# THE SEMIGROUP GENERATED BY A TEMPLE CLASS SYSTEM WITH NON–CONVEX FLUX FUNCTION

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Abstract. We consider the Cauchy problem for a nonlinear $n \times n$ system of conservation laws of Temple class, i.e. with coinciding shock and rarefaction curves and with a coordinate system made of Riemann invariants. Without any assumption on the convexity of the flux function, we prove the existence of a semigroup made of weak solutions of the equations and depending Lipschitz continuously on the initial data with bounded total variation.

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## Introduction

In this paper we consider the Cauchy problem for a strictly hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x & = & 0 \\ u(0,x) & = & u_0(x) \end{cases} \tag{$*$}$$

with $f(\cdot) : \Omega \to \mathbb{R}^n$ sufficiently smooth, $\Omega \subseteq \mathbb{R}^n$ open. We assume that there exists a systems of coordinates consisting of Riemann invariants; we also assume that shock and rarefaction curves coincide and are straight lines.

Systems of this type were studied in [10] and [11]. In particular there it is proved that for any initial datum $u_0 \in L^1(\mathbb{R};\mathbb{R}^n)$ with values in the set

$$E = \{u = (w_1, \ldots, w_n) : w_i \in [a_i, b_i]\ i = 1, \ldots, n\} \subseteq \Omega$$

and with bounded Total Variation, there exists a weak entropic solution $u(t) = u(t,x)$ of $(*)$, defined for all $t \geq 0$ and with values in $E$.

In [1], for a convex flux function $f$, it is proved that, if $\mathcal{D}_M$ is the domain defined as

$$\mathcal{D}_M \doteq \left\{ u : \mathbb{R} \to E : \sum_i \text{Tot.Var.}(w_i(u)) \leq M \right\} \subseteq L^1(\mathbb{R}; E), \tag{$**$}$$

then there exists a Lipschitz continuous semigroup $S_t \cdot : [0,+\infty) \otimes \mathcal{D}_M \longrightarrow \mathcal{D}_M$ with the following properties:

- $\|S_t u - S_s v\|_{L_1} \leq C_M(|t-s| + \|u-v\|_{L_1})$;
- each trajectory $S_t u_0$ is a weak entropic solution of the Cauchy problem $(*)$ with initial datum $u_0$;
- if $u_0$ is a piecewise constant function, then, for small $t$, $S_t u_0$ coincides with the function obtained piecing together the solutions of the corresponding Riemann problems.

In this paper we suppress the assumption of convex flux function, i.e. we do not assume that each characteristic field is either linearly degenerate or genuinely nonlinear. Precisely we will prove the following Theorem:

**Theorem.** *For each $M > 0$ there exists a constant $C_M$ and a semigroup $S : [0,+\infty) \otimes \mathcal{D}_M \to \mathcal{D}_M$, where $\mathcal{D}_M$ is defined as in $(**)$, such that:*

*a)* $\|S_t u - S_s v\|_{L_1} \leq C_M(|t-s| + \|u-v\|_{L_1})$;

*b)* *each trajectory $S_t u_0$ is a weak entropic solution of the Cauchy problem $(*)$ with initial datum $u_0$;*

*c)* *if $u_0$ is a piecewise constant function, then, for small $t$, $S_t u_0$ coincides with the function obtained piecing together the solutions of the corresponding Riemann problems.*

From the results of [3], it follows that the above semigroup is unique, and its trajectories can be characterized as “Viscosity solutions” of the system $(*)$. Our approach resembles the approach used in [1]. The paper is organized as follows.

In section 1 we give the fundamental definitions and we show how to solve a Riemann problem with a non–convex flux function. In particular we show that the entropic solution is obtained patching together the entropic solutions of $n$ scalar equations. In section 2 we describe the wave–front tracking algorithm used to obtain piecewise constant solutions of $(*)$ defined for all $t \geq 0$. We also introduce the fundamental notions of Total Variation Tot.Var.$(u)$ and Interaction Potential $Q(u)$ for a vector function $u(x)$.
In section 3, using some results of [3], we obtain a useful estimate for scalar conservation laws: we give a bound for the $L^1$–distance between solutions of two decoupled scalar equations, bound used heavily in section 5. In section 4 we adapt the results in [2] concerning the Shift–differentials to this special case; we show in particular how we measure the $L^1$–distance between two solution of $(*)$ with different initial data generated by our algorithm. In section 5 we evaluate the increment of this distance when there is an interaction between shocks, and using this estimate we prove in section 6 that this distance is bounded by a constant, depending only on $E$ and the Total Variation of the initial datum, times the $L^1$–distance at time $t = 0$. Finally in section 7, using the standard technique developed in [4] and adapted to this problem in [1], we conclude with the proof of the above Theorem.

## 1. The Riemann problem

Consider a strictly hyperbolic system of $n$ conservation laws:

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $u \in \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$, $\Omega$ being an open subset of $\mathbb{R}^n$ and $f(u)$ sufficiently smooth. We will assume the following hypotheses:

(H1) The system is strictly hyperbolic in $\Omega$, i.e. the matrix $A(u) = [a_{ij}] = [\partial f_i(u)/\partial u_j]$ has $n$ distinct eigenvalues $\{\lambda_i\}_{i=1,\dots,n}$ and the following holds:

$$\sup_{u \in \Omega} \lambda_i(u) < \inf_{u \in \Omega} \lambda_{i+1}(u)$$

for all $i = 1, \dots, n-1$.

(H2) If $r_i(u)$ is the normalized right eigenvector corresponding to the $i$–th eigenvalue, then $r_i \nabla r_i = 0$, i.e. the directional derivative of $r_i$ along $r_i$ is zero. This implies that the $i$-th rarefaction curve is a straight line and coincides with the $i$–th shock curve.

(H3) The rarefaction curves form a system of Riemann coordinates: there exists a system of coordinates $(w_1, \dots, w_n)$ such that the vector $\dfrac{\partial u}{\partial w_i}$ is parallel to $r_i$ for all $i = 1, \dots, n$.

Consider two points $u(-) = (w_1(-), \dots, w_n(-))$ and $u(+) = (w_1(+), \dots, w_n(+))$ in $\Omega$. In the following we call Riemann problem $[u(-), u(+)]$ the Cauchy problem for (1.1) with initial datum

$$u(0,x) = u_0(x) = \begin{cases} u(-) & x < 0 \\ u(+) & x > 0. \end{cases}$$

Let $E \subseteq \Omega$ be a set defined as

$$E = \{u = (w_1, \dots, w_n) : w_i \in [a_i, b_i] \ i = 1, \dots, n\}$$

for some intervals $[a_i, b_i]$. Since this set is compact, there exists a constant $C_1$ such that

$$\frac{1}{C_1} \sum_{i=1}^{n} |w_i(+) - w_i(-)| \leq \|u(-) - u(+)\| \leq C_1 \sum_{i=1}^{n} |w_i(+) - w_i(-)|.$$

