_- \mapsto e^{\bar{A}^- \tau} \underline{h}^-$, while the Frechét differential of f^0 at $\underline{X}^0 = \vec{0}$ is the functional $\underline{h}_0 \mapsto e^{\bar{A}^0 \tau} \underline{h}^0$. □

4 Invariant manifolds for a singular ODE

In the present section we focus on the analysis of the singular ODE (1.1) and we use the tools introduced in Section 3 to extend to the general nonlinear case the analysis done in Section 1.1 in the linear case.

The exposition is organized as follows: we proceed by following Steps 1,...,4 that are outlined in the second part of Section 1.1. In Section 4.1 we introduce the same change of variables as in Step 1 and then we introduce a further change of variables which allows to write system (2.2) in a nicer form. The details concerning this change of variables are actually carried on in Section 4.4. In Section 4.2 we define the manifolds of the slow and the fast dynamics and hence we follow Steps 3 and 4 respectively. Finally, in Section 4.3 we focus on Step 4 and we extend to the general nonlinear case the definition of *uniformly stable space* given in (1.14). We also extend to the case of the singular ODE (1.1) the definition of center-stable manifold.

4.1 Changes of variables

Let us introduce the change of variables $\tau(t)$ defined by the Cauchy problem (1.6), which transforms system (1.1) into (1.5). However, in the nonlinear case the change of variable is not *a priori* well defined since the function $\zeta[U(t)]$ could in principle attain the value 0. Hence we proceed as follows: as in the linear case we carry on the analysis by referring to system (1.5) and then we show that *a posteriori* the change of variables (1.6) is well defined.

In this section we also introduce Proposition 4.1, which loosely speaking states that in a small enough neighborhood of $\vec{0}$ we can define a further change of variables which allows to write system (2.2) in a nicer form. Before giving the precise statement we have to introduce some notations. Let N denote the dimension of U. Also, n_- is the number of eigenvalues of $DF(\vec{0})$ with strictly negative real part, while (n_0+1) is the number of eigenvalues of $DF(\vec{0})$ with zero real part. Each eigenvalue is counted according to its multiplicity. Due to Hypothesis 3, $N = n_- + n_0 + 1$.

Proposition 4.1. *Let Hypotheses 1 ... 8 hold. If the constant δ in Hypothesis 2 is sufficiently small, then in the ball with radius δ and center at the origin we can define a continuously differentiable diffeomorphism Υ satisfying the following properties. Write $\Upsilon(U) = \tilde{U}$ as a column vector:*

$$\tilde{U} = \begin{pmatrix} \zeta \\ u_0 \\ u_- \end{pmatrix},$$

where $\zeta \in \mathbb{R}$, $u_0 \in \mathbb{R}^{n_0}$ and $u_- \in \mathbb{R}^{n_-}$. If U satisfies (2.2), then $\tilde{U}$ satisfies

$$\begin{cases} d\zeta/d\tau = G_{10}(\zeta,\, u_0)u_0\zeta^2 + G_{1-}(\zeta,\, u_0,\, u_-)u_-\zeta \\ du_0/d\tau = \Big\{G_{01}(\zeta,\, u_0) + \big[G_{0-}(\zeta,\, u_0,\, u_-) - G_{0-}(\zeta,\, u_0,\, \vec{0})\big]\Big\}\zeta u_0 \\ du_-/d\tau = G_s(\zeta,\, u_0,\, u_-)u_- \end{cases} \tag{4.1}$$

In the previous expression, G_{10} is a row vector belonging to $\mathbb{R}^{n_0}$, G_{1-} is a row vector in $\mathbb{R}^{n_-}$, the matrices G_{01} and G_{0-} belong to $\mathbb{M}^{n_0 \times n_0}$ and the matrix G_s belongs to $\mathbb{M}^{n_- \times n_-}$.

A center manifold of system (4.1) is the subspace $\{(\zeta,\, u_0,\, \vec{0}) :\ u_- = \vec{0}\}$, the stable manifold is the subspace $\{(0,\, \vec{0},\, u_-) :\ \zeta = 0,\ u_0 = \vec{0}\}$. Let $\mathcal{M}^{us}_E$ be the uniformly stable manifold relative to the manifold $E = \{(\zeta,\, \vec{0},\, \vec{0}) :\ u_0 = \vec{0},\ u_- = \vec{0}\}$, which is entirely constituted by equilibria. Then $\mathcal{M}^{us}_E = \{(\zeta,\, \vec{0},\, u_-) :\ u_0 = \vec{0}\}$.

In the statement of Proposition 4.1 by *uniformly stable manifold relative to E* we mean the manifold defined by Proposition 3.1. Also, note that by construction all the eigenvalues of the matrix $G_s(0,\, \vec{0},\, \vec{0})$ have strictly negative real part.

