SBV REGULARITY FOR HAMILTON-JACOBI EQUATIONS IN $\mathbb{R}^n$.

STEFANO BIANCHINI, CAMILLO DE LELLIS, AND ROGER ROBYR

ABSTRACT. In this paper we study the regularity of the solutions of viscosity solutions of the following Hamilton-Jacobi equations
$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset \mathbb{R} \times \mathbb{R}^n \,.$$
In particular, under the assumption that the Hamiltonian $H \in C^2(\mathbb{R}^n)$ is uniformly convex, we prove that the gradient $D_x u$ belongs to the class $SBV_{loc}(\Omega)$.

1. INTRODUCTION

Theorem 1.1. *Let $H \in C^2(\mathbb{R}^n)$ with*
$$c_H^{-1} Id_n \leq D^2 H \leq c_H Id_n. \tag{1}$$
Let u be a viscosity solution of the following Hamilton-Jacobi equation
$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n, \tag{2}$$
where $D_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ denotes the spatial gradient of u. Then, the set of times
$$S := \{t : D_x u(t, .) \notin SBV_{loc}(\Omega_t)\} \tag{3}$$
is at most countable, where $\Omega_t := \{x \in \mathbb{R}^n : (t, x) \in \Omega\}$. In particular $D_x u, \partial_t u \in SBV_{loc}(\Omega)$.

Corollary 1.1. *Under the same convexity assumptions on the Hamiltonian H as in Theorem 1.1, the gradient of any viscosity solution u of*
$$H(D_x u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{4}$$
belongs to $SBV_{loc}(\Omega)$.

2. PRELIMINARIES: THE THEORY OF MONOTONE FUNCTIONS

Definition 2.1. *Let $A \subset \mathbb{R}^n$ be an open set. We say that a continuous function $u : A \to \mathbb{R}$ is* semiconcave *if, for any convex $K \subset\subset A$, there exists $C_K > 0$ such that*
$$u(x + h) + u(x - h) - 2u(x) \leq C_K |h|^2, \tag{5}$$
for all $x, h \in \mathbb{R}^n$ with $x, x - h, x + h \in K$. The smallest nonnegative costant C_K such that (5) holds on K will be called semiconcavity constant *of u on K.*

Next, we introduce the concept of superdifferential.

Definition 2.2. *Let $u : A \to \mathbb{R}$ be a measurable function. The set $\partial u(x)$, called the* superdifferential *of u at point $x \in A$, is defined as*
$$\partial u(x) := \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}. \tag{6}$$

Using the above definition we can describe some properties of the semiconcave functions (see Proposition 1.1.3 of [5]):

Proposition 2.1. *Let $A \subset \mathbb{R}^n$ be an open and convex set. Let $u : A \to \mathbb{R}$ be a semiconcave function in A with semiconcavity constant $C \geq 0$. Then, the function*

$$\tilde{u} : x \mapsto u(x) - \frac{C}{2}|x|^2 \qquad \text{is concave in } A. \tag{7}$$

In particular, for any given $x, y \in A$, $p \in \partial \tilde{u}(x)$ and $q \in \partial \tilde{u}(y)$ we have that

$$\langle q - p, y - x \rangle \leq 0. \tag{8}$$

From now on, when u is a semi–concave function, we will denote the set-valued map $x \to \partial \tilde{u}(x) + Cx$ as ∂u. An important observation is that, being $\tilde{u}$ concave, the map $x \to \partial \tilde{u}(x)$ is a maximal monotone function.

2.1. **Monotone functions in $\mathbb{R}^n$.** Following the work of Alberti and Ambrosio [1] we introduce here some results about the theory of the monotone functions in $\mathbb{R}^n$. Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued map (or multifunction), i.e. a map which maps every point $x \in \mathbb{R}^n$ into some set $B(x) \subset \mathbb{R}^n$. For all $x \in \mathbb{R}^n$ we define:

- the *domain* of B, $Dm(B) := \{x : B(x) \neq \emptyset\}$,
- the *image* of B, $Im(B) := \{y : \exists x, y \in B(x)\}$,
- the *graph* of B, $\Gamma B := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in B(x)\}$,
- then *inverse* of B, $[B^{-1}](x) := \{y : x \in B(y)\}$.

Definition 2.3. *Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a multifunction, then*

(1) *B is a* monotone *function if*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0 \qquad \forall x_i \in \mathbb{R}^n, y_i \in B(x_i), i = 1, 2. \tag{9}$$

(2) *A monotone function B is called* maximal *when it is maximal with respect to the inclusion in the class of monotone functions, i.e. if the following implication holds:*

$$A(x) \supset B(x) \text{ for all } x, A \text{ monotone } \Rightarrow A = B. \tag{10}$$

Observe that in this work we assume $\leq$ in (9) instead of the most common $\geq$. However, one can pass from one convention to the other by simply considering $-B$ instead of B. The observation of the previous subsection is then summarized in the following Theorem.