To solve the Riemann problem $[u(-), u(+)]$, where $u(-), u(+) \in E$, we consider the intermediate states $u_i = (w_1(+), \dots, w_{i-1}(+), w_i(-), \dots, w_n(-))$, $i = 1, \dots, n+1$, $u_1 = u(-)$ and $u_{n+1} = u(+)$ and we solve the corresponding Riemann problems $[u_i, u_{i+1}]$. As the Riemann coordinate $w_i$ goes from $w_i(-)$ to $w_i(+)$, we consider the line $u(l_i) = u_i + \sigma_i r_i(u_i)$ and the scalar flux function

$$f_i(\sigma_i) = \int_0^{\sigma_i} \lambda_i\big(u_i + s r_i(u_i)\big) ds.$$

We note that $\sigma_i$ is the signed Euclidean length along the segment $L_i$ determined by

$$\begin{aligned} L_i = \{u =&(w_1, \dots, w_n) : w_i(-) \leq w_i \leq w_i(+) \\ &w_1 = w_1(+), \dots, w_{i-1} = w_{i-1}(+), w_{i+1} = w_{i+1}(-), \dots, w_n = w_n(-)\}, \end{aligned} \tag{1.2}$$

whose length is indicated as $\|L_i\|$; we choose the positive direction setting $u_{i+1} = u_i + \|L_i\| r_i(u_i)$. Next we consider the Cauchy problem for the scalar equation

$$\begin{cases} [\sigma_i]_t + [f_i(\sigma_i)]_x &= \quad 0 \\ \quad \sigma_{i,0} &= \quad \sigma_i(0, x) \quad = \quad \|L_i\| \chi_{(0,+\infty)}(x). \end{cases} \tag{1.3}$$

Here and in the following $\chi_A(\cdot)$ denotes the characteristic function of the set $A$. Suppose that $\sigma_i(t, x)$ is the unique entropic solution of (1.3) and let $u_i(t, x) = u_i + \sigma_i(t, x) r_i(u_i)$. By the assumption $r_i \nabla r_i = 0$, it is an easy exercise to prove that this gives a weak entropic solution to the Riemann problem $[u_i, u_{i+1}]$. Now we piece together the solutions $u_i(t, x)$, thanks to (H1), and this obviously is a weak entropic solution to the initial Riemann problem.

## 2. Wave–front tracking algorithm

In this section we construct an algorithm to obtain piecewise constant solutions of the Cauchy problem

$$\begin{cases} u_t + f(u)_x &= \quad 0 \\ \quad u(0, x) &= \quad u_0. \end{cases} \tag{2.1}$$

We base our construction mainly on [3]. Fixed a $\nu \in \mathbb{N}$ we define the set

$$G^\nu = \Big\{ u = (w_1, \ldots, w_n) : w_i \in [a_i, b_i] \cap 2^{-\nu}\mathbb{Z} \Big\} \subseteq E,$$

and we consider initial data $u_0 \in G^\nu$ with $\sum_i \text{Tot.Var.}(w_i(u)) \leq M$, $M$ being a fixed constant. This set of vector functions forms a domain $\mathcal{D}^\nu_M$,

$$\mathcal{D}^\nu_M = \Big\{ u : \mathbb{R} \to G^\nu : \sum_i \text{Tot.Var.}(w_i(u)) \leq M \Big\}. \tag{2.2}$$

We now specify the Riemann solver: given $u(-)$, $u(+)$ in $G^\nu$, we solve the Riemann problem $[u(-), u(+)]$ piecing together the solutions of the Riemann problems for scalar conservation laws

$$\begin{cases} [\sigma^\nu_i]_t + [f^\nu_i(\sigma^\nu_i)]_x & = & 0 \\ \sigma^\nu_{i,0} & = & \|L^\nu_i\| \chi_{(0,+\infty)}(x) \end{cases} \tag{2.3}$$

where $f^\nu_i(\sigma_i)$ is the piecewise linear function that coincides with $f_i(\sigma_i)$ on $G^\nu$: precisely, given a scalar function $h(\cdot)$ defined on a segment $[m, n]$ and a finite subset

$$K^J = \{m, k_1, \ldots, k_J, n\} \subseteq [m, n],$$

we define the piecewise linear function $h^J$ that coincides with $h$ on $K^J$ as

$$h^J(z) = \frac{(z - k_j)h\big(k_{j+1}\big) - (z - k_{j+1})h\big(k_j\big)}{k_{j+1} - k_j} \quad \text{if} \quad z \in [k_j, k_{j+1}]. \tag{2.4}$$

Since each solution $\sigma^\nu_i(t, x)$ is a solution of the scalar equation, we have an exact solution to the Riemann problem, but this solution is in general not entropic.

At time $t = 0$ we solve locally the Riemann problems generated by $u_0$. Patching together these solutions, we have a piecewise constant solution of (2.1), until two or more wave–fronts interact. At each time of interaction, i.e. where two or more shocks collide, we solve again the corresponding Riemann problem according to (2.2). Since our Riemann solver takes values in $G^\nu$, our piecewise constant solution is in $G^\nu$. We denote this solution as $u^\nu(t, x)$. Now we prove that this solution is defined for all times and has a finite number of shocks.

Given a discontinuity $[u(-), u(+)]$, we will use two equivalent quantities to measure the strength of the jump $[u(-), u(+)]$. Namely, let $\sigma_i$ be the signed Euclidean length along the segment $L_i$, defined in (1.2), for the Riemann problem $[u(-), u(+)]$, while $\tau_i$ is the Riemann $i$–th coordinate of the vector $w(u(+)) - w(u(-))$. Next, given a piecewise constant function $u(t, x)$, we define the following two quantities:

– the Total Variation of $u(t,x) \in \mathbb{R}^n$ is the Total Variation of $w(t,x) = w(u(t,x))$, i.e.

$$\text{Tot.Var.}(u(t)) = \sum_{\alpha \in A} \|w(\alpha)_- - w(\alpha)_+\| = \sum_{\alpha \in A} \sum_i |\tau_i(\alpha)|,$$

where $A$ is the set of discontinuities and $w(\alpha)_\pm$ are the values of $w$ across the discontinuity $\alpha$;

– the Interaction Potential of the solution $u(t,x) \in \mathbb{R}^n$ is

$$Q(t) = \sum_{i<j} \sum_{x<y} |\tau_i(y)||\tau_k(x)|,$$

where in this case $\tau_i(x)$ is the $i$–th coordinate of the discontinuity located at $x$, i.e. we consider two wave $\tau_i(y)$, $\tau_k(x)$ as approaching if $i < j$ and $x < y$.

We note that, with our Riemann solver, there are three types of interaction:

a) either two or more shocks of different families collide and $Q$ decreases at least of $2^{1-\nu}$;

b) or two or more shocks of the same family with $\sigma$ of different sign collide and the Total Variation of the solution, measured along Riemann coordinates, decrease at least of $2^{1-\nu}$;

b) or two or more shocks of the same family with $\sigma$ of the same sign collide generating a single shock.

Since at any interaction neither the Total Variation nor the Interaction Potential increase, the constructed solution $u^\nu(t,x)$ of (2.1) at time $t$ is in $\mathcal{D}^\nu_M$ and has a finite number of shocks, because each shock, measured in Riemannian coordinates $(w_1, \dots, w_n)$, is bigger than $2^{-\nu}$. Moreover $Q$ is a priori bounded, so that case a) can occur only a finite number of times. The same remarks can be applied to case b), using in this case the Total Variation, and then also case c) can occur a finite number of times. Thus our solution will have a finite number of points of interaction and it is defined for all times.