4.2 Slow and fast dynamics

Let E denote, as before, the manifold of equilibria $\{(\zeta, \vec{0}, \vec{0}): \ u_0 = \vec{0}, \ u_- = \vec{0}\}$.

Definition 4.1. A manifold of slow dynamics is a center manifold of (4.1). In the following we fix the manifold of the slow dynamics $\{u_- = \vec{0}\}$ and we denote it by $\mathcal{M}^0$.

The manifold of fast dynamics of system (4.1) is the uniformly stable manifold relative to E, namely the subspace $\{u_0 = \vec{0}\}$.

Note that both these manifolds are invariant for system (4.1). Also, for every point $(\underline{\zeta}, \vec{0}, \underline{u}_-)$ belonging to the manifold of fast dynamics, denote by $\big(\zeta(\tau), \vec{0}, u_-(\tau)\big)$ the solution of (4.1) such that

$$\big(\zeta(0), \vec{0}, u_-(0)\big) = (\underline{\zeta}, \vec{0}, \underline{u}_-).$$

Combining (3.48) and (3.49) we get that this solution decays exponentially fast to an equilibrium point. Namely, there exists ζ_∞ such that

$$\lim_{\tau \to +\infty} e^{c\tau/4} |u_-(\tau)| = 0 = \lim_{\tau \to +\infty} e^{c\tau/4} |\zeta(\tau) - \zeta_\infty|,$$

where the positive constant c satisfies $Re\lambda < -c$ for every λ eigenvalue of $G_s(0, \vec{0}, \vec{0})$.

We now consider system (4.1) restricted on the manifold of slow dynamics, namely

$$\begin{cases} d\zeta/d\tau = \zeta^2 G_{10}(\zeta, u_0) u_0 \\ du_0/d\tau = G_{01}(\zeta, u_0) u_0 \zeta \\ u_- \equiv 0. \end{cases} \tag{4.2}$$

If one goes back to the original variable t obtains

$$\begin{cases} d\zeta/dt = \zeta G_{10}(\zeta, u_0) u_0 \\ du_0/dt = G_{01}(\zeta, u_0) u_0 \\ u_- \equiv 0, \end{cases} \tag{4.3}$$

namely an equation with no singularity. Note that (4.2) and (4.3) are equivalent. Indeed, by the uniqueness of the solution of a Cauchy problem associated to (4.3), the following holds. If $\zeta(0) > 0$ then $\zeta(t) > 0$ for every t. Thus, the Cauchy problem (1.6) admits a global solution $\tau : [0, +\infty[\to [0, +\infty[$ whose derivative is always different from 0. Thus, $\tau(t)$ defines a change of variables and (4.2) is equivalent to (4.3).

One of our original goals is to study the solutions of (1.1) that lie on a center manifold. Let $\mathcal{M}^{00}$ be a center manifold for (4.3) of the equilibrium point $(0, \vec{0}, \vec{0})$. Then $\mathcal{M}^{00}$ is a center manifold of

$$\begin{cases} \dfrac{d\zeta}{dt} = \zeta G_{10}(\zeta, u_0) u_0 + G_{1-}(\zeta, u_0, u_-) u_- \\[2ex] \dfrac{du_0}{dt} = \Big\{ G_{01}(\zeta, u_0) + \big[G_{01}(\zeta, u_0, u_-) - G_{0-}(\zeta, u_0, \vec{0})\big] \Big\} u_0 \\[2ex] \dfrac{du_-}{dt} = \dfrac{1}{\zeta} G_s(\zeta, u_0, u_-) u_- \end{cases} \tag{4.4}$$

We collect these results in the following theorem.

Theorem 4.1. *Assume that Hypotheses 1 ... 8 are satisfied. There exists an invariant center manifold $\mathcal{M}^{00}$ of the equilibrium point $(0, \vec{0}, \vec{0})$ of system (4.4). The manifold $\mathcal{M}00$ is contained in the manifold of the slow dynamics, equation (4.4) restricted to $\mathcal{M}^{00}$ is non singular and every solution satisfies the following property: if $\zeta(0) > 0$, then $\zeta(t) > 0$ for every t.*

Remark 4.1. Hypothesis 8 ensures that the manifold $\{U : \zeta(U) = 0\}$ is invariant with respect to the slow dynamics. This hypothesis is not necessary to define an invariant center manifold $\mathcal{M}^{00}$ contained in the manifold of the slow dynamics. However, it is necessary if we want that (4.2) is equivalent to (4.3), namely that the change of variables defined by (1.6) is well defined. To see this, we can proceed as follows.