Theorem 2.1. *The supergradient ∂u of a concave function is a maximal monotone function.*

An important tool of the theory of maximal monotone functions, which will play a key role in this paper, is the Hillle-Yosida approximation. (see Chapters 6 and 7 of [1]):

Definition 2.4. *For every $\varepsilon > 0$ we set $\Psi_\varepsilon(x, y) := (x - \varepsilon y, y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and for every maximal monotone function B we define B_ε as the multifunction whose graph is $\Psi_\varepsilon(\Gamma B)$, that is, $\Gamma B_\varepsilon = \{(x - \varepsilon y, y) : (x, y) \in \Gamma B\}$. Hence*

$$B_\varepsilon := (\varepsilon Id - B^{-1})^{-1}. \tag{11}$$

In the next Theorems we collect some properties of maximal monotone functions B and their approximations B_ε defined above.

Theorem 2.2. *For a.e. $x \in Dm(B)$, the set $B(x)$ is single valued. Therefore, there exists a classical measurable function $\tilde{B} : Dm(B) \to \mathbb{R}^n$ which coincides with B a.e.. If $Dm(B)$ is open, then DB is a measure, i.e. B is a function of locally bounded variation.*

Maximal monotone functions B enjoy the following continuity property. If K_n is a sequence of compact sets contained in the interior of $Dm(B)$ and such that $K_n \downarrow K$, then $B(K_n) \to B(K)$ in the Hausdorff sense.

Finally, if $Dm(B)$ is open and $B = \partial u$ for some concave function $u : Dm(B) \to \mathbb{R}$, then $\tilde{B}(x) = Du(x)$ for a.e. x (recall that u is locally Lipschitz, and hence the distributional derivative of u coincides a.e. with the classical differential).

In this paper, since we will always consider monotone functions that are the supergradients of some concave functions, we will use ∂u for the supergradient and Du for the distributional gradient. A corollary of Theorem 2.2 is that

Corollary 2.1. *If $u : \Omega \to \mathbb{R}$ is semiconcave, then $\partial u(x) = \{Du(x)\}$ for a.e. x, and at any point where ∂u is single–valued, Du is continuous. Moreover $D^2 u$ is a symmetric matrix of Radon measures.*

Next we state the following important convergence theorem. For the notion of current and the corresponding convergence properties we refer to the work of Alberti and Ambrosio. However, we remark that very little of the theory of currents is needed in this paper: what we actually need is a simple corollary of the convergence in (ii), which is stated and proved in Subsection 5.2. In (iii) we follow the usual convention of denoting by $|\mu|$ the total variation of a (real-, resp. matrix-, vector- valued) measure μ.

Theorem 2.3. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and let B be a monotone function such that $\Omega \subset Dm(B)$. Let B_ε be the approximations given in Definition 2.4. Then, the following properties hold.*

(i) *B_ε is a $1/\varepsilon$-Lipschitz maximal monotone function on $\mathbb{R}^n$ for every $\varepsilon > 0$. Moreover, if $B = Du$, then $B_\varepsilon = Du_\varepsilon$ for some concave function u_ε.*

(ii) *ΓB and ΓB_ε have a natural structure as integer rectifiable currents, and $\Gamma B_\varepsilon \llcorner \Omega \times \mathbb{R}^n$ converges to $\Gamma B \llcorner \Omega \times \mathbb{R}^n$ in the sense of currents as $\varepsilon \downarrow 0$.*

(iii) *$DB_\varepsilon \rightharpoonup^* D\tilde{B}$ and $|DB_\varepsilon| \rightharpoonup^* |D\tilde{B}|$ in the sense of measures on Ω.*

2.2. **BV and SBV functions.** We conclude the section by introducing the basic notations related to the space SBV (for a complete survey on this topic we address the reader to [2]). If $B \in BV(A, R^k)$, then it is possible to split the measure DB into three mutually singular parts:

$$DB = D_a B + D_j B + D_c B.$$

$D_a B$ denotes the absolutely continuous part (with respect to the Lebesgue measure). $D_j B$ denotes the jump part of DB. When A is a 1-dimensional domain, $D_j B$ consists of a countable sum of weighted Dirac masses, and hence it is also called the atomic part of DB. In higher dimensional domains, $D_j B$ is concentrated on a rectifiable set of codimension 1, which corresponds to the measure-theoretic jump set JB of B. $D_c B$ is called the Cantor part of the gradient and it is the "diffused part" of the singular measure $D_s B := D_j B + D_c B$. Indeed $D_c B(E) = 0$ for any rectifiable set E of codimension 1.

Definition 2.5. *Let $B \in BV(A)$, then B is a special function of bounded variation, and we write $B \in SBV(B)$, if $D_c B = 0$, i.e. if the measure DB has no Cantor part. The more general space $SBV_{loc}(\Omega)$ is defined in the obvious way.*

In what follows, when u is a (semi)-concave function, we will denote by $D^2 u$ the distributional hessian of u. Since Du is, in this case, a BV map, the discussion above applies. In this case we will use the notation $D_a^2 u$, $D_j^2 u$ and $D_c^2 u$. An important property of $D_c^2 u$ is the following regularity property.

