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REDUCTION ON CHARACTERISTICS FOR CONTINUOUS SOLUTIONS OF A SCALAR BALANCE LAW

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Abstract. We consider continuous solutions u to the balance equation

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(t,x) \qquad f \in C^2(\mathbb{R}),\ g \in L^\infty(\mathbb{R})$$

for a bounded source term g. Continuity improves to Hölder continuity when f is *uniformly* convex, but it is not more regular in general. We discuss the reduction to ODEs on characteristics, mainly based on the joint works [5, 1]. We provide here local regularity results holding in the region where $f'(u)f''(u) \neq 0$ and only in the simpler case of autonomous sources $g = g(x)$, but for solutions $u(t,x)$ which may depend on time. This corresponds to a local regularity result, in that region, for the system of ODEs

$$\begin{cases} \dot\gamma(t) = f'(u(t,\gamma(t))) \\ \frac{d}{dt} u(t,\gamma(t)) = g(t,\gamma(t)). \end{cases}$$

1. **Introduction.** In the context of classical solutions, the balance law

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(t,x), \qquad f \in C^2(\mathbb{R}), \tag{1.1}$$

can be reduced to ordinary differential equations along characteristic curves, defined as those curves $t \mapsto (t,\gamma(t))$ satisfying $\dot\gamma(t) = f'(u(t,\gamma(t)))$. Indeed,

$$\begin{aligned} g(t,\gamma(t)) &= \partial_t u(t,\gamma(t)) + \partial_x [f(u(t,\gamma(t)))] \\ &= \partial_t u(t,\gamma(t)) + f'(u(t,\gamma(t)))\partial_x u(t,\gamma(t)) \\ &= \partial_t u(t,\gamma(t)) + \dot\gamma(t)\partial_x u(t,\gamma(t)) \qquad = \frac{d}{dt} u(t,\gamma(t)). \end{aligned}$$

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This more generally allows a parallel between the Cauchy problem for a scalar quasi-linear first order PDE and for a system of ODEs, which is known as the method of characteristics (see for instance [10], where it is also provided an application to determine local existence).

If one interprets of $f'(u)$ as a velocity, this is just the change of variable from the Eulerian (PDE) to the Lagrangian (ODEs) formulation.

We discuss here what remains of this equivalence when u is just continuous and g is bounded. We prove then in Section 2 that when g depends only on x, but not on the time t, then $u(t,x)$ is locally Lipschitz continuous on the open set where $f'(u)f''(u)$ is nonvanishing. This is sensibly better than the general case, where u is only Hölder continuous. It is based on proving the corresponding result for the system of ODEs. As we are discussing local issues, we will fix for simplicity the domain $\mathbb{R}^2$ and we will assume u bounded.

1.1. **A motivation for a different setting.** The development of Geometric Measure Theory in the context of the sub-Riemannian Heisenberg group $\mathbb{H}^n$ brought the attention to *continuous* solutions to the equation

$$\partial_t u(t,x) + \partial_x \left[\frac{u^2(t,x))}{2}\right] = g(t,x). \tag{1.2}$$

Continuity is natural from the fact that u parametrizes a surface. As one studies surfaces that have differentiability properties in the intrinsic structure of the Heisenberg group, but not in the Euclidean structure, then it is not natural assuming more regularity of u than continuity [13], which for bounded sources improves to $1/2$-Hölder continuity [4, 5]. Notice that with u continuous the second term of the equation cannot even be rewritten as $u\partial_x u$, because $\partial_x u$ is only a distribution and u is not a suitable test function.

The PDE arises if one wants to show the equivalence between a pointwise, metric notion of differentiability and a distributional one: for $n = 1$ the distributional definition is precisely (1.2), while for $n > 1$ it is a related multi-D system of PDEs. The correspondence was introduced first in [3, 4] for intrinsic regular hypersurfaces, which are the analogue of what are C^1-hypersurfaces in the Euclidean setting. It was extended in [5, 7] when considering intrinsic Lipschitz hypersurfaces, analogue of Lipschitz hypersurfeces in the Euclidean setting. The source term g, in $\mathbb{H}^1$, turns out to be what is called the intrinsic gradient of u, which is the counterpart of the gradient in Euclidean geometry; in $\mathbb{H}^n$ it is one if its components: u locally parametrizes an intrinsic regular hypersurface if and only if (1.2) holds locally with g continuous; it parametrizes an intrinsic regular hypersurface if and only if (1.2) holds locally with g bounded. As the notion of differentiability they provide in the intrinsic structure of $\mathbb{H}^n$ is closer to the Lagrangian formulation, the equivalence between Lagrangian and Eulerian formulation arises as intermediate step of this characterization.