At this point, using the finite speed of propagation of shocks, it is easy to prove that the map:

$$\begin{array}{rcccc} u^\nu & : & [0,+\infty) & \longrightarrow & \mathcal{D}^\nu_M \\ & & t & \longrightarrow & u^\nu(t), \end{array} \tag{2.5}$$

is Lipschitz continuous. We define the semigroup $S^\nu_t \cdot$ as

$$\begin{array}{rcccc} S^\nu & : & [0,+\infty) \otimes \mathcal{D}^\nu_M & \longrightarrow & \mathcal{D}^\nu_M \\ & & (t,u_0) & \longrightarrow & S^\nu_t u_0 = u^\nu(t). \end{array} \tag{2.6}$$

## 3. $L_1$ STABILITY OF SCALAR CONSERVATION LAWS

In this section we want to give an estimate of the stability of the solution if we have a perturbed scalar function. Precisely, we consider the following two Cauchy problems:

$$\begin{cases} u_t + f(u)_x &= 0 \\ u(0) &= u_0 \end{cases}, \qquad \begin{cases} v_t + g(v)_x &= 0 \\ v(0) &= v_0, \end{cases} \tag{3.1}$$

with $f$, $g$ sufficiently smooth. It is well known (see [8]) that equations (3.1) generate two semigroups

$$\begin{array}{rccc} S^f & : & [0,+\infty)\otimes L^1 & \longrightarrow & L^1 \\ & & (t,u_0) & \longrightarrow & S^f_t u_0 = u(t,x), \end{array} \tag{3.2}$$

and

$$\begin{array}{rccc} S^g & : & [0,+\infty)\otimes L^1 & \longrightarrow & L^1 \\ & & (t,v_0) & \longrightarrow & S^g_t v_0 = v(t,x), \end{array} \tag{3.2$'$}$$

where $u(t,x)$ and $v(t,x)$ are the entropic solution of (3.1), respectively. These semigroups are contractive in $L^1$: if $\|u_0 - v_0\|_{L^1} < +\infty$, then

$$\|S^f_t u_0 - S^f_t v_0\|_{L^1} \le \|u_0 - v_0\|_{L^1}, \quad \|S^g_t u_0 - S^g_t v_0\|_{L^1} \le \|u_0 - v_0\|_{L^1}. \tag{3.3}$$

We consider here initial data with values in some compact interval $[m,n]$ of the real line; it is well known that, by maximum principle, also the solution of (3.1) will have values in the same interval. We define the quantity

$$\Gamma = \max_{z\in[m,n]} |f'(z) - g'(z)|, \tag{3.4}$$

where $(\cdot)'$ stands for the derivative. We can prove the following Theorem:

**Theorem 3.1.** *Let $u(t) = u(t,x) = S^f_t u_0$, $v(t) = v(t,x) = S^g_t v_0$ be the entropic solutions of equations (3.1); for any time $t \ge 0$ we have*

$$\|u(t) - v(t)\|_{L^1} \le \|u_0 - v_0\|_{L^1} + t\,\Gamma \min\Big\{ \text{Tot.Var.}(u_0), \text{Tot.Var.}(v_0) \Big\}. \tag{3.5}$$

*Proof.* We can consider wave–front approximations of the semigroups $S^f_t\,\cdot$ and $S^g_t\,\cdot$ as in [3], because it is standard to obtain the same estimates for the entropic solutions. Thus in the following we will consider the semigroups $S^{f^\nu}_t\,\cdot$ and $S^{g^\nu}_t\,\cdot$, defined as in (2.6) for the scalar conservation laws (3.1). Here $\mathcal{D}^\nu_M$ is the set

$$\mathcal{D}^\nu_M = \Big\{ u \in L^1\big(\mathbb{R}; 2^{-\nu}\mathbb{Z} \cap [m,n]\big) : \text{Tot.Var.}(u) \le M \Big\},$$

and we assume that $u_0$, $v_0$ belong to $\mathcal{D}_M^\nu$. Formula (3.3) holds also for these semigroups.

Suppose that $u_0$ is the initial datum with lowest Total Variation:

$$\min\big\{ \text{ Tot.Var.}(u_0),\ \text{Tot.Var.}(v_0) \ \big\} = \text{Tot.Var.}(u_0) = \Upsilon.$$

We recall that for the Lipschitz semigroups $S_t^{f^\nu}\ \cdot$ and $S_t^{g^\nu}\ \cdot$, generated respectively by $f$ and $g$, we have the estimate (see [3])

$$\|S_t^{f^\nu}u_0 - S_t^{g^\nu}u_0\|_{L^1} \le \int_0^t ds\ \liminf_{h\to 0^+} \|S_h^{g^\nu} S_s^{f^\nu}u_0 - S_{s+h}^{f^\nu}u_0\|_{L^1}. \tag{3.6}$$

By continuity of $S_t^{f^\nu}\ \cdot$ and $S_t^{g^\nu}\ \cdot$ with respect to time, we have just to evaluate

$$\liminf_{h\to 0^+} \|S_h^{g^\nu} S_s^{f^\nu}u_0 - S_{s+h}^{f^\nu}u_0\|_{L^1}$$

outside the times of interaction, i.e. when $S_s^{f^\nu}u_0$ is piecewise constant. Consider a single shock in $u(t,x)$, with strength $\bar b - \bar a$, with $\bar b, \bar a \in 2^{-\nu}\mathbb{Z}\cap[m,n]$, and speed $\bar\Lambda$:

$$u(t,x) = \bar a\chi_{(-\infty,\bar\Lambda t)} + \bar b\chi_{[\bar\Lambda t,+\infty)},$$

where $\chi_A$ is the characteristic function of the set $A \in \mathbb{R}$. Assume for example that $\bar b > \bar a$, the other case being similar. The Riemann problem $[\bar a, \bar b]$ is solved by $S_t^{g^\nu}\ \cdot$ with $N$ shocks, $N \ge 1$:

$$v(t,x) = \bar a\chi_{(-\infty,\Lambda_0 t)} + \sum_{i=1}^{N-1} v_i\chi_{[\Lambda_{i-1}t,\Lambda_i t)} + \bar b\chi_{[\Lambda_N t,+\infty)},$$

where $v_i$ is the value of $v(t,x)$ after the $i$–th shock. For this simple case, $v_i > v_{i-1}$ and denoting $\bar a$, $\bar b$ with $v_0$ and $v_N$ respectively, we have

$$\begin{aligned}
\|v(t,x) - u(t,x)\|_{L^1} =& \sum_{i=1}^{N}(v_i - v_{i-1})|\Lambda_i - \bar\Lambda|t \\
\le& t \sum_{\substack{i=1\\ \Lambda_i<\bar\Lambda}}^{N} (v_i - v_{i-1})(\bar\Lambda - \Lambda_i) + t \sum_{\substack{i=1\\ \Lambda_i\ge\bar\Lambda}}^{N} (v_i - v_{i-1})(\Lambda_i - \bar\Lambda) \\
\le& t\sum_{i=1}^{N_1}(v_i - v_{i-1})(\bar\Lambda - \Lambda_i) + t \sum_{i=N_1+1}^{N} (v_i - v_{i-1})(\Lambda_i - \bar\Lambda),
\end{aligned} \tag{3.7}$$

for some $1 \leq N_1 \leq N$. Since the shock $[\bar{a}, \bar{b}]$ is admissible for $S_t^{f^\nu}\ \cdot$ , if we denote with $\bar{\Lambda}(-)$ and $\bar{\Lambda}(+)$ respectively the speeds of the shocks $[\bar{a}, v_{N_1}]$ and $[v_{N_1}, \bar{b}]$, the following inequality holds:

$$\bar{\Lambda}(+) \leq \bar{\Lambda} \leq \bar{\Lambda}(-).$$

Using the above inequality and the Rankine–Hugoniot condition, we can write

$$\begin{aligned}
&\sum_{i=1}^{N_1}(v_i - v_{i-1})(\bar{\Lambda}-\Lambda_i) + t\sum_{i=N_1+1}^{N}(v_i - v_{i-1})(\Lambda_i - \bar{\Lambda}) \\
&\qquad=\sum_{i=1}^{N_1}(v_i - v_{i-1})(\bar{\Lambda}(-) - \Lambda_i) + (v_{N_1} - v_0)(\bar{\Lambda} - \bar{\Lambda}(-)) + \\
&\qquad\qquad t\sum_{i=N_1+1}^{N}(v_i - v_{i-1})(\Lambda_i - \bar{\Lambda}(+)) + (v_N - v_{N_1})(\bar{\Lambda}(+) - \bar{\Lambda}) \\
&\qquad\leq -\sum_{i=1}^{N_1}\big(g(v_i) - g(v_{i-1})\big) + f(v_{N_1}) - f(v_0) + \\
&\qquad\qquad\sum_{i=N_1+1}^{N}\big(g(v_i) - g(v_{i-1})\big) - f(v_N) + f(v_{N_1}) \\
&\qquad=\big(f(v_{N_1}) - g(v_{N_1}) - f(v_0) + g(v_0)\big) + \\
&\qquad\qquad\big(-f(v_N) + g(v_N) + f(v_{N_1}) - g(v_{N_1})\big) \\
&\qquad=\int_{v_0}^{v_{N_1}} dz\ \big(f'(z) - g'(z)\big) + \int_{v_{N_1}}^{v_N} dz\ \big(g'(z) - f'(z)\big) \\
&\qquad\leq (v_N - v_0)\Gamma = (\bar{b} - \bar{a})\Gamma.
\end{aligned}$$

