On a Lyapunov Functional Relating Shortening Curves and Viscous Conservation Laws

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Abstract. We study a non linear functional which controls the area swept by a curve moving in the plane in the direction of curvature. In turn, this yields a priori estimates on solutions to a class of parabolic equations and of scalar viscous conservation laws. A further application provides an estimate on the "change of shape" of a BV solution to a scalar conservation law.

Keywords: Area Functional, Parabolic Estimates, Glimm Functional.

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1 - The area swept by a shortening curve

Fix two points A, B in the plane $I\!R^2$ and consider the family $\mathcal{F}_{AB}$ of all polygonal lines joining A with B. Given $\gamma \in \mathcal{F}_{AB}$, say with vertices $A = P_0, P_1, \ldots, P_n = B$, define $v_i \doteq P_i - P_{i-1}$ and consider the functional

$$Q(\gamma) \doteq \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n} |v_i \wedge v_j|, \tag{1.1}$$

where $\wedge$ stands for the external product in $I\!R^2$. Let γ' be obtained from γ by replacing the two segments $P_{\ell-1}P_\ell$ and $P_\ell P_{\ell+1}$ by one single segment $P_{\ell-1}P_{\ell+1}$, as in fig. 1. The area of the triangle with vertices $P_{\ell-1}$, P_ℓ, $P_{\ell+1}$ satisfies

$$\text{Area}(P_{\ell-1}P_\ell P_{\ell+1}) \;=\; \frac{1}{2}|v_{\ell+1} \wedge v_\ell| \;\leq\; Q(\gamma) - Q(\gamma'). \tag{1.2}$$

Indeed,

$$\begin{aligned} Q(\gamma) - Q(\gamma') = \frac{1}{2} \sum_{i=0}^{\ell-1} \Big(|v_i \wedge v_\ell| + |v_i \wedge v_{\ell+1}| - |v_i \wedge (v_\ell + v_{\ell+1})| \Big) + \frac{1}{2}|v_\ell \wedge v_{\ell+1}| \\ + \frac{1}{2} \sum_{j=\ell+2}^{n} \Big(|v_\ell \wedge v_j| + |v_{\ell+1} \wedge v_j| - |(v_\ell + v_{\ell+1}) \wedge v_j| \Big) \geq \frac{1}{2}|v_\ell \wedge v_{\ell+1}|. \end{aligned}$$

Next, assume that γ' is obtained from γ by a finite sequence of consecutive cuts (fig. 2). In other words, let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be a sequence of polygonals with $\gamma_0 = \gamma$, $\gamma_n = \gamma'$ and such that each γ_i is obtained from γ_{i-1} by replacing two adjacent segments with a single one. By (1.2), an inductive argument yields

$$\text{Area}(\gamma, \gamma') \leq \sum_{i=1}^{n} \text{Area}(\gamma_i, \gamma_{i-1}) \leq Q(\gamma) - Q(\gamma'). \tag{1.3}$$

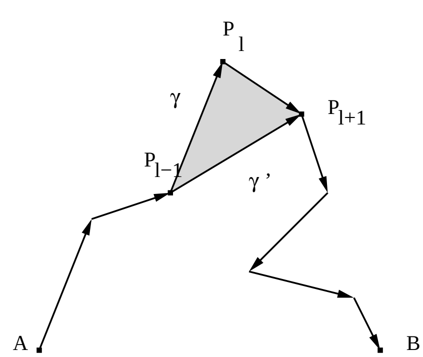

Figure 1.

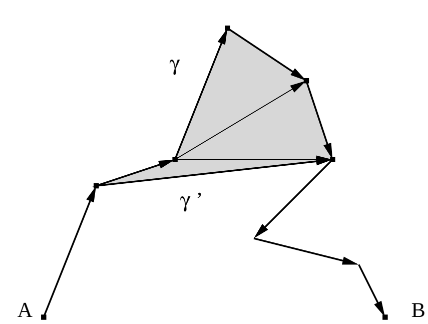

Figure 2.

More generally, instead of polygonals, we can define the functional Q on the family of parametric curves in the plane. Following [2], by a parametric curve we mean a continuous map

$\gamma : [a, b] \mapsto I\!R^2$. Two parametric curves γ_1 and γ_2 are regarded as equivalent if, for every $\varepsilon > 0$, there exists a homeomorphism $h^\varepsilon : [a_1, b_1] \mapsto [a_2, b_2]$ (possibly order–reversing) such that

$$\sup_{x\in[a_1,b_1]} \left|\gamma_1(x) - \gamma_2(h^\varepsilon(x))\right| < \epsilon.$$

Let $\mathcal{F}$ be the set of all parametric curves with finite length, i.e. such that

$$L(\gamma) \doteq \sup \sum_{i=1}^{n} \left|\gamma(x_i) - \gamma(x_{i-1})\right| < +\infty, \tag{1.4}$$

where the supremum is taken w.r.t. all partitions $a = x_0 < \ldots < x_n = b$. We recall that (1.4) implies the existence of an absolutely continuous parameterization. In the following, up to a homeomorphism h, we can assume that each $\gamma \in \mathcal{F}$ is absolutely continuous and parametrized by $x \in [0, 1]$, and we denote with γ_x the derivative of γ w.r.t. x. We recall that $\gamma_x \in L^1([0, 1]; I\!R^2)$ and

$$L(\gamma) = \int_0^1 \left|\gamma_x\right| dx.$$

We can introduce a metric d on $\mathcal{F}$ defined as

$$d(\gamma_1, \gamma_2) \doteq \inf \left\|\gamma_1 - \gamma_2 \circ h\right\|_{C^0}, \tag{1.5}$$