Given equation (2.2), we proceed as in the proof of Lemma 4.2 and we use Hypotheses 1 … 7 but we do *not* use Hypothesis 8. The system we eventually get, restricted on the subspace $\{u_- = \vec{0}\}$, is

$$\begin{cases} d\zeta/d\tau = \zeta g_1(\zeta,\, u_0,\, \vec{0}) \\ du_0/d\tau = G_{01}(\zeta,\, u_0)u_0\zeta \\ u_- \equiv 0 \end{cases} \tag{4.5}$$

where g_1 is the same function as in (4.20) and satisfies

$$g_1(z,\, \vec{0},\, \vec{0}) = 0 \quad \forall\, z.$$

Going back to the original variable t, (4.5) becomes

$$\begin{cases} d\zeta/d\tau = g_1(\zeta,\, u_0,\, \vec{0}) \\ du_0/d\tau = G_{01}(\zeta,\, u_0)u_0 \\ u_- \equiv 0 \end{cases} \tag{4.6}$$

Thus, even if we do *not* assume Hypothesis 8, the ODE (1.1) restricted on the manifold of the slow dynamics $\{u_- = \vec{0}\}$ is non singular. Also, one can define an invariant center manifold $\mathcal{M}^{00}$ which contains only slow dynamics.

Note, however, that if Hypothesis 8 is not satisfied it may happen that for a solution U lying on $\mathcal{M}^{00}$ $\zeta\big(U(0)\big) > 0$ but $\zeta(U)$ touches 0 in finite time. An example is the following.

Consider the equation

$$\begin{cases} du_1/dt = -u_2 \\ du_2/dt = u_2^2(1-u_2) \\ du_3/dt = -u_3/u_1 \end{cases}$$

and set

$$\zeta(U) = u_1 \qquad F(U) = \Bigg(-u_1u_2,\; u_1u_2^2(1-u_2),\; -u_3 \Bigg)^T.$$

Then Hypotheses 1, 3 … 7 are satisfied, but Hypothesis 8 is violated. The manifold of slow dynamics is the subspace $\{u_3 = 0\}$ and it coincides with the center manifold $\mathcal{M}^{00}$. Restrict to this subspace and consider the equation

$$du_2/dt = u_2^2(1-u_2).$$

If $0 < u_2(0) < 1$, then $0 < u_2(t) < 1$ for every t. Also, $du_2/dt > 0$ for every t and hence $u_2(0) < u_2(t) < 1$ for every t. Since

$$du_1/dt = -u_2,$$

then by a comparison argument $u_1(t) \le u_1(0) - u_2(0)t$ for every $t > 0$. In other words, if $u_1(0) > 0$ then $u_1(t)$ attains the value 0 for some $t \le u_1(0)/u_2(0)$.

4.3 Applications of the uniformly stable manifold to the analysis of a singular ordinary differential equation

First, we recall a result we need in the following.

Lemma 4.1. *Let $\zeta(\tau)$ be a real valued, continuous and bounded function satisfying $\zeta(\tau) > 0$ for every $\tau \in [0, +\infty[$. Let $t(\tau)$ be the maximal solution of the forward Cauchy problem*

$$\begin{cases} dt/d\tau = \zeta(\tau) \\ t(0) = 0. \end{cases} \tag{4.7}$$

Then $t(\tau)$ is defined on the whole interval $[0, +\infty[$. Also, the following statements are equivalent:

1. $t(\tau)$ *is a continuously differentiable diffeomorphism* $t: [0, +\infty[\to [0, +\infty[$.

2. $\displaystyle\int_0^{+\infty} \zeta(\tau) d\tau = +\infty.$

Condition 2 guarantees, in particular, that the inverse map $\tau(t)$ is defined on the whole interval $[0, +\infty[$ and that it is continuously differentiable there. Also, note that $\zeta(t) = \zeta\big(\tau(t)\big)$ is automatically strictly bigger than 0 for every t.

Before stating the main result in this section we introduce some notations. As before, $c > 0$ denotes a positive constant satisfying $Re\lambda < -c$ for any λ which is either an eigenvalue of $G_s(0, \vec{0}, \vec{0})$ or an eigenvalue with strictly negative real part of of $G_{01}(0, \vec{0})$. We denote by V^{0-} the subspace

$$V^{0-} = \{(0, \vec{\xi}, \vec{0})\},$$

where $\vec{\xi} \in \mathbb{R}^{n_0}$ belongs to the eigenspace of $G_{10}(0, \vec{0})$ associated to the eigenvalues with strictly negative real part. Also,

$$V^{00-} = \{(0, \vec{\xi}, \vec{0})\},$$

where $\vec{\xi} \in \mathbb{R}^{n_0}$ belongs to the eigenspace of $G_{10}(0, \vec{0})$ associated to the eigenvalues with non positive real part. Clearly, $V^{0-} \subseteq V^{00-}$. We denote by V^{--} the stable manifold, namely

$$V^{--} = \{(0, \vec{0}, u_-) : \ u_- \in \mathbb{R}^{n-}\},$$

Finally, as in Section 4.2 we denote by E the manifold of equilibria $\{(\zeta, \vec{0}, \vec{0}) : \ \zeta \in \mathbb{R}\}$.