Using the above formula and (3.7), we have for $\rho$ and $h$ enough small

$$\|S_h^{g^\nu} S_s^{f^\nu} u_0 - S_{s+h}^{f^\nu} u_0\|_{L^1} = \sum_{\alpha \in A}\int_{x_\alpha-\rho}^{x_\alpha+\rho} |S_h^{g^\nu} S_s^{f^\nu} u_0 - S_{s+h}^{f^\nu} u_0| \leq h\Gamma \sum_{\alpha\in A}|\omega_\alpha| \leq h\Gamma\Upsilon,$$

where $A$ is the set of discontinuities of $S_s^{f^\nu} u_0$, $x_\alpha$ and $\omega_\alpha$ are respectively the position and strength of the $\alpha$–th shock. We conclude, using (3.6) and the $L^1$ contraction of both semigroups, that

$$\begin{aligned}
\|u(t)-v(t)\|_{L^1} &= \|S_t^{f^\nu} u_0 - S_t^{g^\nu} v_0\|_{L^1} \leq \|S_t^{f^\nu} u_0 - S_t^{g^\nu} u_0\|_{L^1} + \|S_t^{g^\nu} u_0 - S_t^{g^\nu} v_0\|_{L^1} \\
&\leq \|u_0 - v_0\|_{L^1} + \int_0^t ds\ \liminf_{h\to 0^+} \|S_h^{g^\nu} S_s^{f^\nu} u_0 - S_{s+h}^{f^\nu} u_0\|_{L^1} \leq \|u_0 - v_0\|_{L^1} + t\Gamma\Upsilon. \quad \square
\end{aligned}$$

## 4. The pseudo–polygonal path

Aim of this section is to provide a way to evaluate the $L_1$–distance between two solutions $S_t^\nu u, S_t^\nu v \in \mathcal{D}_M^\nu$. We use the machinery of the Shift–Differentials, adapting the results in [5].

First we define an elementary path $\zeta(\theta)$, $\theta \in [s,t]$, in $L_1$: this is a path parametrized by $\theta$ of the form

$$
\begin{array}{cccc}
\zeta(\theta) & : & [s,t] & \longrightarrow & L^1 \\
 & & \theta & \longrightarrow & \begin{cases} x_\alpha^\theta = x_\alpha + \xi_\alpha \theta, \ x_\alpha^\theta \le x_{\alpha+1}^\theta \ \forall \theta \in [s,t] \\ \\ u^\theta = \displaystyle\sum_{\alpha=1}^{N} u_\alpha \chi_{(x_{\alpha-1}^\theta, x_\alpha^\theta]}(x). \end{cases}
\end{array}
$$

We define a pseudo–polygonal path

$$
\begin{array}{ccccc}
\gamma(\theta) & : & \mathbb{R} \ni & [a,b] & \longrightarrow & L^1 \\
 & & & \theta & \longrightarrow & u^\theta
\end{array}
$$

as a finite concatenation of elementary paths $\zeta_i(\theta)$, $\theta \in [a_i, b_i]$, $i = 1, \ldots, I$:

$$
\gamma(\theta) = \zeta_i(\theta) \quad \text{if} \quad \theta \in [a_i, b_i],
$$

with $a_1 = 1$, $b_I = b$, $a_{i+1} = b_i$, $\zeta_{i+1}(a_{i+1}) = \zeta_i(b_i)$. Now we compute the length of the the path in a metric equivalent to the $L^1$–distance, defined as

$$
\begin{aligned}
\|\gamma\|_W = \sum_{i=1}^{I} \|\zeta_i\|_W &= \sum_{i=1}^{I} \int_{a_i}^{b_i} d\theta \sum_{\alpha \in A} |u_{\alpha-1} - u_\alpha| \left| \frac{dx_\alpha^\theta}{d\theta} \right| W_\alpha \\
&= \sum_{i} \int_{a_i}^{b_i} d\theta \sum_{\alpha \in A} |\omega_\alpha| |\xi_\alpha| W_\alpha,
\end{aligned} \tag{4.1}
$$

where $1 \le W_\alpha \le C$ is a weight, $C$ is a constant, $A$ is the set of discontinuities and $\omega_\alpha$ is the strength of the shock $\alpha$.

Since the initial data for the approximate algorithm are with compact support, with a finite number of shocks, given two initial data $u_0$ and $v_0$ the pseudo–polygonal path we consider is defined as

$$
\gamma^0(\theta) = u_0 \xi_{(-\infty,\theta]} + v_0 \xi_{(\theta,+\infty)}, \tag{4.3}
$$

where $\theta$ ranges over the compact set $[a,b]$, union of the supports of $u_0$ and $v_0$.

If we recall that the times of interactions and the number of waves for the solutions generated by our algorithm are finite, an easy application of the techniques in [1] and [5] gives the following Theorem:

**Theorem 4.1.** *Consider two different semigroup trajectories, $S_t^\nu u$ and $S_t^\nu v$. If $\gamma_0(\theta)$ is the pseudo–polygonal path (4.3) connecting $u_0 = S_0^\nu u$ and $v_0 = S_0^\nu v$, then the path $\gamma_t(\theta)$ defined as*

$$\begin{array}{rcrcl} \gamma^t(\theta) & : & \mathbb{R} \ni \ [a,b] & \longrightarrow & L^1 \\ & & \theta & \longrightarrow & S_t^\nu \gamma_0(\theta) \end{array}$$

*is still pseudo–polygonal for any time $t \geq 0$.*

## 5. The Interaction Estimate

In this section we estimate the change in $L_1$–length of the path $\gamma_t(\theta) = S_t^\nu \gamma_0(\theta)$, computes as in (4.1) but with $W_\alpha = 1$, when the wave–front configuration changes: this happens when there is an interaction among wave–fronts.

We recall the properties of the semigroup trajectories: $S_t^\nu u_0 \in \mathcal{D}_M^\nu$ is an exact solution of (1.1) with the following properties:

a) the number of interactions is finite, as well as the number of shocks;
b) the Total Variation of each Riemann coordinate is decreasing;
c) $u_\nu$ is Lipschitz continuous w.r.t. time with values in $L^1$;
d) each shock of the $i$–th family satisfies the following property:
either $l_i^\nu(-) < l_i^\nu(+)$ and

$$f_i^\nu(l_i) \geq \frac{\big(l_i^\nu(+) - l_i\big) f_i^\nu\big(l_i^\nu(-)\big) + \big(l_i - l_i^\nu(-)\big) f_i^\nu\big(l_i^\nu(+)\big)}{l_i^\nu(+) - l_i^\nu(-)}$$

for $l_i \in [l_i^\nu(-), l_i^\nu(+)]$, or $l_i^\nu(-) > l_i^\nu(+)$ and

$$f_i^\nu(l_i) \leq \frac{\big(l_i^\nu(+) - l_i\big) f_i^\nu\big(l_i^\nu(-)\big) + \big(l_i - l_i^\nu(-)\big) f_i^\nu\big(l_i^\nu(+)\big)}{l_i^\nu(+) - l_i^\nu(-)}$$