where the infimum is taken over all homeomorphisms $h : [0, 1] \mapsto [0, 1]$. We can now define a functional $Q : \mathcal{F} \mapsto I\!R$ by setting

$$\begin{aligned} Q(\gamma) &\doteq \frac{1}{2} \sup \left\{ \sum_{\substack{i,j=1\\ j>i}}^{n} \left| \big(\gamma(x_i) - \gamma(x_{i-1})\big) \wedge \big(\gamma(x_j) - \gamma(x_{j-1})\big) \right| \right\} \\ &= \frac{1}{2} \int_0^1 \int_x^1 \left|\gamma_x(x) \wedge \gamma_x(y)\right| dy dx \le \frac{1}{2} L(\gamma)^2, \end{aligned} \tag{1.6}$$

where, as above, the supremum is taken w.r.t. all partitions $0 = x_0 < \ldots < x_n = 1$. Observe that the definition (1.6) is the natural extension of (1.1).

It is well known that the length $L(\gamma)$ of a curve is lower semi continuous w.r.t. the distance (1.5). A similar result holds for the functional Q.

Lemma 1. *The functional Q is lower semi continuous in $(\mathcal{F}, d)$. Namely, if $d(\gamma_\nu, \gamma) \to 0$, then*

$$Q(\gamma) \le \liminf_{\nu\to\infty} Q(\gamma_\nu).$$

Proof. For every $\varepsilon > 0$ there exists a $\bar\nu$ such that $d(\gamma_\nu, \gamma) < \varepsilon$ for all $\nu \ge \bar\nu$, and hence

$$\begin{aligned} &\sum_{\substack{i,j=1\\ i<j}}^{n} \left| \big(\gamma(x_i) - \gamma(x_{i-1})\big) \wedge \big(\gamma(x_j) - \gamma(x_{j-1})\big) \right| \\ &\qquad \le \sum_{\substack{i,j=1\\ i<j}}^{n} \left| \big(\gamma_\nu(x_i) - \gamma_\nu(x_{i-1})\big) \wedge \big(\gamma_\nu(x_j) - \gamma_\nu(x_{j-1})\big) \right| + 2n^2\varepsilon^2 \le Q(\gamma_\nu) + 2n^2\varepsilon^2. \end{aligned}$$

Letting $\varepsilon \to 0$ first and then taking the supremum w.r.t. all partitions $0 = x_0 < \ldots < x_n = 1$, the conclusion follows.

Given $\gamma \in \mathcal{F}$, by a *cut* we mean the replacement of the portion of the curve $\{\gamma(x); x \in [x_1, x_2] \subseteq [0,1]\}$ with the segment connecting $\gamma(x_1)$ to $\gamma(x_2)$, for some $x_1, x_2 \in [0,1]$. We say that γ' *follows* γ, and write $\gamma \prec \gamma'$, if there exists a sequence of curves γ_n converging to γ' in $(\mathcal{F}, d)$ such that each γ_n is obtained from γ by a finite sequence of consecutive *cuts* (fig. 3). Note that, as a consequence of this definition, γ' must have the same endpoints of γ. It is easy to see that $\prec$ defines a partial order relation.

Given $\gamma, \gamma' \in \mathcal{F}$ with $\gamma \prec \gamma'$, we consider the closed curve $\gamma \cup \gamma' : [0,2] \mapsto \mathbb{R}^2$ as

$$\Big(\gamma \cup \gamma'\Big)(x) \doteq \begin{cases} \gamma(x) & x \in (0,1] \\ \gamma'(2-x) & x \in (1,2] \end{cases} \tag{1.7}$$

By the *area between* γ *and* γ', denoted by $\text{Area}[\gamma, \gamma']$, we mean the area of the regions where the winding number of the curve $\gamma \cup \gamma'$ is odd. For the definition and basic properties of the winding number of a closed curve in the plane we refer to [8]. We remark that the above definition of area between curves is not affected by a change in the parameterizations.

Lemma 2. *If $\gamma \prec \gamma'$, then the area between the two curves satisfies*

$$\text{Area}[\gamma, \gamma'] \leq Q(\gamma) - Q(\gamma'). \tag{1.8}$$

Proof. In the case where γ is a polygonal and γ' is obtained from γ with a finite sequence of cuts, the result was already proved in (1.3). The general case follows by approximation, using Lemma 1.

Remark 1. One can give an equivalent definition of the partial order relation "$\prec$" by setting $\gamma \prec \gamma'$ if the following holds:

There exists a sequence of parabolic problems on the plane:

$$\begin{cases} \xi^\nu_t + \lambda_\nu(t,x)\xi^\nu_x - c_\nu(t,x)\xi^\nu_{xx}(t,x) = 0, & \\ \xi_\nu(0,x) = \gamma(x) & x \in [0,1], \\ \xi_\nu(t,0) = \gamma(0), \quad \xi_\nu(t,1) = \gamma(1), & t \in [0,1], \end{cases} \tag{1.9}$$

whose solutions at time $t = 1$ converge to γ', i.e.

$$\lim_{\nu \to \infty} d\big(\xi^\nu(1,\cdot), \gamma'\big) = 0.$$

Here λ_ν, c_ν are smooth functions from $[0,1] \times [0,1] \mapsto \mathbb{R}$, with c_ν strictly positive. Note that by a change of variable $y = y(t,x)$, where y is the unique positive solution to the uniformly parabolic system

$$\begin{cases} y_t + \lambda_\nu(t,x) y_x - c_\nu(t,x) y_{xx} = 0, & \\ y(0,x) = x & x \in [0,1], \\ y(t,0) = 0, \quad y(t,1) = 1, & t \in [0,1], \end{cases}$$

equation (1.9) becomes

$$\xi_t^\nu = c_\nu(t,x)\big(y_x(t,x)\big)^2 \xi_{yy}^\nu = c'_\nu(t,y)\xi_{yy}^\nu, \tag{1.9'}$$

whit $c'_\nu(t,x) \geq c > 0$ for all $(t,x) \in [0,1] \times [0,1]$.