Proposition 2.2. *Let u be a (semi)-concave function. If D denotes the set of points where ∂u is not single–valued, then $|D_c^2 u|(D) = 0$.*

3. HAMILTON-JACOBI EQUATIONS

In this section we collect some definitions and well-known results about Hamilton-Jacobi equations. For a complete survey on this topic we redirect the reader to the vast literature. For an introduction to the topic we suggest the following sources [4],[5],[7]. In this paper we will consider the following Hamilton-Jacobi equations

$$\partial_t u + H(D_x u) = 0, \qquad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n , \tag{12}$$

$$H(D_x u) = 0, \qquad \text{in } \Omega \subset \mathbb{R}^n , \tag{13}$$

under the assumption that

A1: The Hamiltonian $H \in C^2(\mathbb{R}^n)$ satisfies:

$$p \mapsto H(p) \text{ is convex and } \lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty.$$

We will often consider $\Omega = [0, T] \times \mathbb{R}^n$ in (12) and couple it with the initial condition

$$u(0, x) = u_0(x) \tag{14}$$

under the assumption that

A2: The initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous and bounded.

Definition 3.1 (Viscosity solution). *A bounded, uniformly continuous function u is called a* viscosity solution *of (12) (resp. (13)) provided that*

(1) *u is a* viscosity subsolution *of (12) (resp. (13)): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a maximum at (t_0, x_0) (resp. x_0),*

$$v_t(t_0, x_0) + H(D_x v(t_0, x_0)) \leq 0 \qquad (\text{resp. } H(Dv(x_0)) \leq 0); \tag{15}$$

(2) *u is a* viscosity supersolution *of (12) (resp. (13)): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a minimum at (t_0, x_0) (resp. x_0),*

$$v_t(t_0, x_0) + H(D_x v(t_0, x_0)) \geq 0 \qquad (\text{resp. } H(Dv(x_0)) \geq 0). \tag{16}$$

In addition, we say that u solves the Cauchy problem (12)-(14) on $\Omega = [0, T] \times \mathbb{R}^n$ if (14) holds in the classical sense.

Theorem 3.1 (The Hopf-Lax formula as viscosity solution)**.** *The unique viscosity solution of the initial-value problem (12)-(14) is given by the Hopf-Lax formula*

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left\{ u_0(y) + tL\left(\frac{x-y}{t}\right) \right\} \qquad (t > 0, x \in \mathbb{R}^n), \qquad (17)$$

where L is the Legendre transform of H:

$$L(q) := \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(p)\} \qquad (q \in R^n). \qquad (18)$$

In the next Proposition we collect some properties of the viscosity solution defined by the Hopf-Lax formula:

Proposition 3.1. *Let $u(t,x)$ be the viscosity solution of (12)-(14) and defined by (17), then*

(i) ***A functional identity:*** *For each $x \in \mathbb{R}^n$ and $0 \le s < t \le T$, we have*

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left\{ u(s,y) + (t-s)L\left(\frac{x-y}{t-s}\right) \right\}. \qquad (19)$$

(ii) ***Semiconcavity of the solution:*** *For any fixed $\tau > 0$ there exists a constant $C(tau)$ such that the function defined by*

$$u_t(.) : \mathbb{R}^n \to \mathbb{R}^n \ \text{ with } u_t(x) := u(t,x), \qquad (20)$$

is semiconcave with constant less than C for any $t \ge \tau$.

(iii) ***Characteristics:*** *The minimum point y in (17) is unique if and only if $\partial u_t(x)$ is single valued. Moreover, in this case we have $y = x - tDH(Du(t,x))$.*

(iv) ***The linear programming principle:*** *Let $t > s > 0$, $x \in \mathbb{R}^n$ and assume that y is a minimum for (17). Let $z = \frac{s}{t}x + (1 - \frac{s}{t})y$. Then y is the* unique *minimum for $u_0(w) + sL((z-w)/s)$.*

Remark 3.1. *For a detailed proof of Theorem 3.1 and Proposition 3.1 we address the reader to Chapter 6 of [5] and Chapters 3, 10 of [7] .*

Next, we state a useful locality property of the solutions of (12).

Proposition 3.2. *Let u be a viscosity solution of (12) in Ω. Then u is locally Lipschitz. Moreover, for any $(t_0, x_0) \in \Omega$, there exists a neighborhood U of (t_0, x_0), a positive number ε and a Lipschitz function v_0 on $\mathbb{R}^n$ such that*

(Loc) *u coincides on U with the viscosity solution of*

$$\begin{cases} \partial_t v + H(D_x v) = 0 & in \ [t_0 - \varepsilon, t_0 + \varepsilon] \times \mathbb{R}^n \\ v(t_0 - \varepsilon, x) = v_0(x). \end{cases}$$

4. Proof of the main Theorem

4.1. **Preliminary remarks.** Let u be a viscosity solution of (12). By Proposition 3.2 and the time invariance of the equation, we can, without loss of generality, assume that u is a solution on $[0, T] \times \mathbb{R}^n$ of the Cauchy-Problem (12)-(14) under the assumptions A1, A2. Clearly, it suffices to show that, for every $j > 0$, the set of times $S \cap]1/j, +\infty[$ is countable.

Therefore, by Proposition 3.1 and the time–invariance of the Hamilton–Jacobi equations, we can restrict ourselves to the following case:

$$\exists C \text{ such that } u_\tau \text{ is semiconcave with constant less than } C \text{ for any } \tau \in [0, T]. \tag{21}$$

Arguing in the same way, we can further assume that

$$T \text{ is smaller than some constant } \varepsilon(C) > 0, \tag{22}$$

where the choice of the constant $\varepsilon(C)$ will be specified later.