We are now read to state our main result.

Theorem 4.2. *Let Hypotheses 1 ... 8 hold. If the constant* δ *in Hypothesis 2 is sufficiently small, then in the ball with radius* δ *and center at the origin one can define two manifolds,* $\mathcal{M}^s$ *and* $\mathcal{M}^{cs}$*, satisfying the following properties:*

1. *both* $\mathcal{M}^s$ *and* $\mathcal{M}^{cs}$ *are locally invariant for* (4.1)*, namely, if the initial datum lies on the manifold, then the solution* $\big((\zeta(\tau), u_0(\tau), u_-(\tau)\big)$ *of* (4.1) *also lies on the manifold for* $|\tau|$ *sufficiently small.*

2. $\mathcal{M}^s$ *is contained in* $\mathcal{M}^{cs}$.

3. $\mathcal{M}^s$ *is parameterized by* $E \oplus V^{0-} \oplus V^{--}$ *and it is tangent to this subspace at the origin. Also,* $\mathcal{M}^{cs}$ *is parameterized by* $E \oplus V^{00-} \oplus V^{--}$ *and it is tangent to this subspace at the origin.*

4. *Let* $U(\tau) = \Big(\zeta(\tau),\ u_0(\tau),\ u_-(\tau)\Big)$ *be an orbit lying either on* $\mathcal{M}^s$ *or on* $\mathcal{M}^{cs}$ *and satisfying* $\zeta(0) > 0$. *Then the maximal solution of the forward Cauchy problem*

$$\begin{cases} dt/d\tau = \zeta(\tau) \\ t(0) = 0 \end{cases} \tag{4.8}$$

defines a continuously differentiable diffeomorphism $t: [0, +\infty[\to [0, +\infty[$. *Let* $\tau(t)$ *denotes its inverse. Then the function* $U(t) = U\big(\tau(t)\big)$ *is a solution of* (1.1) *and satisfies* $\zeta(t) > 0$ *for every* $t \geq 0$.

5. *Any orbit lying on* $\mathcal{M}^s$ *can be decomposed as*

$$U(\tau) = U^-(\tau) + U^{sl}(\tau) + U^p(\tau), \tag{4.9}$$

where $U^-(\tau)$ *satisfies*

$$|U^-(\tau)| \leq k_- e^{-c\tau/2} |U^-(0)| \tag{4.10}$$

for a suitable constant k_-. *Conversely, the component* $U^{sl}(\tau) = \big(\zeta^{sl}(\tau),\ u_0^{sl}(\tau),\ \vec{0}\big)$ *lies on the manifold of the slow dynamics. Also, if we use the variable* t *defined as the maximal solution of the Cauchy problem* (4.8)*, we have that there exists a point* $(\zeta_\infty, \vec{0})$ *such that*

$$\lim_{t \to +\infty} \big(|\zeta(t) - \zeta_\infty| + |u_0(t)|\big) e^{ct/2} = 0. \tag{4.11}$$

Finally, the perturbation term is small in the sense that

$$|U^p(\tau)| \le k_p|\zeta^{sl}(0)||U^-(0)|e^{-c\tau/4} \tag{4.12}$$

for a suitable constant $k_p > 0$.

6. *Any orbit* $U(\tau)$ *lying on* $\mathcal{M}^{cs}$ *can be decomposed as*

$$U(\tau) = U^-(\tau) + U^{sl}(\tau) + U^p(\tau), \tag{4.13}$$

where U^- *and* U^p *satisfy* $|U^-(\tau)| \le k_- e^{-c\tau/2}|U^-(0)|$ *and* $|U^p(\tau)| \le k_p|\zeta^{sl}(0)||U^-(0)|e^{-c\tau/4}$ *respectively. Here* k_- *and* k_p *denote the same constants as in* (4.10) *and* (4.12). *The component* $U^{sl}(\tau) = \big(\zeta^{sl}(\tau),\, u^{sl}(\tau),\, \vec{0}\big)$ *lies on the manifold of the slow dynamics. More precisely, the following holds. Consider the maximal solution of the Cauchy problem*

$$\begin{cases} dt/d\tau = \zeta^{sl}(\tau) \\ t(0) = 0 \end{cases}$$

and set $\zeta^{sl}(t) = \zeta^{sl}\big(\tau(t)\big)$ *and* $u^{sl}(t) = u^{sl}\big(\tau(t)\big)$. *Then* $\big(\zeta^{sl}(t),\, u^{sl}(t)\big)$ *is a solution lying on a center-stable manifold of*

$$\begin{cases} d\zeta/dt = \zeta G_{10}(\zeta,\, u_0)u_0 \\ du_0/dt = G_{01}(\zeta,\, u_0)u_0 \\ u_- \equiv 0. \end{cases}$$

Note that, strictly speaking, in (4.9) and in (4.13) the component U^- does *not* lie on the manifold of the fast dynamics. Indeed, as we will see in the proof, U^- is a solution of (3.31) and hence does not lie on $\{(0,\, \vec{0},\, u_-)\}$. However, loosely speaking it can be regarded as a fast dynamic because of its exponential decay.