Consider now any cut in the curve γ, say obtained by replacing the portion $\{\gamma(x); x \in [a,b] \subseteq [0,1]\}$ by a single segment. This can be uniformly approximated by solutions of (1.9), letting the viscosity coefficient c_ν tend to $+\infty$ inside the interval $[a,b]$ and to 0 outside $[a,b]$.

Conversely, any solution of (1.9') can be uniformly approximated by a forward difference scheme

$$\xi^\nu\big((n+1)\Delta t, k\Delta x\big) - \xi^\nu\big(n\Delta t, k\Delta x\big) = \\ \frac{\Delta t}{\Delta x^2} c_\nu\big(n\Delta t, k\Delta x\big)\Big[\xi^\nu\big(n\Delta t,(k+1)\Delta x\big) - 2\xi^\nu\big(n\Delta t, k\Delta x\big) + \xi^\nu\big(n\Delta t,(k-1)\Delta x\big)\Big].$$

In fact, if

$$\frac{\Delta t}{\Delta x^2} \max_{0 \leq t,x \leq 1} \big\{c^\nu\big\} < \frac{1}{2},$$

it is easy to prove that the quantities $|\xi^\nu(x+\Delta x)|$, $|\xi^\nu(x+\Delta x) - \xi^\nu(x))/\Delta x|$ remain uniformly bounded. If we consider the polygonal line $\gamma^\nu(n\Delta t)$ obtained by connecting the points $\xi^\nu(n\Delta t, k\Delta x)$ with n fixed, then it follows that γ^n converges in C^0 to the unique classical solution of (1.9'). At each time step, the approximate solutions γ^ν thus constructed will satisfy $\gamma^\nu(n\Delta t) \prec \gamma^\nu((n+1)\Delta t)$. From the relations $\gamma^\nu(0) \prec \gamma^\nu(1)$, and the convergence $\gamma^\nu(0) \to \gamma$, $\gamma^\nu(1) \to \gamma'$ we conclude $\gamma \prec \gamma'$.

We also observe that (1.9) implies that the vector ξ_t^ν lies in the half plane $\{v \in I\!R^2,\ v \cdot \xi_{xx}^\nu \geq 0\}$. Hence, by possibly modifying the function λ (which has the only effect of changing the parameterization), we obtain a solution of where the vectors two ξ_t^ν and ξ_{xx}^ν are parallel and have the same orientation. In other words, ξ^ν will move in the direction of curvature.

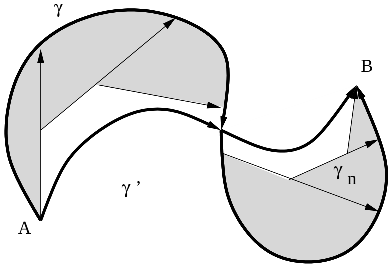

Figure 3.

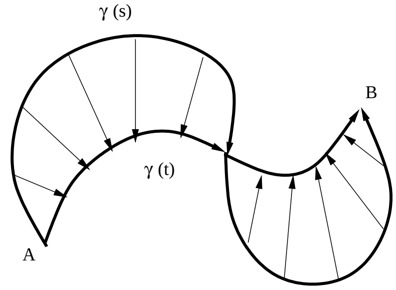

Figure 4.

More generally, consider a path varying with time. This is described by a map $\gamma : [t_1, t_2] \mapsto \mathcal{F}$. We say that γ *moves in the direction of curvature* if $\gamma(s) \prec \gamma(t)$ for all $s < t$, $s, t \in [t_1, t_2]$ (fig. 4). Observe that, from our definitions, it follows that the endpoints of $\gamma(t)$ remain constant in time. The *area swept* by $\gamma(t)$ during the time interval $[t_1, t_2]$ is defined as

$$\mathrm{Area}\big(\gamma;\ [t_1,\ t_2]\big) \doteq \sup\left\{\sum_{i=1}^{n} \mathrm{Area}\big[\gamma(s_i),\ \gamma(s_{i-1})\big];\quad t_1 = s_0 < \cdots < s_n = t_2, \quad n \geq 1\right\}. \tag{1.10}$$

An immediate consequence of the above definitions is

Theorem 1. *Let $t \mapsto \gamma(t) \in \mathcal{F}$ denote a curve in the plane, moving in the direction of the curvature. Then, for every $t_1 < t_2$ one has*

$$\mathrm{Area}\big(\gamma;\ [t_1, t_2]\big) \leq Q\big(\gamma(t_1)\big) - Q\big(\gamma(t_2)\big). \tag{1.11}$$

Remark 2. It is possible to state Theorem 1 in more generality by enlarging the class of equivalent curves. Consider two curves $\gamma_i \in \mathcal{F}$, $i = 1, 2$, parametrized by arc length, and define $\gamma_1 \sim \gamma_2$ if there exists a bijection $h : \big[0,\ L(\gamma_1)\big] \mapsto \big[0,\ L(\gamma_2)\big]$ such that

$$\gamma_1 = \gamma_2 \circ h. \tag{1.12}$$

Using (1.4) and (1.6), it is easy to verify that $L(\gamma_1) = L(\gamma_2)$ and $Q(\gamma_1) = Q(\gamma_2)$. In essence, we identify two curves γ_1, γ_2 of finite length if the sets $\big\{\gamma_i(x); x \in [a_i, b_i]\big\}$, $i = 1, 2$, are equal.