Next we consider a ball $B_R(0) \subset \mathbb{R}^n$ and a bounded convex set $\Omega \subset [0, T] \times \mathbb{R}^n$ with the properties that:

- $B_R(0) \times \{s\} \subset \Omega$ for every $s \in [0, T]$;
- For any $(t, x) \in \Omega$ and for any y reaching the minimum in the formulation (17), $(0, y) \in \Omega$ (and therefore the entire segment joining (t, x) to $(0, y)$ is contained in Ω).

This set exists because Du is bounded. Our goal is now to show the countability of the set S in (3).

4.2. **A function depending on time.** For any $s < t \in [0, T]$, we define the set–valued map

$$X_{t,s}(x) \;:=\; x - (t - s)DH(\partial u_t(x)). \tag{23}$$

Moreover, we will denote by $\chi_{t,s}$ the restriction of $X_{t,s}$ to the points where $X_{t,s}$ is single-valued. According to Theorem 2.2 and Proposition 3.1(iii), the domain of $\chi_{t,s}$ consists of those points where $Du_t(\cdot)$ is continuous, which are those where the minimum point y in (19) is unique. Moreover, in this case we have $\chi_{t,s}(x) = \{y\}$.

Clearly, $\chi_{t,s}$ is defined a.e. on Ω_t. With a slight abuse of notation we set

$$F(t) \;:=\; |\chi_{t,0}(\Omega_t)|, \tag{24}$$

meaning that, if we denote by U_t the set of points $x \in \Omega_t$ such that (17) has a unique minimum point, we have $F(t) = |X_{t,0}(U_t)|$.

The proof is then split in the following three lemmas:

Lemma 4.1. *The functional F is nonincreasing,*

$$F(\sigma) \geq F(\tau) \qquad \text{for any } \sigma, \tau \in [0, T] \text{ with } \sigma < \tau. \tag{25}$$

Lemma 4.2. *If ε in (22) is small enough, then the following holds. For any $t \in]0, T[$ and $\delta \in]0, T - t]$ there exists a Borel set $E \subset \Omega_t$ such that*

(i) *$|E| = 0$, and $|D_c^2 u_t|(\Omega_t \setminus E) = 0$;*
(ii) *$X_{t,0}$ is single valued on E (i.e. $X_{t,0}(x) = \{\chi_{t,0}(x)\}$ for every $x \in E$);*
(iii) *and*

$$\chi_{t,0}(E) \cap \chi_{t+\delta,0}(\Omega_{t+\delta}) = \emptyset. \tag{26}$$

Lemma 4.3. *If ε in (22) is small enough, then the following holds. For any $t \in]0, \varepsilon]$ and any Borel set $E \subset \Omega_t$, we have*

$$|X_{t,0}(E)| \geq c_0|E| - c_1 t \int_E d(\Delta u_t), \tag{27}$$

where c_0 and c_1 are positive constants and Δu_t is the Laplacian of u_t.

4.3. **Proof of Theorem 1.1.** The three key lemmas stated above will be proved in the next two sections. We now show how to complete the proof of the Theorem. First of all, note that F is a bounded function. Since F is, by Lemma 4.1, a monotone function, its points of discontinuity are, at most, countable. We claim that, if $t \in]0, T[$ is such that $u_t \notin SBV_{loc}(\Omega_t)$, then F has a discontinuity at t.

Indeed, in this case we have

$$|D_c^2 u_t|(\Omega_t) > 0. \qquad (28)$$

Consider any $\delta > 0$ and let B be the set of Lemma 4.2. Clearly, by Lemma 4.2(i) and (ii), (26) and (27),

$$F(t + \delta) \ \leq \ F(t) + c_1 t \int_B \Delta_s u_t \ \leq \ F(t) + c_1 t \int_{\Omega_t} \Delta_c u_t, \qquad (29)$$

where the last inequality follows from $\Delta_s u_t = \Delta_c u_t + \Delta_j u_t$ and $\Delta_j u_t \leq 0$ (because of the semiconcavity of u).

Next, consider the Radon–Nykodim decomposition $D_c^2 u_t = M|D_c^2 u_t|$, where M is a matrix–valued Borel function with $|M| = 1$. Since we are dealing with second derivatives, M is symmetric, and since u_t is semiconcave, $M \leq 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $-M$. Then $1 = |M|^2 = \lambda_1^2 + \ldots + \lambda_n^2$ and $-TrM = \lambda_1 + \ldots + \lambda_n$. Since $\lambda_i \geq 0$, we easily get $-TrM \geq 1$. Therefore,

$$-\Delta_c u_t(.) \ = \ -TrM|D_c^2 u_t| \ \geq \ |D_c^2 u_t|. \qquad (30)$$

Hence

$$F(t + \delta) \ \overset{(29)+(30)}{\leq} \ F(t) - c_1 t |D_c^2 u_t|(\Omega_t).$$

Letting $\delta \downarrow 0$ we conclude

$$\limsup_{\delta \downarrow 0} F(t + \delta) \ < \ F(t).$$

Therefore t is a point of discontinuity of F, which is the desired claim.