for $l_i \in [l_i^\nu(+), l_i^\nu(-)]$, where $f_i^\nu$ and $l_i^\nu$ are the quantities introduced in (2.3) and (2.4). As usual, from now on we will suppress the index $\nu$. We recall that each shock of the $i$–th family satisfies the reduced Rankine–Hugoniot condition $f_i(l_i^-) - f_i(l_i^+) = \lambda(l_i^- - l_i^+)$, where $\lambda$ is the speed of the shock. We want to study the relation between the shifts of the incoming and the outgoing wave–fronts. Consider a generic point of interaction in which $N_i$ shocks of the $i$–th family interact, with sizes $\sigma_i^j(-)$, $\tau_i^j(-)$ and shifts $\xi^j(-)$, and let $\sigma_i^{j'}(+)$, $\tau_i^{j'}(+)$ and $\xi^{j'}(+)$ be the sizes and shifts of the outgoing fronts of the $i$–th family. The quantities $\sigma_i$ and $\tau_i$ were defined in section 2. We note that the following relation holds:

$$\sum_{j'} \tau_i^{j'}(+) = w_i(+) - w_i(-) = \sum_j \tau_i^j(-)$$

by the assumption of the existence of Riemann coordinates, hypotheses (H1) in section 1.

Let $v = (v_1, v_2)$ the shift of the interaction point; we have obviously

$$\xi_i^{j/j'}(-/+) = v_1 - v_2 \Lambda_i^{j/j'}(-/+),$$

where $\Lambda_i^{j/j'}(-/+)$ is the speed of the incoming/outgoing front. We will denote with a $\tilde{\cdot}$ the shifted quantities: for example $P = (t_1, x_1)$ is the original point of interaction and $\tilde{P} = (t_2, x_2)$ is the shifted one. We consider here 4 scalar functions which measure exactly the shifts and strengths of the incoming and outgoing waves.

Before interaction, the shocks of the $i$–th family lie on the Riemann invariant

$$R_i(-) = \Big\{ u = u(w_1, \dots, w_n) : a_i \leq w_i \leq b_i \\ w_1 = w_1(-), \dots, w_{i-1} = w_{i-1}(-), w_{i+1} = w_{i+1}(+), \dots, w_n = w_n(+) \Big\},$$

and if $l_i^-$ is the signed Euclidean distance from

$$u_i(-) = u(w_1(-), \dots, w_i(-), w_{i+1}(+), \dots, w_n(+))$$

to

$$u_{i+1}(-) = u(w_1(-), \dots, w_{i-1}(-), w_i(+), \dots, w_n(+)),$$

we have for the shock of the $i$–th family

$$u(t,x) = u_i(-) + l_i^-(t,x) r_i(u_i(-)),$$

where $l_i^-(t,x)$ satisfies the scalar conservation law

$$[l_i^-]_t + [f_i^-(l_i^-)]_x = 0.$$

and $f_i^-(l_i^-)$ is the piecewise linear flux function along $R_i(-)$. We define the functions $p_i = l_i^-(t,x)$ and $\tilde{p}_i = \tilde{l}_i^-(t,x)$, respectively corresponding to $u$ and $\tilde{u}$, the shifted one. We assume that $p_i$, $\tilde{p}_i$ are prolonged for $t \in (-\infty, t_{1/2})$ by continuing the shocks and for $t \geq t_1$, $t_2$ by solving the Riemann problem as if the waves of the other families were not present. Precisely, if we define $L_i(-) = \|u_{i+1}(-) - u_i(-)\|$, at point $(t_1, x_1)$ we solve

$$\begin{cases} [p_i]_t + [f_i^-(p_i)]_x &= 0 \\ p(t_1, x) &= L_i(-) \chi_{(x_1, +\infty)}(x) \end{cases}$$

and at point $(t_2, x_2)$ the corresponding equation for $\tilde{p}$

$$\begin{cases} [\tilde{p}_i]_t + [f_i^-(\tilde{p}_i)]_x &= 0 \\ \tilde{p}(t_2, x) &= L_i(-) \chi_{(x_2, +\infty)}(x) \end{cases}$$

where, as usual, $\chi_A(x)$ is the characteristic function of the set $A$. After the interaction in $P$ or $\tilde{P}$, the outgoing shocks of the $i$–th family lie on the Riemann invariant

$$R_i(+) = \Big\{u = u(w_1, \ldots, w_n) : a_i \leq w_i \leq b_i \\ w_1 = w_1(+), \ldots, w_{i-1} = w_{i-1}(+), w_{i+1} = w_{i+1}(-), \ldots, w_n = w_n(-)\Big\},$$

and if $l_i^+$ is the signed distance from $u_i(+) = u(w_1(+), \ldots, w_{i-1}(+), w_i(-), \ldots, w_n(-))$ to $u_{i+1}(-) = u(w_1(+), \ldots, w_i(+), w_{i+1}(-), \ldots, w_n(-))$, we have for the shock of the $i$–th family and for $t \geq t_1$

$$u(t,x) = u_i(+) + l_i^+(t,x) r_i(u_i(+)),$$

where $l_i^+(t,x)$ is the solution to the Riemann problem

$$\begin{cases} [l_i^+]_t + [f_i^+(l_i^+)]_x &= \quad 0 \\ l_i^+(t_1,x) &= \quad L_i(+)\chi_{(x_1,+\infty)} \end{cases}$$

and $L_i(+) = \|u_{i+1}(+) - u_i(+)\|$, whereas $f_i^+(l_i^+)$ is the piecewise linear flux function along $R_i(+)$. We define $s(x,t) = l_i^+(x,t)$ for $t \geq t_1$ and the corresponding $\tilde{s}(x,t) = \tilde{l}_i^+(x,t)$ for $t \geq t_2$ for the shifted solution $\tilde{u}$: in particular, for $t = t_1$ we assume $s(x,t) = L_i(+)\chi_{(x_1,+\infty)}(x)$ and, for $t = t_2$, $\tilde{s}(x,t) = L_i(+)\chi_{(x_2,+\infty)}(x)$. Suppose without any loss of generality that $t_2 \geq t_1$. We note that after the interaction $s$, $\tilde{s}$ are monotone increasing, and then for any $t \geq t_2$ we have

$$\|s - \tilde{s}\|(t) = \sum_{j'} |\sigma_i^{j'}(+)\xi_i^{j'}(+)|.$$

Moreover at time $t = t_2$

$$\tilde{s}(t_2) = \frac{L_i(+)}{L_i(-)}\tilde{p}(t_2),$$

and then we can write

$$\begin{aligned} \sum_{j'} |\sigma_i^{j'}(+)\xi_i^{j'}(+)| =& \|s - \tilde{s}\|(t_2) = \left\| s - \frac{L_i(+)}{L_i(-)}\tilde{p} \right\|(t_2) \\ &\leq \left\| s - \frac{L_i(+)}{L_i(-)}p \right\|(t_2) + \frac{L_i(+)}{L_i(-)}\|p - \tilde{p}\|(t_2). \end{aligned} \tag{5.1}$$

We define $q(t,x) = \frac{L_i(+)}{L_i(-)}p(t,x)$, so that $q$ satisfies the Cauchy problem for the scalar equation

$$\begin{cases} [q]_t + [g(q)]_x &= \quad 0 \\ q(t_1,x) &= \quad L_i(+)\chi_{(x_1,+\infty)}(x) \end{cases}$$

with $g(\,\cdot\,) = \frac{L_i(+)}{L_i(-)} f_i^- \big(\frac{L_i(-)}{L_i(+)}\,\cdot\,\big)$. At this point, we evaluate separately the two terms.