When considering intrinsic Lipschitz hypersurfaces the fact that g is only bounded gives rise to new subtleties. In particular, one already knows by an intrinsic Rademacher theorem [11] that the intrinsic differential exists and it is unique $\mathcal{L}^2$-a.e. However, for the ODE formulation this is not enough: as one needs to restrict this L^∞ function on curves, a precise representative is needed also at points where u is not intrinsically differentiable. Viceversa, if one chooses badly the representative of the source of the ODE formulation a priori it differs on a positive measure set from the source of the ODE. There is however a canonical choice for defining the two sources, which makes the formulations equivalent when the inflection points of f are negligible.

1.2. **Summary of the equivalence.** When u is Lipschitz, the ODEs

$$\begin{cases} \dot\chi(t,x) = f'(u(t,\chi(t,x))) \\ \chi(0,x) = x \end{cases} \qquad x \in \mathbb{R}, f \in C^2$$

provides a local diffeomorphism by the classical theory on ODE. If u is instead continuous, Peano's theorem ensures local existence of solutions, but more characteristics may start at one point and characteristics from different points may collapse (see in [5] the classical example of the square-root). This makes clearly impossible to have a local diffeomorphism, or even having a Lagrangian flow in the sense by Ambrosio-DiPerna-Lions [9, 2]. A recent result about this can be obtained for u not depending on time [6], but it is clearly not our assumption. Dropping out injectivity, it is however possible to construct a continuous change of variables with bounded variation.

Let u be a continuous, bounded function.

Lemma 1.1. *There exists a continuous function $\chi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

- *$\tau \mapsto \chi(t,\tau)$ is nondecreasing for every t and surjective;*
- *$\partial_t \chi(t,\tau) = f'(u(t,\tau))$.*

We call it Lagrangian parameterization. This function is not unique.

See [1, 5] for the proof. See also [12] for a similar change of variable, for a $1D$-system. In general one cannot have that χ is SBV [1].

Consider now u continuous distributional solution to (1.1) with g bounded.

Lemma 1.2. *Assume that $\mathcal{L}^1(\text{clos}(\{\text{Inflection points of } f\})) = 0$. Then u is Lipschitz continuous along every characteristic curve.*

The proof follows a computation by Dafermos [8]. For general fluxes, there are cases when u is not Lipschitz along some Lagrangian parameterization [1]. The counterexample holds also for continuous autonomous sources $g(t,x) = g_0(x)$. What we find more striking is the following equivalence.

Theorem 1.3. *Assume that $\mathcal{L}^1(\text{clos}(\{\text{Inflection points of } f\})) = 0$. Then there exists a pointwise defined function $\hat g(t,x)$ a.e. equal to g such that*

$$\frac{d}{dt} u(t,\gamma(t)) = \hat g(t,\gamma(t)) \qquad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for every characteristic curve } \gamma.$$

The proof is based on a selection theorem as a technical device, but $\hat g$ is essentially uniquely defined as the derivative of u along some characteristic.

Remark 1.4. There is a substantial difference between the uniformly convex case and the case of negligible inflection points: in the former at almost every (t,x) there exists a unique value for $\frac{d}{dt}u(t,\gamma(t))$, $\gamma(t)=x$, and it does not depend on which characteristic $\gamma(s)$ one has chosen (for quadratic flux, this is Rademacher theorem in [11]). That value is the most natural choice of g at those points. For non convex fluxes, instead, there may be a set of positive $\mathcal{L}^2$-measure made of points where $\frac{d}{dt}u(t,\gamma(t))$ does not exist at the point, independently of which characteristic γ one choses through the point.

The converse also holds. We give here a weaker statement without the negligibility condition on the inflection points. Identifying sources is delicate.