4.4. **Easy corollaries.** The conclusion that $D_x u \in SBV$ follows from the slicing theory of BV functions, whereas to prove the same property for $\partial_t u$ we apply the Volpert chain rule to $\partial_t u = -H(D_x u)$. We refer in both cases to the book [2].

As for Corollary 1.1, let u be a viscosity solution of (13) and set $\widetilde{u}(t, x) := u(x)$. Then $\tilde{u}$ is a viscosity solution of

$$\partial_t \widetilde{u} + H(D_x \widetilde{u}) = 0$$

in $\mathbb{R} \times \Omega$. By our main Theorem 1.1 the set of times for which $D_x \widetilde{u}(t, .) \notin SBV_{loc}(\Omega)$ is at most countable. Since $D_x \widetilde{u}(t, \cdot) = Du$, for every t, we conclude that $Du \in SBV_{loc}(\Omega)$.

Remark 4.1. *The special case of this Corollary for $\Omega \subset \mathbb{R}^2$ was already proved in [3] (see Corollary 1.4 therein). We note that the proof proposed in [3] was more complicated than the one above. This is due to the power of Theorem 1.1. In [3] the authors proved the 1–dimensional case of Theorem 1.1. The proof above reduces the 2–dimensional case of Corollary 1.1 to the $2 + 1$ case of Theorem 1.1. In [3] the 2-dimensional case of Corollary 1.1 was reduced to the $1 + 1$ case of Theorem 1.1, which requires a subtler argument.*

5. ESTIMATES

In this section we prove two important estimates. The first is the one in Lemma 4.3. The second is an estimate which will be useful in proving Lemma 4.2 and will be stated here.

Lemma 5.1. *If $\varepsilon(C)$ in (22) is sufficiently small, then the following holds. For any $t \in]0, T]$, any $\delta \in [0, t]$ and any Borel set $E \subset \Omega_t$ we have*

$$\left| X_{t,\delta}(E) \right| \geq \frac{(t - \delta)^n}{t^n} \left| X_{t,0}(E) \right|. \tag{31}$$

5.1. Injectivity.

In the proof of both lemmas, the following remark plays a fundamental role.

Proposition 5.1. *For any $C > 0$ there exists $\varepsilon(C) > 0$ with the following property. If v is a semiconcave function with constant less than C, then the map $x \mapsto x - tDH(\partial v)$ is injective for every $t \in [0, \varepsilon(C)]$.*

Here the injectivity of a set–valued map B is understood in the following natural way

$$x \neq y \qquad \Longrightarrow \qquad B(x) \cap B(y) = \emptyset \,.$$

Proof. We assume by contradiction that there exist $x_1, x_2 \in \Omega_t$ with $x_1 \neq x_2$ and such that:

$$[x_1 - tDH(\partial v(x_1))] \cap [x_2 - tDH(\partial v(x_2))] \neq \emptyset.$$

This means that there is a point y such that

$$\begin{cases} \frac{x_1-y}{t} \in DH(\partial v(x_1)), \\ \frac{x_2-y}{t} \in DH(\partial v(x_2)); \end{cases} \Rightarrow \begin{cases} DH^{-1}(\frac{x_1-y}{t}) \in \partial v(x_1), \\ DH^{-1}(\frac{x_2-y}{t}) \in \partial v(x_2). \end{cases} \tag{32}$$

By the semiconcavity of v we get:

$$M(x_1, x_2) := \left\langle DH^{-1}\left(\frac{x_1 - y}{t}\right) - DH^{-1}\left(\frac{x_2 - y}{t}\right), x_1 - x_2 \right\rangle \leq C|x_1 - x_2|^2. \tag{33}$$

On the other hand, $D(DH^{-1})(x) = (D^2H)^{-1}(DH(x))$. Therefore $D(DH^{-1})(x)$ is a symmetric matrix, with $D(DH^{-1})(x) \geq c_H^{-1} Id_n$. It follows that

$$M(x_1, x_2) = t\left\langle DH^{-1}\left(\frac{x_1 - y}{t}\right) - DH^{-1}\left(\frac{x_2 - y}{t}\right), \frac{x_1 - y}{t} - \frac{x_2 - y}{t} \right\rangle \geq$$

$$\geq \frac{t}{2c_H}\left|\frac{x_1 - y}{t} - \frac{x_2 - y}{t}\right|^2 \geq \frac{1}{2tc_H}|x_1 - x_2|^2 \geq \frac{1}{2\varepsilon c_H}|x_1 - x_2|^2. \tag{34}$$

But if $\varepsilon > 0$ is small enough, or more precisely if it is chosen to satisfy $2\varepsilon c_H < \frac{1}{C}$ the two inequalities (33) and (34) are in contradiction. □

5.2. Approximation.

We next consider u as in the formulations of the two lemmas, and $t \in [0, T]$. Then the function $\tilde{v}(x) := u(x) - C|x|^2/2$ is concave. Consider the approximations B_η (with $\eta > 0$) of $\partial\tilde{v}$ given in Definition 2.4. By Theorem 2.3(i), $B_\eta = D\tilde{v}_\eta$ for some concave function $\tilde{v}_\eta$ with Lipschitz gradient. Consider therefore the function $v_\eta(x) = \tilde{v}_\eta(x) + C|x|^2/2$. The semiconcavity constant of v_η is not larger than C.