By scalar $L^1$–contraction principle, we have for all $t \leq t_2$

$$\|p - \tilde{p}\|(t_2) \leq \|p - \tilde{p}\|(t) \leq \sum_j |\sigma_i^j(-)\xi_i^j(-)| \tag{5.2}$$

because for $t \to -\infty$ we have $\|p - \tilde{p}\|(t) \to \sum_j |\sigma_i^j(-)\chi_i^j(-)|$, the speeds of the waves being different.

To evaluate $\|s - q\|(t_2)$ we apply the estimate (3.5) and we and get

$$\|s - q\|(t_2) \leq L_i(+)\Gamma(t_2 - t_1) = L_i(+)\Gamma|v_2|. \tag{5.3}$$

We recall that by definition (see (3.4))

$$\Gamma = \sup_{z \in [0, L_i(+)]} \big|(f_i^+)'(z) - g'(z)\big| = \sup_{z \in [0, L_i(+)]} \left|\big(f_i^+\big)'(z) - \big(f_i^-\big)'\Big(\frac{L_i(-)}{L_i(+)} z\Big)\right|.$$

To evaluate $\Gamma$, we write $\forall z \in [0, L_i(+)]$

$$\begin{aligned}\big(f_i^+\big)'(z) - g'(z) =& \big(f_i^+\big)'(z) - \big(f_i^-\big)'\Big(\frac{L_i(-)}{L_i(+)} z\Big) \\ =& \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} ds\ \lambda_i\Big(u_i(+) + sr_i\big(u_i(+)\big)\Big) - \\ & \qquad \frac{1}{b_{i+1} - b_i} \int_{b_i}^{b_{i+1}} ds\ \lambda_i\Big(u_i(-) + sr_i\big(u_i(-)\big)\Big)\end{aligned}$$

where the $a_i$'s are the point in which $f_i^{+,\nu} = f_i^+$ and $z \in [a_i, a_{i+1}]$, whereas for any $b_i$ we have $f_i^{-,\nu} = f_i^-$ and $\frac{L_i(-)}{L_i(+)} z \in [b_i, b_{i+1}]$. At this point we make the linear transformation $s = \frac{a_{i+1} - a_i}{b_{i+1} - b_i}(t - b_i) + a_i$ and we get

$$\begin{aligned}\big(f_i^+\big)'(z) - g'(z) =& \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} ds\ \Big[\lambda_i\Big(u_i(+) + sr_i\big(u_i(+)\big)\Big) \\ & \qquad - \lambda_i\Big(u_i(-) + \Big(\frac{b_{i+1} - b_i}{a_{i+1} - a_i}\big(t - a_i\big) + b_i\Big) r_i\big(u_i(-)\big)\Big)\Big] \\ \leq& C \max_{s \in [0.L_i(+)]} \left\| u_i(+) + sr_i\big(u_i(+)\big) - u_i(-) - \frac{L_i(-)}{L_i(+)} sr_i\big(u_i(-)\big)\right\| \\ \leq& C \sum_{k \neq i} |w_k(+) - w_k(-)|\,,\end{aligned} \tag{5.4}$$

by smoothness of the eigenvalues $\lambda_i$, compactness of $E$ and the fact that the maximum is assumed at the extremals of the segments $L_i(+)$ and $L_i(-)$. Here and in the following, $C$ will denote a suitable constant. We note also that

$$L_i(+) = \left|\sum_j \sigma_i^j(+)\right| = \frac{L_i(+)}{L_i(-)}\left|\sum_j \sigma_i^j(-)\right|, \quad |w_k(+) - w_k(-)| \leq C\left|\sum_l \sigma_k^l(-)\right|. \tag{5.5}$$

By a trivial geometrical argument, if $\Lambda_i^j(\pm)$ is the speed of the $j$–th incoming/outcoming wave of the $i$–th family, we have $v_2 = \frac{\xi_i^j(-)-\xi_k^l(-)}{\Lambda_k^l(-)-\Lambda_i^j(-)}$ for any couple of approaching waves of different families. Thus, using (5.3), (5.4) and (5.5), we have

$$\begin{aligned}
\|s-q\|(t_2) \leq{}& C L_i(+)\left(\sum_{k\neq i}\left|\sum_l \sigma_k^l(-)\right|\right)|v_2| \\
\leq{}& C\frac{L_i(+)}{L_i(-)}\left(\sum_{k\neq i}\left|\sum_l \sigma_k^l(-)\right|\right)|\sum_j \sigma_i^j(-)||v_2| \\
={}& C\frac{L_i(+)}{L_i(-)}\left(\sum_{k\neq i}\left|\sum_l \sigma_k^l(-)\right|\left|\sum_j \sigma_i^j(-)\right|\left|\frac{\xi_i^j(-)-\xi_k^l(-)}{\Lambda_k^l(-)-\Lambda_i^j(-)}\right|\right) \\
\leq{}& C\frac{L_i(+)}{L_i(-)}\sum_{k\neq i}\left|\sum_{l,j}\sigma_k^l(-)\sigma_i^j(-)\big(\xi_i^j(-)-\xi_k^l(-)\big)\right| \\
\leq{}& C\frac{L_i(+)}{L_i(-)}\sum_{k\neq i}\left(\left|\sum_l \sigma_k^l(-)\right|\left|\sum_j \sigma_i^j(-)\xi_i^j(-)\right| + \left|\sum_l \sigma_k^l(-)\xi_k^l(-)\right|\left|\sum_j \sigma_i^j(-)\right|\right).
\end{aligned} \tag{5.6}$$

In the previous formula we used the bound $\left|\frac{1}{\Lambda_k^l(-)-\Lambda_i^j(-)}\right| \leq C$ for some constant $C$, consequence of hypotheses (H1) of section 1. We note at this point that

$$\frac{L_i(+)}{L_i(-)} \leq \left(1 + C\sum_{k\neq i}\left|\sum_j \tau_k^l(-)\right|\right) \leq C \tag{5.7}$$

for some constant, because $E$ is compact and the two systems of coordinates are equivalent.

Using (5.1), (5.2), (5.6) and (5.7), we have finally

$$
\begin{aligned}
\sum_{j'} |\sigma_i^{j'}(+)\xi_i^{j'}(+)| = \|s-\tilde{s}\| &= \left\| s - \frac{L_i(+)}{L_i(-)}\tilde{p} \right\| \leq \left\| s - \frac{L_i(+)}{L_i(-)} p \right\| + \frac{L_i(+)}{L_i(-)} \|p - \tilde{p}\| \\
\leq & C\frac{L_i(+)}{L_i(-)} \sum_{k\neq i} \left( \left| \sum_l \sigma_k^l(-) \right| \left| \sum_j \sigma_i^j(-)\xi_i^j(-) \right| + \left| \sum_l \sigma_k^l(-)\xi_k^l(-) \right| \left| \sum_j \sigma_i^j(-) \right| \right) \\
& \qquad + \frac{L_i(+)}{L_i(-)} \left| \sum_j \sigma_i^j(-)\xi_i^j(-) \right| \\
\leq & \frac{L_i(+)}{L_i(-)} \left( 1 + C \sum_{k\neq i} \left| \sum_l \sigma_k^l(-) \right| \right) \sum_j \left| \sigma_i^j(-)\xi_i^j(-) \right| \\
& \qquad + C\frac{L_i(+)}{L_i(-)} \sum_{k\neq i} \left| \sum_l \sigma_k^l(-)\xi_k^l(-) \right| \left| \sum_j \sigma_i^j(-) \right| \\
\leq & \left( 1 + C \sum_{k\neq i} \left| \sum_l \tau_k^l(-) \right| \right)^2 \sum_j \left| \sigma_i^j(-)\xi_i^j(-) \right| + C \left| \sum_j \tau_i^j(-) \right| \sum_{k\neq i} \sum_l \left| \sigma_k^l(-)\xi_k^l(-) \right|,
\end{aligned}
$$

possibly increasing the values of constants, and using again the equivalence of the two systems of coordinates. We summarize the result in the following Proposition.