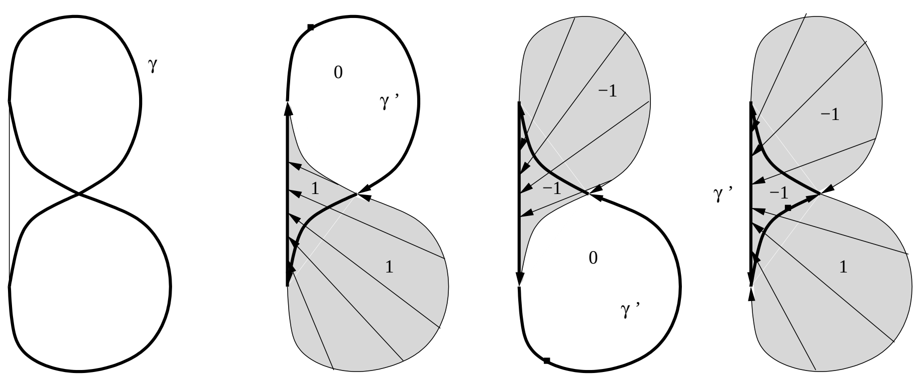

Figure 5.

Let now a continuous parameterization $x \to \gamma(x) \in I\!\!R^2$ be given. As before, by a *cut* we mean the replacement of the part of the curve γ, $\{\gamma(x); x \in [a, b] \subseteq [0, 1]\}$ with the segment connecting $\gamma(a)$ to $\gamma(b)$. Note however that this definition now depends on the parameterization. For example, consider the curve in fig. 5. Given two points on the curve γ, we can obtain many different curves γ' depending on which part of the original curve is replaced by the cut.

When $\gamma \prec \gamma'$, the *area between* γ *and* γ' can be again defined as the area of the regions in which the *winding number of the curve* $\gamma \cup \gamma'$ is odd. Recall that $\gamma \cup \gamma'$ was defined at (1.7). It is not difficult to see that the estimate (1.11) remains valid also in this more general case.

Remark 3. It is possible to generalize Theorem 1 to a curve $\gamma(t)$ in $\mathbb{R}^n$: the definition of motion in the direction of curvature remains the same. However when computing the area swept by γ and the functional $Q(\gamma)$ in (1.6) we should use the external product on $\mathbb{R}^n$.

2 - Estimates for parabolic equations

In this section we consider two applications of Theorem 1 to parabolic equations. The basic idea is to associate to a solution $u(t)$ a parametric curve $\gamma(t)$ in such a way that $\gamma(t)$ moves in the direction of curvature.

Theorem 2. *Consider a scalar parabolic equation on an open interval* $]a, b[$, *with fixed boundary conditions:*

$$u_t = c(t,x)u_{xx}, \qquad \begin{cases} u(t,a) = \bar{u}_1, \\ u(t,b) = \bar{u}_2. \end{cases} \tag{2.1}$$

Assume that the function $c(t,x)$ *is continuous and positive on* $[0,T] \times [a,b]$. *Then every solution of (2.1) satisfies the a priori estimate*

$$\int_0^T \int_a^b \big|u_t(t,x)\big|\, dxdt \le \frac{1}{4} \int_a^b \int_a^b \big|u_x(0,x) - u_x(0,y)\big|\, dxdy. \tag{2.2}$$

Proof. We can assume that $c(t,x)$ is smooth: otherwise we can approximate $c(t,x)$ with a sequence of positive smooth functions $c^\nu(t,x)$ and apply classical convergence results to get that the solution $u^\nu(t,x)$ tends to the solution $u(t,x)$ of (2.1) in $C^0([0,T] \times [a,b])$ (see for example [4]).

Observe that the graph of every solution of (2.1) is a curve moving in the direction of the curvature. Indeed, setting $\gamma(t,x) \doteq \big(x, u(t,x)\big)$, one has

$$\gamma_t = \big(0,\ c(t,x)u_{xx}\big) = c(t,x)\gamma_{xx}.$$

Applying Theorem 1 to the curve γ we obtain the a priori estimate

$$\begin{aligned} \text{Area}\big(\gamma;\, [0,T]\big) &= \int_0^T \int_a^b \Big|\gamma_t(s,y) \wedge \gamma_x(s,y)\Big|\, dyds \int_0^T \int_a^b \big|u_t(t,x)\big|\, dxdt \\ &\le Q\big(\gamma(0)\big) - Q\big(\gamma(T)\big) \le \frac{1}{2} \iint_{a<x<y<b} \Big|\big(1,\, u_x(0,x)\big) \,\wedge\, \big(1,\, u_x(0,y)\big)\Big|\, dxdy. \end{aligned}$$

This yields (2.2).

Remark 4. Recalling Remark 2, one easily checks that the same estimate (2.2) holds in the periodic case, i.e. for solutions of

$$u_t = \phi(t,x)u_{xx}, \qquad \begin{cases} u(t,a) = u(t,b), \\ u_x(t,a) = u_x(t,b), \end{cases} \tag{2.3}$$

assuming that the continuous function ϕ satisfies $\phi(t,a) = \phi(t,b)$ for every t.