Theorem 1.5. *Assume that a continuous function u has a Lagrangian parameterization χ for which there exists a bounded function $\tilde{g}$ s.t. it satisfies*

$$\frac{d}{dt}u(t,\chi(t,\tau))=\tilde{g}(t,\chi(t,\tau)) \qquad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for every } \tau\in\mathbb{R}. \tag{1.3}$$

Then this same relation is satisfied for a function $g(t,x)$ s.t. (1.1) holds.

Viceversa, if (1.1) holds then there exists a *Lagrangian parameterization χ and function $\tilde{g}$ s.t. (1.3) holds.*

We are not stating the compatibility of the two sources g, $\tilde{g}$, which at least under the negligibility condition on inflection points of Theorem 1.3 holds. We finally mention that continuous distributional solutions to this simple equation do not dissipate entropy.

Theorem 1.6. *Let u be a continuous distributional solution to (1.1) with bounded source g. Then for every smooth function η and q satisfying $q'=\eta' f'$*

$$\partial_t\left[\eta(u(t,x))\right]+\partial_x\left[q(u(t,x))\right]=\eta'(u(t,x))g(t,x).$$

2. **Some Local Regularity with Autonomous Sources.** We mention a local regularity result holding in the case of autonomous sources: the continuous function $u(t,x)$ is locally Lipschitz continuous in the (open) complementary of the 0-level set of the product $f'(u)f''(u)$. For $f(u)=u^2/2$, this means $u\neq 0$. When the source is not autonomous, then this fails to be true, indeed characteristics may bifurcate also at points where u is not vanishing.

We remind [1] that when f has inflection points of positive measure, then a priori u may not be Lipschitz along some characteristics, even with $g=g(x)$.

Lemma 2.1. *There may be locally multiple solutions to the ordinary differential equation*

$$\begin{cases}\dot\gamma(t)=u(t,\gamma(t))\\ \ddot\gamma(t)=g(\gamma(t))\end{cases} \qquad \gamma(\bar t)=\bar x \qquad u(t,x) \text{ continuous, } g(x) \text{ bounded}$$

only if $u(\bar t,\bar x)=0$ but it does not identically vanish in a whole neighborhood.

Remark 2.2. We are not stating existence. The lemma is however still not obvious because we do not have differentiability properties of u, which follow a posteriori by the next corollary in the region where u does not vanish. As a consequence, we do not have now the differentiability of the map $\gamma(t)$ w.r.t. the initial data of the ODE. The lemma asserts indeed the continuity in this variable in that region, provided it exists. We remind that when g depends on t bifurcations may easily occur also if $u \neq 0$.

In particular, if u is not locally Lipschitz where nonvanishing the above system cannot have solutions through each point of the plane.

Proof. We just prove that if u does not vanish at some point $(\bar{t}, \bar{x})$, at that point there is at most one solution of the ODE, as an effect of the autonomous source. The reason is that if $u(\bar{t}, \bar{x})$ does not vanish, then any Lipschitz characteristic $x = \gamma(t)$, with $\bar{x} = \gamma(\bar{t})$, is a diffeomorphism in some neighborhood of $(\bar{t}, \bar{x})$, and we can invert it. This allows to have the space variable as a parameter: the characteristic can be expressed as $t = \theta(x)$. However, the second order relation $\ddot{\gamma}(t) = g(\gamma(t))$, once expressed in the x variable, can be integrated determining the function θ.

By elementary arguments, it suffices to show that there exists (locally) only one characteristic passing through $(\bar{t}, \bar{x}) = (0, 0)$ with slope $u(0, 0) = 1$. Focus the attention on a neighborhood of the origin where u is bigger than some $\varepsilon > 0$.

Let $x = \gamma(t)$ be any Lipschitz continuous solution of the ODE. Since $\dot{\gamma}(0) = u(0, 0) > 0$, by the inverse function theorem there exists $\delta > 0$ and a function

$$\theta : (\gamma(-\delta), \gamma(\delta)) \to (-\delta, \delta) \quad : \quad \theta(\gamma(t)) = t,\ \gamma(\theta(x)) = x.$$

Moreover, it is continuously differentiable with derivative

$$\dot{\theta}(x) = \frac{1}{\dot{\gamma}(\theta(x))} = \frac{1}{u(\theta(x), x)} \in \left[\frac{1}{\max|u|}, \frac{1}{\varepsilon}\right]. \tag{2.1}$$