Proof. We first define $\mathcal{M}^s$.

Consider system (4.1) restricted on the manifold of the slow dynamics. Due to the analysis in Section 4.2 the variables t and τ are then equivalent. Using the variable t, we get

$$\begin{cases} d\zeta/dt = \zeta G_{10}(\zeta,\, u_0)u_0 \\ du_0/dt = G_{01}(\zeta,\, u_0)u_0 \\ u_- \equiv 0. \end{cases} \tag{4.14}$$

The manifold $E = \{(\zeta,\, \vec{0},\, \vec{0}) :\ \zeta \in \mathbb{R}\}$ is then entirely constituted by equilibria. Applying Proposition 3.1 to system (4.14) with $\mathcal{S}_0 = E$, we then obtain M^{us}_E, the uniformly stable manifold relative to E, which is parameterized by $E \oplus V^{0-}$. Note that so far we have used only the variable t: M^{us}_E is a uniformly stable manifold for (4.14) with respect to the variable t and by construction it is included in $\{u_- = \vec{0}\}$, a center manifold for (4.1) with respect to the variable τ. The manifold $\mathcal{M}^s$ is then obtained by using the variable τ and applying Proposition 3.1 to system (4.1) with $\mathcal{S}_0 = M^{us}_E$. Also, the set

$$\mathcal{Z}_0 = \{(0,\, u_0,\, u_-) :\ u_0 \in \mathbb{R}^{n_0},\ u_- \in \mathbb{R}^{n_-}\}$$

satisfies (3.8). Properties 1, 3 and estimates (4.10) and (4.12) in the statement of Theorem 4.2 are then automatically satisfied, so we are left to prove estimate (4.11) and property 4.

To show that estimate (4.11) holds we apply Lemma 4.1. By using (4.9) we get

$$\zeta(\tau) = \zeta^{sl}(\tau) + \zeta^p(\tau),$$

where $U^{sl}(\tau) = \big(\zeta^{sl}(\tau),\, u_0^{sl}(\tau),\, \vec{0}\big)$ lies on the manifold of the slow dynamics and ζ^p is the first component of the perturbation term U^p. Let $\tilde{t}$ be defined as the maximal solution of

$$\begin{cases} d\tilde{t}/d\tau = \zeta^{sl}(\tau) \\ \tilde{t}(0) = 0, \end{cases}$$

Then there exits $(\zeta_\infty, \vec{0})$ such that

$$\lim_{\tilde{t}\to+\infty} \big(|\zeta^{sl}(\tilde{t}) - \zeta_\infty| + |u_0^{sl}(\tilde{t})|\big)e^{c\tilde{t}/2} = 0. \tag{4.15}$$

Since $|\zeta^p(\tau)| \le k_p\delta^2 e^{-c\tau/4}$, then for every τ

$$|\tilde{t}(\tau) - t(\tau)| \le \mathcal{O}(1)\delta^2$$

where $t(\tau)$ is defined by (4.8). Since also $|u_0^{sl}(\tau) - u_0(\tau)| \le k_p\delta^2 e^{-c\tau/4}$, we conclude that (4.11) implies (4.15).

Concering the proof of property 4, we apply Lemma 4.1. Since $U^{sl}(\tau) = \big(\zeta^{sl}(\tau),\, u_0^{sl}(\tau),\, \vec{0}\big)$ lies on the manifold of the slow dynamics, then by the analysis in Section 4.2 it satisfies condition 1 in the statement of Lemma 4.1 and hence

$$\int_0^{+\infty} \zeta^{sl}(\tau)d\tau = +\infty.$$

Since $|\zeta^p(\tau)| \le \delta^2 e^{-c\tau/4}$, then

$$\int_0^{+\infty} \Big(\zeta^{sl} + \zeta^p\Big)(\tau)d\tau = +\infty.$$

Applying again Lemma 4.1 we get property 4.

To define the manifold $\mathcal{M}^{cs}$ we proceed as follows. Consider M^{cs}, a center-stable manifold for (4.14). This manifold is parameterized by $E \oplus V^{00-}$ and it is tangent to this space at the origin. The manifold $\mathcal{M}^{cs}$ is defined by applying Proposition 3.1 to system (4.1) with $\mathcal{S}_0 = M^{cs}$ and by using that the set $\mathcal{Z}_0 = \{(0,\, u_0,\, u_-) : \; u_0 \in \mathbb{R}^{n_0},\; u_- \in \mathbb{R}^{n_-}\}$ satisfies (3.8). Proceeding as before one gets that properties 1, 3, 4 and 6 are satisfied.