Therefore we can apply Proposition 5.1 and choose $\varepsilon(C)$ sufficiently small in such a way that the maps

$$x \mapsto A(x) = x - tDH(\partial u_t) \qquad x \mapsto A_\eta(x) = x - tDH(Dv_\eta) \tag{35}$$

are both injective. Consider next the following measures:

$$\mu_\eta(E) \ := \ |(Id - tDH(Dv_\eta))(E)| \qquad \mu(E) \ := \ |(Id - tDH(\partial u_t))(E)| \, . \qquad (36)$$

These measures are well-defined because of the injectivity property proved in Proposition 5.1.

Now, according to Theorem 2.3, the graphs ΓDv_η and $\Gamma\partial u_t$ are both rectifiable currents and the first are converging, as $\eta \downarrow 0$, to the latter. We denote them, respectively, by T_η and T. Similarly, we can associate the rectifiable currents S and S_η to the graphs ΓA and ΓA_η of the maps in (35). Note that these graphs can be obtained by composing $\Gamma\partial u_t$ and ΓDv_η with the following global diffeomorphism of $\mathbb{R}^n$:

$$(x, y) \ \mapsto \ \Phi(x, y) = x - tDH(y) \, .$$

In the language of currents we then have $S_\eta = \Phi_\sharp T_\eta$ and $S = \Phi_\sharp T$. Therefore, $S_\eta \to S$ in the sense of currents.

We want to show that

$$\mu_\eta \ \rightharpoonup^* \ \mu \, . \qquad (37)$$

First of all, note that S and S_η are rectifiable currents of multiplicity 1 supported on the rectifiable sets $\Gamma A = \Phi(\Gamma\partial u_t)$ and $\Gamma A_\eta = \Phi(\Gamma B_\eta) = \Phi(\Gamma Dv_\eta)$. Since B_η is a Lipschitz map, the approximate tangent plane $\vec{v}$ to S_η in (a.e.) point $(x, A_\eta(x))$ is spanned by the vectors $e_i + DA_\eta(x) \cdot e_i$.

Now, by the calculation of Proposition 5.1, it follows that $\det DA_\eta \geq 0$. Hence

$$\langle dy_1 \wedge \ldots \wedge dy_n, \vec{v} \rangle \ \geq \ 0 \, . \qquad (38)$$

By the convergence $S_\eta \to S$, (38) holds for the tangent planes to S as well.

Next, consider a $\varphi \in C_c^\infty(\Omega_t)$. Since both ΓA and ΓA_η are bounded sets, consider a ball $B_R(0)$ such that $\mathrm{supp}\,(\Gamma A), \mathrm{supp}\,(\Gamma A_\eta) \subset \mathbb{R}^n \times B_R(0)$ and let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function with $\chi|_{B_R(0)} = 1$. Then, by standard calculations on currents, the injectivity property of Proposition 5.1 and (38) imply that

$$\int \varphi d\mu \ = \ \langle S, \varphi(x)\chi(y)dy_1 \wedge \ldots \wedge dy_n \rangle, \qquad (39)$$

$$\int \varphi d\mu_\eta \ = \ \langle S_\eta, \varphi(x)\chi(y)dy_1 \wedge \ldots \wedge dy_n \rangle \, . \qquad (40)$$

Therefore, since $S_\eta \to S$, we conclude that

$$\lim_{\eta\downarrow 0} \int \varphi d\mu_\eta \ = \ \int \varphi d\mu \, .$$

This shows (37).

5.3. **Proof of Lemma 5.1.** First of all we choose ε so small that the conclusions of Proposition 5.1 and those of Subsection 5.2 hold.

We consider therefore, the approximations v_η of Subsection 5.2, we define the measures μ and μ_η as in (36) and the measures $\hat{\mu}$ and $\hat{\mu}_\eta$ as

$$\hat{\mu}(E) \ := \ |(Id - (t-\delta)DH(\partial u_t))(E)| \qquad \hat{\mu}_\eta(E) \ := \ |(Id - (t-\delta)DH(Dv_\eta))(E)| \, . \quad (41)$$

By the same arguments as in Subsection 5.2, we necessarily have $\hat{\mu}_\eta \rightharpoonup^* \hat{\mu}$.

The conclusion of the Lemma can now be formulated as

$$\hat{\mu} \geq \frac{(t-\delta)^n}{t^n} \mu \,. \tag{42}$$

By the convergence of the measures μ_η and $\hat{\mu}_\eta$ to μ and $\hat{\mu}$, it suffices to show

$$\hat{\mu}_\eta \geq \frac{(t-\delta)^n}{t^n} \mu_\eta \,. \tag{43}$$

On the other hand, since the maps $x \mapsto x - tDH(Dv_\eta)$ and $x \mapsto x - (t-\delta)DH(Dv_\eta)$ are both injective and Lipschitz, we can use the area formula to write:

$$\hat{\mu}_\eta(E) = \int_E \det\left(Id_n - (t-\delta)D^2H(Dv_\eta(x))D^2v_\eta(x)\right) dx, \tag{44}$$

$$\mu_\eta(E) = \int_E \det\left(Id_n - tD^2H(Dv_\eta(x))D^2v_\eta(x)\right) dx \tag{45}$$