From the Lipschitz continuity of $u(t, \gamma(t))$ and the fact that γ is a local diffeomorphism with inverse θ we deduce that the composite function $u(\theta(x), x)$ is Lipschitz continuous. At points of differentiability by the classical chain rule

$$\lim_{h\downarrow 0} \frac{\dot{\gamma}(\theta(x+h)) - \dot{\gamma}(\theta(x))}{h} = \frac{\dot{\gamma}(\theta(x+h)) - \dot{\gamma}(\theta(x))}{\theta(x+h) - \theta(x)} \frac{\theta(x+h) - \theta(x)}{h} = \ddot{\gamma}(\theta(x))\dot{\theta}(x)$$

and (2.1) we have that $\dot{\theta}$ is differentiable at $x \in X$ with derivative

$$\ddot{\theta}(x) = -\frac{\ddot{\gamma}(\theta(x))\dot{\theta}(x)}{[\dot{\gamma}(\theta(x))]^2} = -\frac{g(\theta(x))}{u^3(\theta(x), x)} \qquad \Leftrightarrow \qquad -\frac{\ddot{\theta}(x)}{[\dot{\theta}(x)]^3} = g(x).$$

For those $x \in X$, the differential equation may be rewritten as

$$\frac{d}{dx}\left[\frac{1}{2[\dot{\theta}(x)]^2}\right] = g(x) \qquad \Leftrightarrow \qquad \frac{d}{dx}\frac{u^2(\theta(x), x)}{2} = g(x).$$

The explicit ODE for $\theta(x)$, with initial data $\theta(0) = 0$, $[\dot\theta(0)]^{-1} = u(0,0) = 1$ is easily solved locally by

$$u^2(\theta(x), x) = \frac{1}{\dot\theta^2(x)} = 1 + 2\int_0^x g(z)dz. \tag{2.2}$$

This shows that the slope of every characteristic through the origin, which is a local diffeomophism, is fixed at each x independently of the characteristic we have chosen: therefore there can be only one characteristic, precisely (in the space parameterization)

$$\theta(x; \bar t, \bar x) = \bar t + \int_{\bar x}^x \frac{1}{\sqrt{u^2(\bar t, \bar x) + 2\int_{\bar x}^w g(z)dz}} dw. \tag{2.3}$$

Notice finally that if u vanishes in a neighborhood, being $\dot\gamma(t) \equiv 0$ there characteristics must be vertical (in that region of the (x,t)-plane). □

Lemma 2.3. *Under the hypothesis of Lemma 2.1, if $g(x)$ is continuous it should also vanish at points where there are more characteristics, but it must not identically vanish in a neighborhood.*

Proof. We show that not only u, but also g must vanish. The argument shows that when two characteristics meet and have both second derivative with the same value, this value must be 0. For simplifying notations, consider two characteristics $\gamma_1(t) \le \gamma_2(t)$ for arbitrarily small $t > 0$ with $\gamma_1(0) = \gamma_2(0) = 0$. If $\gamma_1(t_k)\gamma_2(t_k) \le 0$ for $t_k \downarrow 0$, then

$$0 \le \ddot\gamma_2(0) = g(0) = \ddot\gamma_1(0) \le 0,$$

thus g vanishes. If instead e.g. $g > 0$ near the origin, having excluded the above case there exists $\delta > 0$ such that $0 \le \gamma_1(t) \le \gamma_2(t)$ for $t \in [0, \delta]$. Then (2.2) implies that the two curves coincide: for small $x > 0$ necessarily $\dot\gamma_1(t) > 0$, $\dot\gamma_2(t) > 0$, otherwise we would have a sequence $x_k \downarrow 0$ where g vanishes, and therefore

$$u^2(\gamma_1^{-1}(x), x) + 2\int_x^0 g(z)dz = \dot\gamma_1^2(0)$$
$$= 0 = \dot\gamma_2^2(0) = u^2(\gamma_2^{-1}(x), x) + 2\int_x^0 g(z)dz,$$

showing that $\dot\gamma_1(t) \equiv \dot\gamma_2(t)$ for small times. This implies that the two curves coincide.

Finally, suppose g vanishes in a neighborhood. Then, as $\ddot\gamma(t) = 0$ in that neighborhood, characteristics are straight lines. As by the continuity of u characteristics may only intersect with the same derivative, they must be parallel lines and therefore bifurcation of characteristics does not occur. □

We now show that in case u does not vanish, in the above lemma much more regularity holds.