To verify property 2, we first observe that $M_E^{us} \subseteq M^{cs}$. To obtain $\mathcal{M}^s$ and $\mathcal{M}^{cs}$ we applied Proposition 3.1 to $\mathcal{S}_0 = M_E^{us}$ and $\mathcal{S}_0 = M^{cs}$ respectively. Going back to the proof of Proposition 3.1 one can observe that the way we constructed the uniformly stable manifold with respect to $\mathcal{S}_0$ is we associated to any orbit lying on $\mathcal{S}_0$ the manifold constructed in Theorem 3.1. Thus from the inclusion $M_E^{us} \subseteq M^{cs}$ we infer $\mathcal{M}^s \subseteq \mathcal{M}^{cs}$. □

4.4 Proof of Proposition 4.1

4.4.1 A preliminary result

Before proving Proposition 4.1, we have to introduce a preliminary result, Lemma 4.2.

Let Υ be a continuously differentiable local diffeomorphism. To simplify the exposition, we also assume that $\Upsilon(\vec{0}) = \vec{0}$. Let $\tilde{U} := \Upsilon(U)$ and

$$\tilde{F}(\tilde{U}) := D\Upsilon\big(\tilde{\Upsilon}^{-1}(\tilde{U})\big)F\big(\Upsilon^{-1}(\tilde{U})\big) \tag{4.16}$$

As pointed out in Section 3.1.2, if the function $U(\tau)$ satisfies (2.2), then $\tilde{U}(\tau)$ is a solution of the ODE (3.3). Also, given a real valued function $\zeta(\tilde{U})$, let

$$\tilde{\zeta}(\tilde{U}) := \zeta\big[\Upsilon^{-1}(\tilde{U})\big]. \tag{4.17}$$

By direct check, one can verify that the following holds true.

Lemma 4.2. *Assume that Hypotheses 1, 3 … 8 are satisfied by F and ζ. Also, assume that Hypothesis 2 is satisfied for some δ. Then Hypotheses 1, 3 … 8 are verified by $\tilde{F}$ and $\tilde{\zeta}$ and there exists $\tilde{\delta}$, possibly smaller than δ, such that Hypothesis 2 is as well satisfied.*

4.4.2 Proof of Proposition 4.1: first part

We are now ready to prove Proposition 4.1. The proof actually relies on standard techniques, but we give it for completeness. We proceed in several steps.

- *Step 1:* let $U = (u_1 \dots u_N)^T$ be the components of U. Due to Hypothesis 4, $\nabla\zeta(\vec{0}) \neq \vec{0}$. Just to fix the ideas, we can assume
$$\frac{\partial \zeta}{\partial u_1}(\vec{0}) \neq 0.$$
By a smooth local change of variables we can assume that $\zeta(U) = u_1$. By Lemma 4.2, Hypotheses 1 ... 8 are satisfied by the ODE written using the new variable. To simplify the exposition, we write U and ζ instead of $\bar{U}$ and $\bar{\zeta}$.

- *Step 2:* due to Hypothesis 6, there exists a manifold $\mathcal{M}^{eq}$ which is entirely constituted by equilibria and which is transversal to the manifold $\mathcal{S}$, namely to $\{u_1 = 0\}$. *Via* a smooth local change of variables we can assume that the one-dimensional subspace
$$E := \{\bar{u}_2 = \dots = \bar{u}_N = 0\} \tag{4.18}$$
is entirely contained in $\mathcal{M}^{eq}$. Hypotheses 1 ... 8 are satisfied in the new variables due to Lemma 4.2.

- *Step 3:* let E be as in (4.18) and denote by V^c the eigenspace of $DF(\vec{0})$ associated to eigenvalues with 0 real part. Also, let V^{--} be the eigenspace associated to eigenvalues with strictly negative real part. The dimension of V^c and of V^{--} is $n_0 + 1$ and n_- respectively. Thanks to Hypothesis 3, $N = n_0 + 1 + n_-$. The vector $(1, 0 \dots 0)$ belongs to V^c because $E \subseteq V^c$. Also, we can assume, *via* a linear change of variables, that
$$V^c = \{u_{n_0+2} = \dots u_N = 0\} \qquad V^s = \{\zeta = 0,\, u_2 = \dots u_{n_0+1} = 0\}.$$
Fix a center manifold $\mathcal{M}^c$ of the equilibrium point $\vec{0}$ of system (2.2), then $\mathcal{M}^c$ is parameterized by V^c and it is tangent to this space at the origin $\vec{0}$. Also, let $\mathcal{M}^{us}_E$ be the uniformly stable manifold of system (2.2) relative to the manifold of equilibria E defined by (4.18): this manifold is paramerized by $V^s \oplus E$ and it is tangent to this space at the origin. By a local smooth change of variables we can assume that actually
$$\mathcal{M}^c = \{u_{n_0+2} = \dots u_N = 0\} \qquad \mathcal{M}^{us}_E = \{u_2 = \dots u_{n_0+1} = 0\}.$$
Note that the Hypotheses 1 ... 8 are satisfied because of Lemma 4.2.