Our second application is concerned with viscous scalar conservation laws. Let $f, c : I\!R \mapsto I\!R$ be smooth functions, with $c \geq 0$, and consider the conservation law

$$u_t + f(u)_x - \big(c(u)u_x\big)_x = 0. \tag{2.4}$$

Let $u = u(t,x)$ be a smooth solution of (2.4), with initial data having bounded variation. This of course implies $u(t,\cdot) \in BV$ for all $t \geq 0$, because the total variation cannot increase in time. For each $t \geq 0$, consider the curve $\gamma(t) \in \mathcal{F}$ defined as

$$\gamma(t,x) \doteq \begin{pmatrix} u(t,x) \\ f\big(u(t,x)\big) - c\big(u(t,x)\big)u_x(t,x) \end{pmatrix} \qquad x \in I\!R. \tag{2.5}$$

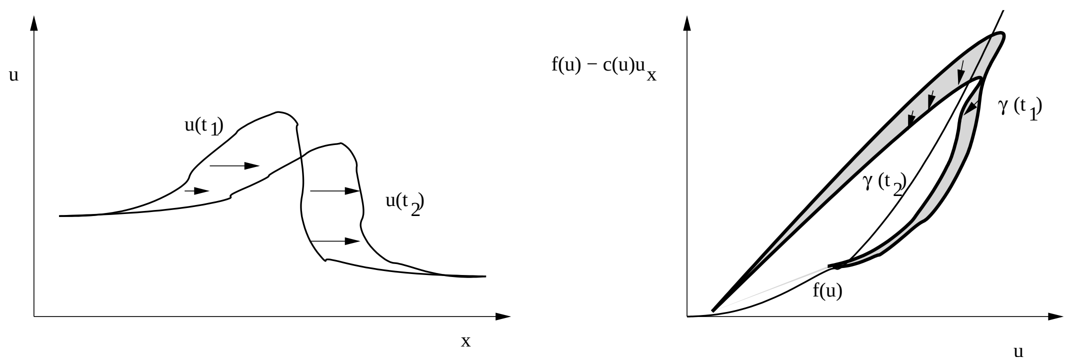

Figure 6.

By a direct computation, one checks that γ moves in the direction of curvature (fig. 6). Indeed, it satisfies the equation

$$\gamma_t + f'(u)\gamma_x - \big(c(u)\gamma_x\big)_x = 0. \tag{2.6}$$

If we define *amount of interaction* $\mathcal{I}(u; [t_1,t_2])$ during the interval $[t_1,t_2]$ as the area swept by $\gamma = \gamma(u)$, then Theorem 1 implies

$$\mathcal{I}\big(u;\ [t_1,t_2]\big) \doteq \text{Area}\big(\gamma;\ [t_1,t_2]\big) \leq \mathcal{Q}\big(u(t_1)\big) - \mathcal{Q}\big(u(t_2)\big), \tag{2.7}$$

where the *interaction potential* $\mathcal{Q}(u)$ is now defined as

$$\begin{aligned} \mathcal{Q}(u) \doteq Q\big(\gamma(u)\big) &= \frac{1}{2}\iint_{x<y} \big|\gamma_x(x) \wedge \gamma_x(y)\big|\,dxdy \\ &= \frac{1}{2}\iint_{x<y} \big|u_x(x)\big|\,\big|u_x(y)\big| \cdot \big|\eta(x) - \eta(y)\big|\,dxdy, \end{aligned} \tag{2.8}$$

$$\eta \doteq f'(u) - \frac{\big(c(u)u_x\big)_x}{u_x}\,. \tag{2.9}$$

Observe that, for a solution of (2.4), the above quantity $\eta = -u_t/u_x$ can be regarded as the speed of a viscous wave. Indeed, for a given $\bar{x} \in I\!\!R$, $\eta(t,\bar{x})$ is the speed of the intersection of the graph of $x \mapsto u(t,x)$ with the horizontal line $u \equiv u(\bar{x})$. Clearly $\mathcal{Q}(u) = 0$ if and only if this speed η is constant. The functional $\mathcal{Q}$ thus seems to be suited for studying the stability of traveling waves. By (2.7), $\mathcal{Q}$ provides a Lyapunov functional for all BV solutions of (2.4). Indeed

$$\frac{d}{dt}\mathcal{Q}\big(u(t,\cdot)\big) \leq 0. \tag{2.10}$$

Remark 5. An entirely similar result holds for periodic solutions of (2.6), say with $u(t,x+p) = u(t,x)$. In this case $\gamma : [0,p] \mapsto I\!\!R^2$, defined as in (2.5), is a closed curve. The inequalities (2.7), (2.10) remain valid in connection with the functional

$$\mathcal{Q}(u) \doteq \frac{1}{2}\iint_{0\leq x<y\leq p} \big|u_x(x)\big|\,\big|u_x(y)\big| \cdot \big|\eta(x) - \eta(y)\big|\,dxdy, \tag{2.11}$$

3 - Application to scalar conservation laws

Consider a scalar conservation law

$$u_t + f(u)_x = 0, \tag{3.1}$$

with f sufficiently smooth. Given an initial data $u_0 \in$ BV, let $u = u(t,x)$ be the corresponding unique entropic solution. In this section we will show that to $u(t,\cdot)$ one can associate a parametric curve $\gamma(t)$ such that $\gamma(t)$ moves in the direction of curvature, i.e. $\gamma(s) \prec \gamma(t)$ whenever $0 \leq s < t$.