**Proposition 5.1.** *Consider a point $P=(t_1,x_1)$ of interaction in which $N_i$–shocks of the $i$–th family collide. Let $u(t,x)$ be the solution defined for $t<t_1$ by the continuation the shocks involved in the interaction at $P$, and for $t \geq t_1$ by solving the Riemann problem with our approximate algorithm. Let $\tilde{u}(t,x)$ be the rigidly shifted solution. Then for some constant $C$ we have*

$$
\begin{aligned}
& \sum_{j'} \left| \sigma_i^{j'}(+)\xi_i^{j'}(+) \right| \leq \\
& \quad \left( 1 + C \sum_{k\neq i} \left| \sum_l \tau_k^l(-) \right| \right)^2 \sum_j \left| \sigma_i^j(-)\xi_i^j(-) \right| + C \left| \sum_j \tau_i^j(-) \right| \sum_{k\neq i} \sum_l \left| \sigma_k^l(-)\xi_k^l(-) \right|,
\end{aligned}
\tag{5.8}
$$

*where, as above, $\sigma_i^j(-)$, $\tau_i^j(-)$, $\xi^j(-)$, and $\sigma_i^{j'}(+)$, $\tau_i^{j'}(+)$ and $\xi^{j'}(+)$ are the sizes, measured in both coordinate systems, and shifts of the incoming/outgoing fronts of the $i$–th family.*

## 6. Weighted path lengths

In this section we give weights to each shock such that the distance between solution, computed as in (4.1), is decreasing in time and we prove that this distance remains uniformly equivalent to the $L^1$–norm. Our technique is the same as in [1].

Since there are at most a finite number of interaction, after a time $T$ the solution does not change the wave–front pattern and we give to each shock weight 1. Next we consider a point $P$ of interaction and we suppose to have assigned weights to the outgoing shocks. We give weights to the incoming shocks as follows:

a) if the waves of the same family disappear, i.e. there are no outgoing waves of that family, we put weight 1 to each of them;

b) if $N_i(-)$ wave–fronts of each family interact, then we set

$$W_i^j(-) = \left(1 + C\sum_{k\neq i}\left|\sum_l \tau_k^l(-)\right|\right)^2 \max_{j'} W_i^{j'}(+) + C\sum_{k\neq i}\left(\left|\sum_l \tau_k^l(-)\right|\right)\max_{l'} W_k^{l'}(+). \tag{6.1}$$

The apostrophe denotes the outgoing quantities.

In this way we assign a weight to each shock, since the number of interactions is finite. With the above choice of weights, recalling (5.8), we have

$$\begin{aligned}
\sum_{i,j'}\left|\sigma_i^{j'}(+)\xi_i^{j'}(+)\right| W_i^{j'}(+) &\leq \sum_{i,j'}\left|\sigma_i^{j'}(+)\xi_i^{j'}(+)\right| \max_{j'} W_i^{j'}(+)\\
&\leq \sum_i\left(1 + C\sum_{k\neq i}\left|\sum_l \tau_k^l(-)\right|\right)^2 \sum_j\left|\sigma_i^j(-)\xi_i^j(-)\right|\max_{j'} W_i^{j'}(+) +\\
&\qquad C\sum_i\left|\sum_j \tau_i^j(-)\right|\max_{j'} W_i^{j'}(+)\sum_{k\neq i}\sum_l\left|\sigma_k^l(-)\xi_k^l(-)\right|\\
&= \sum_{i,j}\left|\sigma_i^j(-)\xi_i^j(-)\right|\left(1 + C\sum_{k\neq i}\left|\sum_l \tau_k^l(-)\right|\right)^2\max_{j'} W_i^{j'}(+) +\\
&\qquad C\sum_{i,j}\left|\sigma_i^j(-)\xi_i^j(-)\right|\sum_{k\neq i}\left|\sum_l \tau_k^l(-)\right|\max_{l'} W_k^{l'}(+) \leq \sum_{i,j}\left|\sigma_i^j(-)\xi_i^j(-)\right| W_i^j(-).
\end{aligned}$$

This shows that with our choice, the length of $\gamma_t(\theta)$, computed as in (4.1), is decreasing. We must verify that this weights do not become infinitely large, as $\nu \to 0$, i.e. when the number of collisions goes to $+\infty$. In the next Lemma we prove that these weights are bounded by a constant which does not depend on $\nu$.

**Lemma 6.1.** *For any semigroup trajectory $S_t^\nu u$, $\nu \in \mathbb{N}$, the weights $W_\alpha$ are bounded by a constant $L_M$, which depends only on the Total Variation of $u_0$, $M$.*

*Proof.* When we say to follow a shock, we will mean that at any future interaction we choose the outgoing shock of the same family with the maximal weight. With this choice, the weight of the shock increases only when it meets shocks of families different from its.

Suppose that at time $\bar{t}$ there is an interaction such that the shocks after interaction have weights less or equal to $e^{5CM}$, where $M$ is the Total Variation of the initial datum and one of the incoming shocks has weight greater that $e^{5CM}$: we indicate the quantities related to this shock as $\bar{\cdot}$. Following the shock, we have by equation (6.1) that

$$\bar{W} \leq e^{2CM}\left(1 + C \sum_{k' \in \bar{J}} \left|\sum_{l'} \tau_{k'}^{l'}\right| \max_{l'} W_{k'}^{l'}(+)\right),$$

where $\bar{J}$ is the set of shocks of different families that after $\bar{t}$ meet the shock. By our hypotheses, we have

$$\sum_{k' \in \bar{J}} |\sum_{l'} \tau_{k'}^{l'}| \max_{l'} W_{k'}^{l'}(+) \geq \frac{e^{3CM} - 1}{C}.$$

We evaluate the weights $W_{k'}^{l'}$ using the same formula and following the shocks back with the maximal weights:

$$\begin{aligned}\sum_{k' \in \bar{J}} |\sum_{l'} \tau_{k'}^{l'}| \max_{l'} W_{k'}^{l'}(+) \leq &\sum_{k' \in \bar{J}} |\sum_{l'} \tau_{k'}^{l'}| e^{2CM}\Big(1 + C \sum_{k'' \in \bar{J}_{k'}} |\sum_{l''} \tau_{k''}^{l''}| \max_{l''} W_{k''}^{l''}(+)\Big) \\ \leq &Me^{2CM} + Ce^{7CM} \sum_{k' \in \bar{J}} |\sum_{l'} \tau_{k'}^{l'}| \sum_{k'' \in J_{k'}} |\sum_{l''} \tau_{k''}^{l''}|.\end{aligned}$$

We note that $|\sum_{l'} \tau_{k'}^{l'}| \sum_{k'' \in J_{k'}} |\sum_{l''} \tau_{k''}^{l''}|$ is the product of the strength in Riemann coordinates of the outcoming shocks of the $k'$–th family with the strength of the shocks of the $k''$–th family that meet one of the previous shocks at a time greater than $\bar{t}$. Obviously we have that the sum of these quantities over all the set $\bar{J}$ is smaller than the Interaction Potential immediately after $\bar{t}$:

$$\sum_{k' \in \bar{J}} |\sum_{l'} \tau_{k'}^{l'}| \sum_{k'' \in \bar{J}_k} |\sum_{l''} \tau_{k''}^{l''}| \leq Q(\bar{t}),$$

where $Q(\bar{t}) - Q(T) = Q(\bar{t})$ is the Interaction Potential of the solution an time $\bar{t}$. Finally

$$\begin{aligned}\sum_{k' \in \bar{J}} |\sum_{l'} \tau_{k'}^{l'}| \max_{l'} W_{k'}^{l'}(+) \leq &Me^{2CM} + Ce^{7CM}(Q(\bar{t}) - Q(T)) \\ = &Me^{2CM} + Ce^{7CM}(Q(\bar{t})).\end{aligned}$$

The above formulas imply

$$Q(\bar{t}) \geq K_M = \frac{e^{3CM} - CMe^{2CM} - 1}{C^2 e^{7CM}} > 0.$$

Suppose now that at time $t_k$ all weights are $\leq e^{5kCM}$ and there exists a wave with weight $> e^{5(k+1)CM}$ at some time $t_{k+1} < t_k$. Then we can repeat the previous computation to prove that $Q$ must decrease at least of $K_M$ over $[t_{k+1}, t_k]$. At this point we divide the interval $[0, T]$, where $T$ is the time of decoupling (see [10]), into a finite number of subintervals $I_k = [t_{k+1}, t_k]$ such that

– either $I_k$ contains a single interaction and the decrease of $Q(t)$ is bigger than $K_M$;

– or $Q(t_{k+1}) - Q(t_k) < K_M$ and $I_k$ is maximal w.r.t. this property.