- *Step 4:* consider the decomposition
$$U = \begin{pmatrix} \zeta \\ u_0 \\ u_- \end{pmatrix} \qquad F(U) = \begin{pmatrix} f_1(\zeta,\, u_0,\, u_-) \\ F_0(\zeta,\, u_0,\, u_-) \\ F_-(\zeta,\, u_0,\, u_-), \end{pmatrix} \tag{4.19}$$
where $\zeta,\ f_1 \in \mathbb{R},\ u_0,\ F_0 \in \mathbb{R}^{n_0}$ and $u_-,\ F_- \in \mathbb{R}^{n_-}$. In the new coordinates, the center manifold $\mathcal{M}^c$ is the subspace $\{u_- = \vec{0}\}$ and the uniformly stable manifold $\mathcal{M}^{us}_E$ is $\{u_0 = \vec{0}\}$.
The center manifold $\{u_- = \vec{0}\}$ is invariant for the ODE (2.2) and hence $F_-(\zeta,\, u_0,\, \vec{0}) = \vec{0}$ for every ζ and u_0. By regularity,
$$F_-(\zeta,\, u_0,\, u_-) = G_s(\zeta,\, u_0,\, u_-)u_-$$
for a suitable matrix $G_s \in \mathbb{M}^{n_- \times n_-}$. Also, the uniformly stable manifold is invariant and hence proceeding as before we get that
$$F_0(\zeta,\, u_0,\, u_-) = G_c(\zeta,\, u_0,\, u_-)u_0$$

for a suitable matrix $G_c \in \mathbb{M}^{n_0 \times n_0}$. Finally, Hypothesis 7 implies that

$$f_1(0,\ u_0,\ u_-) = 0$$

and hence by regularity $f_1(\zeta,\ u_0,\ u_-) = g_1(\zeta,\ u_0,\ u_-)\zeta$. Consider the decomposition

$$G_c(\zeta,\ u_0,\ u_-) = G_c(\zeta,\ u_0,\ \vec{0}) + \big[G_c(\zeta,\ u_0,\ u_-) - G_c(\zeta,\ u_0,\ \vec{0})\big].$$

Due to Hypothesis 5, the subspace $\{\zeta = 0,\ u_- = \vec{0}\}$ is entirely constituted by equilibria and hence

$$G_c(0,\ u_0,\ \vec{0}) = \vec{0}.$$

By regularity, $G_c(\zeta,\ u_0,\ \vec{0}) = G_{01}(\zeta,\ u_0)\zeta$ for a suitable matrix $G_{01} \in \mathbb{M}^{n_0 \times n_0}$. Putting all the previous considerations together, we get that system (2.2) can be written as

$$\begin{cases} d\zeta/d\tau = g_1(\zeta,\ u_0,\ u_-)\zeta \\ du_0/d\tau = \Big\{G_{01}(\zeta,\ u_0)\zeta + \big[G_c(\zeta,\ u_0,\ u_-) - G_c(\zeta,\ u_0,\ \vec{0})\big]\Big\}u_0 \\ du_-/d\tau = G_s(\zeta,\ u_0,\ u_-)u_- \end{cases} \tag{4.20}$$

Consider the decomposition

$$g_1(\zeta,\ u_0,\ u_-) = g_1(\zeta,\ u_0,\ \vec{0}) + \big[g_1(\zeta,\ u_0,\ u_-) - g_1(\zeta,\ u_0,\ \vec{0})\big]$$

By construction $G_s(0,\ \vec{0},\ \vec{0})$ admits only eigenvalues with strictly negative real part, thus $G_s(\zeta,\ u_0,\ u_-)u_- = \vec{0}$ implies $u_- = \vec{0}$. Thus, the set $\{U :\ \zeta(U) = 0,\ F(U) = \vec{0}\}$ is the subspace $\{\zeta = 0,\ u_- = \vec{0}\}$. By Hypothesis 8, we have

$$g_1(0,\ u_0,\ \vec{0}) = 0.$$

By regularity, we thus have

$$g_1(\zeta,\ u_0,\ \vec{0}) = g_{11}(\zeta,\ u_0)\zeta \qquad \big[g_1(\zeta,\ u_0,\ u_-) - g_1(\zeta,\ u_0,\ \vec{0})\big] = G_{1-}(\zeta,\ u_0,\ u_-)u_-$$

for a suitable row vector $G_{1-}(\zeta,\ u_0,\ u_-) \in \mathbb{R}^{n_-}$. Also, since the manifold $\{u_0 = \vec{0},\ u_- = \vec{0}\}$ is entirely constituted by equilibria, then $g_{11}(\zeta,\ \vec{0}) = 0$ for every ζ and hence