Therefore, if we set

$$M_1(x) := Id_n - (t-\delta)D^2H(Dv_\eta(x))D^2v_\eta(x)$$
$$M_2(x) := Id_n - tD^2H(Dv_\eta(x))D^2v_\eta(x) \,,$$

the inequality (42) is equivalent to

$$\det M_1(x) \geq \frac{(t-\delta)^n}{t^n} \det M_2(x) \qquad \text{for a.e. } x. \tag{46}$$

Note next that

$$\det M_1(x) = \det(D^2H(Dv_\eta(x))) \det\left([D^2H(Dv_\eta(x))]^{-1} - (t-\delta)D^2v_\eta(x)\right)$$

$$\det M_2(x) = \det(D^2H(Dv_\eta(x))) \det\left([D^2H(Dv_\eta(x))]^{-1} - tD^2v_\eta(x)\right)$$

Set $A(x) := [D^2H(Dv_\eta(x))]^{-1}$ and $B(x) = D^2v_\eta(x)$. Then it suffices to prove that:

$$\det(A(x) - (t-\delta)B(x)) \geq \frac{(t-\delta)^n}{t^n} \det(A(x) - tB(x)) \,. \tag{47}$$

Note that

$$A - (t-\delta)B = \frac{\delta}{t}A + \frac{t-\delta}{t}(A - tB) \,.$$

By choosing ε sufficiently small (but only depending on c_H and C), we can assume that $A - tB$ is a positive semidefinite matrix. Since A is a positive definite matrix, we conclude

$$A - (t-\delta)B \geq \frac{t-\delta}{t}(A - tB) \,. \tag{48}$$

A standard argument in linear algebra shows that

$$\det(A - (t-\delta)B) \geq \frac{(t-\delta)^n}{t^n} \det(A - tB) \tag{49}$$

which concludes the proof. We include, for the reader convenience, a proof of $(48) \Longrightarrow (49)$. It suffices to show that, if E and D are positive semidefinite matrices with $E \geq D$, then $\det E \geq \det D$. Without loss of generality, we can assume that E is in diagonal form,

i.e. $E = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and that $E > D$. Then each λ_i is positive. Define $G := \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Then

$$\text{Id}_n \geq G^{-1}DG^{-1} = \tilde{D}.$$

Our claim would follow if we can prove $1 \geq \det \tilde{D}$, that is, if we can prove the original claim for E and D in the special case where E is the identity matrix. But in this case we can diagonalize E and D at the same time. Therefore $D = \text{diag}(\mu_1, \ldots, \mu_n)$. But, since $E \geq D \geq 0$, we have $0 \leq \mu_i \leq 1$ for each μ_i. Therefore

$$\det E = 1 \geq \Pi_i \mu_i = \det D.$$

5.4. **Proof of Lemma 4.3.** As in the proof above we will show the Lemma by approximation with the functions v_η. Once again we introduce the measures μ_η and μ of (36). Then, the conclusion of the Lemma can be formulated as

$$\mu \geq c_0 - tc_1 \Delta u_t. \tag{50}$$

Since $\Delta v_\eta \rightharpoonup^* \Delta u_t$ by Theorem 2.3(iii), it suffices to show

$$\mu_\eta \geq c_0 - tc_1 \Delta v_\eta. \tag{51}$$

Once again we can use the area formula to compute

$$\mu_\eta(E) = \int_E \det(D^2 H(Dv_\eta(x))) \det\left([D^2 H(Dv_\eta(x))]^{-1} - tD^2 v_\eta(x)\right) dx \tag{52}$$

Since $D^2 H \geq c_H^{-1} Id_n$ and $[D^2 H]^{-1} \geq c_H^{-1} Id_n$, we can estimate

$$\det(D^2 H(Dv_\eta(x))) \det\left([D^2 H(Dv_\eta(x))]^{-1} - tD^2 v_\eta(x)\right) \geq c_H^{-n} \det\left(\frac{1}{c_H} Id_n - tD^2 v_\eta(x)\right) \tag{53}$$

arguing as in Subsection 5.3. If we choose ε so small that $0 < \varepsilon < \frac{1}{2c_H C}$, then $M(x) := \frac{1}{2c_H} Id_n - tD^2 v_\eta(x)$ is positive semidefinite. Therefore

$$\det(D^2 H(Dv_\eta(x))) \det\left([D^2 H(Dv_\eta(x))]^{-1} - tD^2 v_\eta(x)\right) \geq c_H^{-n} \det\left(\frac{1}{2c_H} Id_n + M(x)\right). \tag{54}$$

Diagonalizing $M(x) = \text{diag}(\lambda_1, \ldots, \lambda_n)$, we can estimate

$$\det\left(\frac{1}{2c_H} Id_n + M(x)\right) = \left(\frac{1}{2c_H}\right)^n \prod_{i=1}^n (1 + 2c_H \lambda_i) \geq \left(\frac{1}{2c_H}\right)^n (1 + 2c_H \text{Tr} M(x))$$
$$= c_2 - c_3 t \Delta v_\eta(x) \tag{55}$$

Finally, by (52), (53), (54) and (55), we get

$$\mu_\eta(E) \geq \int_E (c_0 - c_1 t \Delta v_\eta(x)) \, dx.$$

This concludes the proof.