Given a map $u : I\!\!R \mapsto I\!\!R$ with bounded variation, define the function $U \in$ BV as

$$U(x) \doteq \int_{-\infty}^{x} |Du| = \text{Tot.Var.}\Big\{u;\,(-\infty,x]\Big\}. \tag{3.2}$$

Here Du is the measure corresponding to the distributional derivative of u. For $\theta \in \big]0, \text{Tot.Var.}(u)\big[$, we define $x(\theta)$ to be the point x such that

$$U(x-) \leq \theta \leq U(x+). \tag{3.3}$$

Let now two points $\big(u^-, f(u^-)\big), \big(u^+, f(u^+)\big) \in I\!\!R^2$, be given, with $u^- > u^+$ (or $u^- < u^+$, respectively). We then define the curve $\mathcal{R}\big(\theta; [u^-, u^+]\big)$, where $\theta \in \big[0,\ |u^+ - u^-|\big]$, as the graph of the convex (concave) envelope of the function $f(u)$ on the interval $[u^-, u^+]$. To a function $u \in \mathrm{BV}$ we associate the parametric curve $\gamma : \big[0,\ \mathrm{T.V.}(u)\big] \mapsto \mathbb{R}^2$ defined as

$$\gamma(u;\theta) \doteq \begin{cases} \Big(u(-\infty),\ f\big(u(-\infty)\big)\Big) & \theta = 0, \\ \Big(u(\theta),\ f\big(u(\theta)\big)\Big) & u \text{ is continuous at } x(\theta), \\ \mathcal{R}\Big(\theta - U(x(\theta)-); [u(x(\theta))^-, u(x(\theta))^+]\Big) & \begin{array}{c} u \text{ has a jump in } x(\theta), \\ \theta \in [U(x(\theta)-), U(x(\theta)+)], \end{array} \\ \Big(u(\infty),\ f\big(u(\infty)\big)\Big) & \theta = \mathrm{Tot.Var.}(u). \end{cases} \tag{3.4}$$

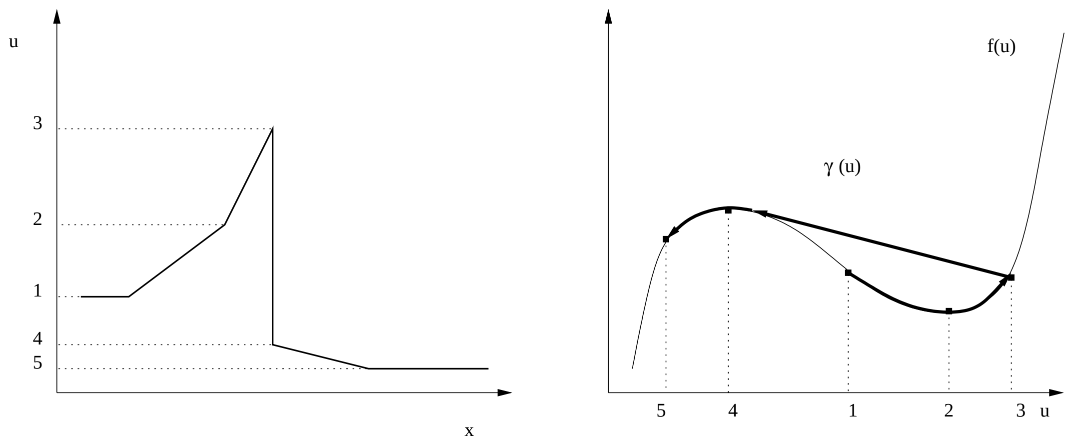

Figure 7.

We claim that $\gamma(u) \in \mathcal{F}$. Indeed, if κ provides a Lipschitz constant for the function f restricted to the range of u, then

$$L\big(\gamma(u)\big) \leq \sqrt{1+\kappa^2} \cdot \ \mathrm{Tot.Var.}(u).$$

We can thus associate to $\gamma(u)$ the functional

$$Q(\gamma(u)) = \frac{1}{2} \iint_{\theta<\theta'} \Big|\gamma(u;\theta) \wedge \gamma(u;\theta')\Big| \, d\theta d\theta'. \tag{3.5}$$

Given $u \in \mathrm{BV}$, let $\gamma(u) \in \mathcal{F}$ be the corresponding parametric curve. We say that γ' is obtained from $\gamma(u)$ by a *Riemann cut* if, for some points $x_1 < x_2$, the curve γ' can be constructed by replacing the part of the curve $\Big\{\gamma(u;\theta); \theta \in \big[U(x_1), U(x_2)\big]\Big\}$ with $\Big\{\mathcal{R}(\theta); \theta \in \big[U(x_1), U(x_2)\big]\Big\}$. Note that this definition depends on the choice of the flux function $f(u)$. It essentially corresponds to the substitution of u with

$$\hat{u}(x) \doteq \begin{cases} u(x) & x < x_1, \\ u(x_1-) & x_1 \leq x \leq \overline{x}, \\ u(x_2+) & \overline{x} < x \leq x_2, \\ u(x) & x > x_2. \end{cases} \tag{3.6}$$

All waves of u located inside the interval $[x_1, x_2]$ are thus collapsed to a single point $\bar{x}$, with $x_1 < \bar{x} < x_2$.