$$g_{11}(\zeta,\ u_0) = G_{10}(\zeta,\ u_0)u_0$$

for a suitable vector $G_{10} \in \mathbb{R}^{n_0}$. In other words, (4.20) reduces to

$$\begin{cases} d\zeta/d\tau = \zeta^2 G_{10}(\zeta,\ u_0)u_0 + \zeta G_{1-}(\zeta,\ u_0,\ u_-)u_- \\ du_0/d\tau = \Big\{G_{01}(\zeta,\ u_0)\zeta + \big[G_c(\zeta,\ u_0,\ u_-) - G_c(\zeta,\ u_0,\ \vec{0})\big]\Big\}u_0 \\ du_-/d\tau = G_s(\zeta,\ u_0,\ u_-)u_- \end{cases} \tag{4.21}$$

- *Step 5:* we introduce a refined change of variables. Consider system (4.20) restricted on the invariant subspace $\{\zeta = 0\}$. One obtains

$$\begin{cases} du_0/d\tau = \Big[G_c(0,\ u_0,\ u_-) - G_c(0,\ u_0,\ \vec{0})\Big]u_0 \\ du_-/d\tau = G_s(0,\ u_0,\ u_-)u_- \end{cases} \tag{4.22}$$

The subspace $\{u_- = \vec{0}\}$ is entirely constituted by equilibria. Also, given a point $(u_0,\ u_-)$ belonging to a small enough neighbourhood of $\vec{0}$, then the solution of (4.22) starting at $(u_0,\ u_-)$ decays exponentially fast to a point in the subspace $\{u_- = \vec{0}\}$. This is a consequence of the fact that $G_s(0, \vec{0}, \vec{0})$ admits only eigenvalues with strictly negative real part.

We can define a change of variables $\tilde{U} = \Upsilon^4(U)$ such that in the new variables $\tilde{U}$ the following holds. For every $\tilde{u}_0(0) \in \mathbb{R}^{n_0}$ and for every $\tilde{u}_-(0) \in \mathbb{R}^{n_-}$, the solution of (4.22) starting at the point $\big(\tilde{u}_0(0),\, \tilde{u}_-(0)\big)$ converges exponentially fast to the point $\big(\bar{u}_0(0),\, \vec{0}\big)$. In other words, the set $\{\bar{u}_0 = \bar{u}_0(0)\}$ is the stable manifold of system (4.22) around the equilibrium point $\big(\tilde{u}_0(0),\, \vec{0}\big)$. Let $\tilde{F}(\bar{U})$ be defined as in (4.16), with $\tilde{\Upsilon} = \Upsilon^4$. Then

$$F(\tilde{U}) = \begin{pmatrix} \tilde{\zeta}^2 \tilde{G}_{10}(\tilde{\zeta},\, \tilde{u}_0)\tilde{u}_0 + \tilde{\zeta}\tilde{G}_{1-}(\tilde{\zeta},\, \tilde{u}_0,\, \tilde{u}_-)\tilde{u}_- \\ \Big\{\tilde{G}_{01}(\tilde{\zeta},\, \tilde{u}_0)\tilde{\zeta} + \big[\tilde{G}_c(\tilde{\zeta},\, \tilde{u}_0,\, \tilde{u}_-) - \tilde{G}_c(\tilde{\zeta},\, \tilde{u}_0,\, \vec{0})\big]\Big\}\tilde{u}_0 \\ G_s(\tilde{\zeta},\, \tilde{u}_0,\, \tilde{u}_-)\tilde{u}_- \end{pmatrix}$$

Because of the way we chose Υ_4, when $\tilde{\zeta} = 0$ then $d\tilde{u}_0/d\tau = 0$ and hence

$$\big[\tilde{G}_c(0,\, \tilde{u}_0,\, \tilde{u}_-) - \tilde{G}_c(0,\, \tilde{u}_0,\, \vec{0})\big]\tilde{u}_0 = \vec{0}.$$

By regularity,

$$\big[\tilde{G}_c(\tilde{\zeta},\, \tilde{u}_0,\, \tilde{u}_-) - \tilde{G}_c(\tilde{\zeta},\, \tilde{u}_0,\, \vec{0})\big] = \big[\tilde{G}_{0-}(\tilde{\zeta},\, \tilde{u}_0,\, \tilde{u}_-) - \tilde{G}_{0-}(\tilde{\zeta},\, \tilde{u}_0,\, \vec{0})\big]\tilde{\zeta}$$

for a suitable function $\tilde{G}_{0-} \in \mathbb{M}^{n_0 \times n_0}$.

- *Step 6:* to conclude the proof of Proposition 4.1 we define the local diffeomorphism Υ as the composition of all the local diffeomorphisms defined at the previous steps.

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