6. Proofs of Lemma 4.1 and Lemma 4.2

6.1. Proof of Lemma 4.1. The claim follows from the following consideration:

$$\chi_{t,0}(\Omega_t) \subset \chi_{s,0}(\Omega_s) \qquad \text{for every } 0 \leq s \leq t \leq T. \tag{56}$$

Indeed, consider $y \in \chi_{t,0}(\Omega_t)$. Then there exists $x \in \Omega_t$ such that y is the unique minimum of (17). Consider $z := \frac{s}{t}x + \frac{t-s}{t}y$. Then $z \in \Omega_s$. Moreover, by Proposition 3.1(iv), y is the unique minimizer of $u_0(w) + sL((z-w)/s)$. Therefore $y = \chi_{s,0}(z) \in \chi_{s,0}(\Omega_s)$.

6.2. Proof of Lemma 4.2. First of all, by Proposition 2.2, we can select a Borel set E of measure 0 such that

- $\partial u_t(x)$ is single-valued for every $x \in E$;
- $|E| = 0$;
- $|D_c^2 u_t|(\Omega_t \setminus E) = 0$.

If we assume that our statement were false, then there would exist a compact set $K \subset E$ such that $|D_c^2 u_t|(K) > 0$ and $X_{t,0}(K) = \chi_{t,0}(K) \subset \chi_{t+\delta,0}(\Omega_{t+\delta})$. Therefore it turns out that $X_{t,0}(K) = \chi_{t+\delta,0}(\tilde{K}) = X_{t+\delta,0}(\tilde{K})$ for some Borel set $\tilde{K}$.

Now, consider $x \in \tilde{K}$ and let $y := \chi_{t+\delta,0}(x) \in X_{t+\delta,0}(\tilde{K})$ and $z := \chi_{t+\delta,t}(x)$. By Proposition 3.1(iv), y is the unique minimizer of $u_0(y) + tL((z-y)/t)$, i.e. $\chi_{t,0}(z) = y$.

Since $y \in \chi_{t,0}(K)$, there exists z' such that $\chi_{t,0}(z')$. On the other hand, by Proposition 5.1, provided ε has been chosen sufficiently small, $\chi_{t,0}$ is an injective map. Hence we necessarily have $z' = z$. This shows that

$$X_{t+\delta,t}(\tilde{K}) \subset K. \tag{57}$$

By Lemma 5.1,

$$|K| \geq |X_{t+\delta,t}(\tilde{K})| \geq \frac{\delta^n}{(t+\delta)^n}|X_{t+\delta,0}(\tilde{K})| = \frac{\delta^n}{(t+\delta)^n}|X_{t,0}(K)|. \tag{58}$$

Hence, by Lemma 4.3

$$|K| \geq c_0|K| - c_1 t \frac{\delta^n}{(t+\delta)^n} \int_K \Delta u_t. \tag{59}$$

On the other hand, recall that $K \subset E$ and $|E| = 0$. Thus,

$$|K| \geq -c_1 t \frac{\delta^n}{(t+\delta)^n} \int_K \Delta_s u_t \geq c_1 t \frac{\delta^n}{(t+\delta)^n} \int_K \Delta_c u_t, \tag{60}$$

where the last inequality follows from $-\Delta_s u_t = -\Delta_c u_t - \Delta_j u_t \geq -\Delta_c u_t$ (by the semiconcavity of u). Arguing as in Subsection 4.3, we can show $-\Delta_c u_t \geq |D_c^2 u_t|$, and hence

$$|K| \geq c_0 t \frac{\delta^n}{(t+\delta)^n} |D_c^2 u_t|(K) > 0. \tag{61}$$

This contradicts the assumptions $K \subset E$ and $|E| = 0$, and hence concludes the proof.

References

[1] G.Alberti & L.Ambrosio: *A geometrical approach to monotone functions in $\mathbb{R}^n$*: Math. Z. 230 (1999), no.2, pp. 259-316.

[2] L.Ambrosio, N.Fusco & D.Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, 2000.

[3] L.Ambrosio & C.De Lellis: *A note on admissible solutions of 1d scalar conservation laws and 2d Hamilton-Jacobi equations.* J. Hyperbolic Diff. Equ., **31(4)**, 2004, pp. 813-826.

[4] A.Bressan: *Viscosity Solutions of Hamilton-Jacobi Equations and Optimal Control Problems (an illustrated tutorial)*, Lecture Notes.

[5] P.Cannarsa & C.Sinestrari: *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Birkhäuser, Boston, 2004.

[6] C.Castaing & M.Valadier: *Convex Analysis and Measurable Multifunction*, Lect. Notes in Math. **580**, Springer-Verlag, Berlin 1977.

[7] L.C.Evans: *Partial differential equations.* Graduate Studies in Mathmathics **319**, AMS, 1991.

[8] G.Minty: *Monotone nonlinear operators on a Hilbert space*, Duke Math. J. **29**, 1962, pp. 341-346.

Stefano Bianchini, SISSA, via Beirut 2-4 34014 Trieste, Italy

Camillo De Lellis, Roger Robyr, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: bianchin@sissa.it, delellis@math.unizh.ch, roger.robyr@math.unizh.ch