Since with this choice $Q(t_{k+1}) - Q(t_{k-1}) \geq K_M$, there are at most $p \leq 2M^2/K_M$ intervals, and in the two cases we have

– if $Q(t_{k+1}) - Q(t_k) < K_M$, then $W(t_{k+1}) \leq W(t_k) e^{5CM}$;

– if $Q(t_{k+1}) - Q(t_k) \geq K_M$, then directly from formula (6.1) we deduce

$$W(t_{k+1}) \leq 2W(t_k) e^{2CM} \leq W(t_k) e^{5CM}.$$

The two cases imply easily:

$$1 \leq W \leq \left[e^{5CM}\right]^{2M^2/K_M} = L_M, \tag{6.2}$$

and this is the desired bound. □

At this point we can prove that the semigroups $S_t^\nu \cdot$ , defined in (2.5), are uniformly Lipschitz, independently of $\nu$. In fact, consider $u_0, v_0 \in \mathcal{D}_M^\nu$ and the pseudo–polygonal path $\gamma_0(\theta)$ defined in (4.3). Since $\gamma_0 \in \mathcal{D}_{2M}^\nu$, we conclude by (4.1), (4.3), (6.2) and Lemma 6.1 that

$$\|S_t^\nu u_0 - S_t^\nu v_0\|_L^1 \leq \|\gamma^t\|_W \leq \|\gamma_0\|_W \leq L_{2M} \|u_0 - v_0\|.$$

## 7. The semigroup over $E$

Now we prove that there exists a Lipschitz continuous semigroup $S_t \cdot$ , defined as

$$\begin{array}{rcccc} S_t \cdot & : & [0, +\infty) \otimes \mathcal{D}_M & \longrightarrow & \mathcal{D}_M \\ & & (t, u_0) & \longrightarrow & S_t u_0 = u(t), \end{array}$$

where $\mathcal{D}_M \subseteq L^1(\mathbb{R}; \mathbb{R}^n)$ is the set

$$\mathcal{D}_M = \left\{ u : \mathbb{R} \to E : \sum_i \text{Tot.Var.}(w_i(u)) \leq M \right\},$$

such that $S_t u_0$ is an entropic solution of the Cauchy problem (2.1). The semigroup is Lipschitz continuous in $L_1$, i.e. there exists a constant $L_{2M}$ such that

$$\|S_t u - S_t v\|_{L^1} \leq L_{2M} \|u - v\|_{L^1}.$$

Since the domain $\mathcal{D}_M^\nu$, defined in (2.2), contains $\mathcal{D}_M^{\nu-1}$ and their union is dense in $\mathcal{D}_M$, we define the semigroup $S_t\,\cdot$ as

$$\begin{aligned} S_t u &= L_1 - \lim_{\nu\to\infty} S_t^\nu u^\nu, \\ \mathcal{D}_M^\nu &\ni u^\nu \xrightarrow[L^1]{} u \in \mathcal{D}_M. \end{aligned} \tag{7.1}$$

Now we prove that this is a Cauchy sequence, so that the limit is well defined. We recall that for the Lipschitz semigroups $S_t^\nu\,\cdot$ , $S_t^\mu\,\cdot$ , with $\mu > \nu$, the following estimate holds (formula (5.3) of [1]):

$$\begin{aligned} \|S_t^\nu u^\nu - S_t^\mu u^\mu\|_{L_1} &\leq \|S_t^\mu u^\nu - S_t^\mu u^\mu\|_{L_1} + \|S_t^\mu u^\nu - S_t^\nu u^\nu\|_{L_1} \\ &\leq L_{2M} \|u^\nu - u^\mu\|_{L_1} + L_{2M} \int_0^t \liminf_{h\to 0^+} \frac{1}{h} \|S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu\|_{L_1}\, d\tau. \end{aligned} \tag{7.2}$$

Since the number of times of interactions between shocks of $S_t^\nu u^\nu$ is finite and the semigroups are continuous in $L^1$, we have just to evaluate $\|S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu\|_{L_1}$ when no interactions occur:

$$\|S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu\|_{L_1} = \sum_{\alpha\in A} \int_{x_\alpha-\rho}^{x_\alpha+\rho} |S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu|,$$

where $A$ is the set of discontinuities, $x_\alpha$ is the position of the shock $\alpha$ and $h$, $\rho$ are small enough. We note that the quantity $|S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu|$ can be different from zero only when the function $S_\tau^\nu u^\nu$ has a non–entropic shock of size $2^{-\nu}$: in fact in this case can happen that $S_h^\mu u^\mu$ solves the Riemann problem splitting that wave–front into smallest ones. The difference in speed of this shock and the new ones is, by the smoothness of the flux function $f$, of the order $2^{-\nu}$. Then, denoting with $\mathcal{R}$ the set of non–entropic shocks of $S_\tau^\nu u^\nu$ and using (3.5), we can conclude that

$$\begin{aligned} \|S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu\|_{L_1} &= \sum_{\alpha\in\mathcal{R}} \int_{x_\alpha-\rho}^{x_\alpha+\rho} |S_h^\mu S_\tau^\nu u^\nu - S_{\tau+h}^\nu u^\nu| \\ &\leq \sum_{\alpha\in\mathcal{R}} hC(2^{-\nu})^2 \leq hC2^{-\nu}M, \end{aligned}$$

for some constant $C$ depending only on $f$ and $E$. At this point (7.2) becomes

$$\|S_t^\nu u^\nu - S_t^\mu u^\mu\|_{L_1} \leq L_{2M}\|u^\nu - u^\mu\|_{L_1} + L_{2M} MC2^{-\nu} t, \tag{7.3}$$

showing that $S_t^\nu u^\nu$ is a Cauchy sequence in $L_1$. We can then state the following Theorem:

**Theorem 7.1.** *For each $M > 0$ there exists a constant $C_M$ and a semigroup $S : [0, +\infty) \otimes \mathcal{D}_M \to \mathcal{D}_M$ such that:*

*a)* $\|S_t u - S_s v\|_{L_1} \leq C_M(|t - s| + \|u - v\|_{L_1})$*;*

*b) each trajectory $S_t u_0$ is a weak entropic solution of the Cauchy problem (2.1) with initial datum $u_0$;*

*c) if $u_0$ is a piecewise constant function, then, for small $t$, $S_t u_0$ coincides with the function obtained piecing together the solutions of the corresponding Riemann problems.*

*Proof.* Since the functions $S^\nu_t u^\nu$ converges in $L_1$, the limit is obviously an entropic weak solution, and the semigroup properties are easily verified. The Lipschitz continuity follows from (7.3) and (2.6), whereas the last property is a consequence of our construction. □

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