Lemma 3. *Let u be a scalar BV function. Fix any three points $x_1 < \bar{x} < x_2$ and construct the function $\hat{u}$ as in (3.6). Then the corresponding curves, defined by (3.4), satisfy $\gamma(u) \prec \gamma(\hat{u})$.*

Proof. Consider first the case where u is piecewise constant. By an inductive argument, it suffices to prove the result in the case where u contains two adjacent jumps, say at $y_1 < y_2$, which are replaced by a single jump of $\hat{u}$. Consider the left, middle and right states

$$u_l = u(y_1-), \qquad u_m = u(y_1+) = u(y_2-), \qquad u_r = u(y_2+).$$

In the various possible configurations, it it easy to check that the portion of the curve $\gamma(\hat{u})$ corresponding to the jump (u_r, u_l) is contained in the convex hull of the portion of the curve $\gamma(u)$ corresponding to the two jumps (u_r, u_m) and (u_m, u_l). The two main cases $u_l < u_m < u_r$ and $u_l < u_r < u_m$ are illustrated in fig. 8.

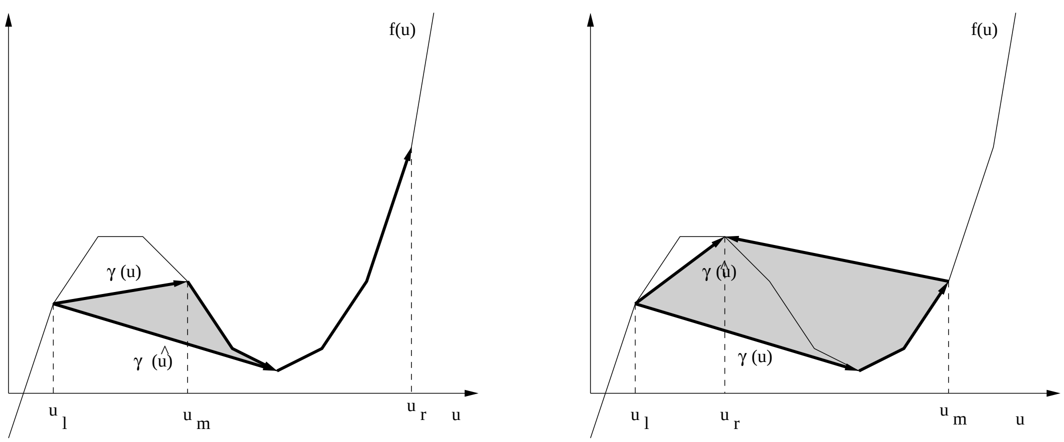

Figure 8.

This proves the lemma in the case of piecewise functions u. The general case is handled by a standard approximation argument.

We now establish a relation between the $\mathbf{L}^1$ convergence of a sequence of functions u_n and the convergence of the corresponding parametric curves $\gamma(u_n)$. The main ideas in the proof are taken from [1], where a similar construction was used to prove the lower semicontinuity of the Glimm functional.

Lemma 4. *Let $(u_n)_{n\geq 1}$ be a sequence of scalar functions with uniformly bounded total variation. Assume that $u_n \to u$ in $\mathbf{L}^1$ and $\gamma(u_n) \to \gamma_0$ in $\mathcal{F}$. Then $\gamma_0 \prec \gamma(u)$.*

Proof. By possibly taking a subsequence, we can assume the pointwise convergence $u_n(x) \to u(x)$

and the weak convergence of measures $|Du_n| \rightharpoonup \mu$, for some positive measure μ. The lemma will be proved by showing that, for every $\varepsilon > 0$, one can construct a sequence of curves $\hat{\gamma}_n$ such that

$$\limsup_{n\to\infty} d\big(\hat{\gamma}_n,\ \gamma_0\big) \leq \varepsilon, \qquad \gamma(u_n) \prec \hat{\gamma}_n \quad \text{for each } n. \tag{3.7}$$

Let $\varepsilon > 0$ be given. Consider first a point $\bar{x}$ such that $\mu\big(\{x\}\big) = 0$. We can then choose $\delta_\varepsilon > 0$ such that

$$\mu\big([\bar{x} - \delta_\varepsilon,\ \bar{x} + \delta_\varepsilon]\big) < \varepsilon, \qquad |Du_n|\big([\bar{x} - \delta_\varepsilon,\ \bar{x} + \delta_\varepsilon]\big) < \varepsilon$$

for all n sufficiently large. Choose a point $y \in]\bar{x} - \delta_\varepsilon,\ \bar{x} + \delta_\varepsilon[$ where u and all u_n are continuous, so that the point $\big(u_n(y), f(u_n(y))\big)$ lies in the range of the map $\gamma\big(u_n\big)$, for each $n \geq 1$. Let $Lip(f)$ be a Lipschitz constant for f on an interval $[u_{min},\ u_{max}]$ containing the range of all functions u_n. From the relations

$$\limsup_{n\to+\infty} \big|u_n(y) - u(\bar{x})\big| \leq \varepsilon, \qquad \limsup_{n\to+\infty} \Big|f\big(u_n(y)\big) - f\big(u(\bar{x})\big)\Big| \leq \varepsilon \cdot Lip(f),$$

it follows that the point $\big(u(\bar{x}), f(u(\bar{x})\big)$ lies in the range of γ_0. In other words, points $\bar{x}$ such that $\mu(\{\bar{x}\}) = 0$ correspond to the same point on the image of γ_0 and on the image of $\gamma(u)$.