Lemma 2.4. *In the setting of Lemma 2.1, $u(t,x)$ is locally Lipschitz in the open set $\{(t,x) : u(t,x) \neq 0\}$.*

Proof. By Lemma 2.1 there is a unique characteristic starting at each point $(\bar{t},\bar{x}) \in \Omega = \{(t,x) : u(t,x) \neq 0\}$, which is given by (2.3). We start comparing the value of u at two points $(0,0)$, $(-t,0)$, $t>0$, in a ball B compactly contained in Ω. In particular, there exists $\delta(B)$ s.t. the two characteristics starting from the points we have chosen do not intersect if $0<x<\delta(B)$, as there u does not vanish. For such small x one has

$$\int_0^x \frac{1}{\sqrt{\lambda_1^2 + 2\int_0^w g(z)dz}}dw > -t + \int_0^x \frac{1}{\sqrt{\lambda_2^2 + 2\int_0^w g(z)dz}}dw, \tag{2.4}$$

where we defined $\lambda_1 = u(0,0)$ and $\lambda_2 = u(-t,0)$. Equivalently

$$t > \int_0^x \left\{ \frac{1}{\sqrt{\lambda_2^2 + 2\int_0^w g(z)dz}} - \frac{1}{\sqrt{\lambda_1^2 + 2\int_0^w g(z)dz}} \right\} dw.$$

Suppose $\lambda_1 > \lambda_2$. By convexity of the graph of $r \mapsto \frac{1}{\sqrt{r}}$, the RHS is more than

$$\begin{aligned} &\int_0^x \frac{d}{dr}\left\{\frac{1}{\sqrt{r}}\right\}\Big|_{r=\lambda_1^2+2\int_0^w g(z)dz} (\lambda_2^2 - \lambda_1^2) dw \\ &= \left\{ \frac{\lambda_2+\lambda_1}{-2} \int_0^x \frac{1}{(\lambda_1^2 + 2\int_0^w g(z)dz)^{3/2}} dw \right\} (\lambda_2 - \lambda_1) \\ &\geq \left\{ \frac{\lambda_2+\lambda_1}{2(\lambda_1^2+2Gx)^{3/2}} x \right\} (\lambda_1 - \lambda_2) \end{aligned}$$

The argument in the last brackets is uniformly continuous and as $t \downarrow 0$ it is more than x/λ_1^2. As the inequalities hold for every positive t, $x < \delta = \delta(B)$, the non-intersecting condition (2.4) implies

$$u(0,0) - u(t,0) = \lambda_1 - \lambda_2 \leq \left(\frac{\lambda_1^2}{\delta} + \varepsilon \right) t,$$

which is half the Lipschitz inequality at the points $(0,0)$, $(-t,0)$. The other half, for $\lambda_1 < \lambda_2$ is similarly obtained considering small negative x.

For comparing two generic close points (t,x) and $(0,0)$, by the finite speed of propagation one can combine the Lipschitz regularity along characteristics and the Lipschitz regularity along vertical lines. □

Corollary 2.5. *Let $u(t,x)$ be a continuous solution to the balance equation*

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(x), \qquad g \in L^\infty(\mathbb{R}).$$

The function $u(t,x)$ is locally Lipschitz in the open set

$$\{(t,x) : f'(u(t,x)) \cdot f''(u(t,x)) \neq 0\}.$$

Proof. We first consider the case of quadratic flux $f(u) = u^2/2$. By Theorem 1.3, there exists a function $\hat{g}(t,x)$ such that we can apply Lemma 2.1, which gives the thesis. If $g \in L^\infty$ they may a priori differ on an $\mathcal{L}^2$-negligible set, but one can prove that $\hat{g}(t,x) = \hat{g}(x)$.

Being u an entropy solution by Theorem 1.6, $f'(u)$ solves the equation

$$\left[f'(u)\right]_t + \left[\frac{f'(u)^2}{2}\right]_x = f''(u)g.$$

By the previous case then $f'(u)$ is Lipschitz in the open set where it does not vanish. If moreover $f''(u)$ does not vanish, then the regularity of u can be proved just by inverting f'. □

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