Next, let $x_1 < \cdots < x_N$ be all the points such that

$$\mu(x_\alpha) \geq \varepsilon \qquad \alpha = 1, \ldots, N.$$

Choose $\delta_\varepsilon > 0$ such that

$$\mu\big([x_\alpha - \delta_\varepsilon, x_\alpha[\cup]x_\alpha, x_\alpha + \delta_\varepsilon]\big) < \varepsilon, \tag{3.8}$$

$$|Du_n|\big([x_\alpha - \delta_\varepsilon, x_\alpha[\cup]x_\alpha, x_\alpha + \delta_\varepsilon]\big) < \varepsilon, \tag{3.9}$$

for every α and all n sufficiently large. Choose two points y_α^-, y_α^+ with $x_\alpha - \delta_\varepsilon < y_\alpha^- < x_\alpha < y_\alpha^+ < x_\alpha + \delta_\varepsilon$, where all u_n are continuous and such that $\mu\big(\{y_\alpha^-\}\big) = \mu\big(\{y_\alpha^+\}\big) = 0$. These conditions imply that the points $\big(u_n(y_\alpha^\pm), f(u_n(y_\alpha^\pm))\big)$, $\big(u(y_\alpha^\pm), f(u(y_\alpha^\pm))\big)$, belong to the range of $\gamma(u_n)$, $\gamma(u)$, respectively. Moreover,

$$\limsup_{n\to\infty} \big|u_n(y_\alpha^-) - u(x_\alpha-)\big| \leq \varepsilon, \qquad \limsup_{n\to\infty} \big|u_n(y_\alpha^+) - u(x_\alpha+)\big| \leq \varepsilon.$$

We now define $\hat{\gamma}_n$ as the curve obtained from $\gamma(u_n)$ by performing N Riemann cuts, in connection with the intervals $[y_\alpha^-, y_\alpha^+]$, $\alpha = 1, \ldots, N$. By construction, $\gamma(u_n) \prec \hat{\gamma}_n$. Moreover, the above analysis yields

$$d\big(\hat{\gamma}_n,\ \gamma(u_n)\big) \leq C\varepsilon,$$

for some constant C independent of ε and all n sufficiently large. This establishes (3.7), proving the lemma.

As a consequence of the above lemma we have:

Theorem 3. *Assume that $f : I\!R \mapsto I\!R$ is locally Lipschitz and let $u(t)$ be the unique entropy solution to (3.1) with initial data $u_0 \in BV$. Call $\gamma(t) \doteq \gamma(u(t))$ the parametric curve associated to $u(t)$, as in (3.4). Then $\gamma(t)$ moves in the direction of curvature, i.e. $\gamma(s) \prec \gamma(t)$ whenever $s < t$.*

Proof. By the semigroup property [7], it is not restrictive to assume $s = 0$. Following [3], the solution u can be obtained as limit of a sequence of front tracking approximations u_n. More precisely, for each $n \geq 1$ we let f_n be a piecewise affine function which coincides with f at each node $p_{i,n} \doteq 2^{-n} i$. Moreover, we choose a piecewise constant approximate initial data $u_{0,n}$ taking values inside the grid $2^{-n} \mathbb{Z}$ so that

$$u_{0,n} \to u_0 \quad \text{in } \mathbf{L}^1_{\text{loc}}, \qquad d\big(\gamma(u_0),\ \gamma(u_{0,n})\big) \to 0. \tag{3.10}$$

Calling $u_n = u_n(t,x)$ the corresponding entropy solution of

$$u_t + f_n(u)_x = 0, \qquad u_n(0,x) = u_{0,n},$$

it is known that each u_n is piecewise constant, with jumps on a finite number of straight lines in the t-x plane, and takes values within the grid $2^{-n}\mathbb{Z}$. The corresponding curves $\gamma_n(t) = \gamma\big(u_n(t)\big)$ all move in the direction of the curvature. Indeed, each γ_n remains constant except at finitely many times $0 < t_1 < t_2 < \cdots < t_{m_n}$ where two fronts interact. At each of these times t_j, the curve $\gamma_n(t_j+)$ is obtained from $\gamma_n(t_j-)$ by a Riemann cut, hence the relation $\gamma_n(0) \leq \gamma_n(t)$ clearly holds for every $t > 0$.

Using Lemma 4 we can now pass to the limit as $n \to \infty$. By (3.10) this yields $\gamma(0) \prec \gamma(t)$, proving the theorem.

Remark 6. As in Section 2, define the *amount of interaction* $\mathcal{I}(u; [t_1, t_2])$ during $[t_1, t_2]$ as the area swept by $\gamma(u(t))$ over the time interval $[t_1, t_2]$. Theorems 1 and 3 then yield

$$\mathcal{I}(u; [t_1, t_2]) \leq \mathcal{Q}(u(t_1)) - \mathcal{Q}(u(t_2)), \tag{3.11}$$

where $\mathcal{Q}(u)$ is the interaction potential defined at (3.5). One can interpret the quantity $\mathcal{I}(u; [t_1, t_2])$ as measuring the change in the shape of the solution $u = u(t,x)$ within the time interval $[t_1, t_2]$. By (3.11), this is bounded in terms of a Glimm type potential, similar to the one introduced by T.P. Liu in his paper on hyperbolic systems [9].

Remark 7. For an arbitrary flux function f, there may not exist a Borel function λ (the wave speed), such that

$$\mathcal{Q}(u) = \frac{1}{2} \iint_{x<y} |\lambda(x) - \lambda(y)| |Du|(x) |Du|(y). \tag{3.12}$$

However, if f is strictly convex and u has no upward jumps, i.e. $u(x-) \geq u(x+)$ for every $x \in \mathbb{R}$, then (3.12) holds with

$$\lambda(x) \doteq \begin{cases} f'\big(u(x)\big) & u \text{ is continuous in } x, \\ \dfrac{f\big(u(x+)\big) - f\big(u(x-)\big)}{u(x+) - u(x-)} & u \text{ has a jump at } x\,. \end{cases} \tag{3.13}$$

In particular the entropic solution of a scalar conservation law with strictly convex flux satisfies (3.12)-(3.13) for every $t > 0$.

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