Quadratic interaction functional and regularity results for conservation laws

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Crash course on 1d hyperbolic systems of conservation laws

Hyperbolic system of conservation laws in 1d

$$\begin{cases} u_t + f(u)_x &= 0 \\ u(0,x) &= u_0(x) \end{cases} \qquad u \in \mathbb{R}^N,\ f : \mathbb{R}^N \to \mathbb{R}^N \tag{1}$$

with $Df(u)$ having N distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u).$$

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The left/right eigenvectors $\ell_i(u)$, $r_i(u)$, $i = 1, \dots, N$, allow to define the *characteristic families/wavefronts* $w_i(t,x)$

$$u_x \approx \sum_i w_i r_i(u), \quad w_i \approx \ell_i(u) \cdot u_x,$$

traveling with speed $\approx \lambda_i(u)$.

Theorem (Existence of entropy solutions)

If $\mathrm{Tot.Var.}(u_0) \ll 1$*, then there exists a unique "entropy" solution* $u(t) = S_t u_0$ *to* (1) *and* S_t *defines a Lipschitz semigroup in* L^1_{loc}.

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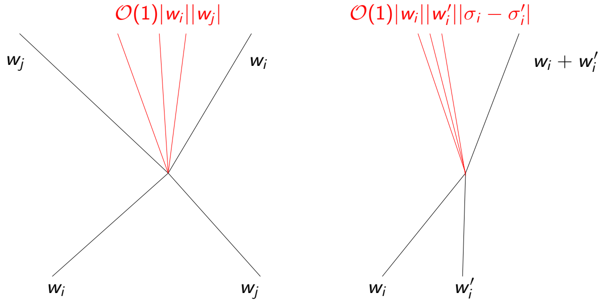
Sketch of the proof.

Main problem: $\mathrm{Tot.Var.}(u(t))$ may increase in time due to the nonlinear interaction among wavefronts w_i, w_j.



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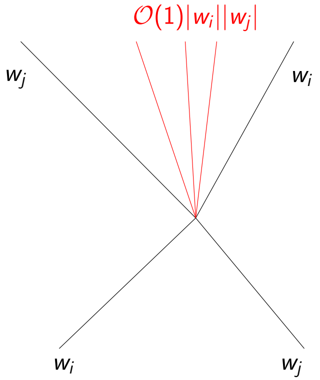
Two types of interaction:

Transversal if $i \neq j$, i.e. the wavefronts belong to two different families;

Non Transversal if $i = j$, i.e. the wavefronts belong to the same family.

Transversal interactions. This is a *linear* phenomenon: wavefronts with different speed cross each other, and never cross again.



Transversal interactions. This is a *linear* phenomenon: wavefronts with different speed cross each other, and never cross again. Hence the *quadratic* functional (*approaching wavefronts*)

$$Q^{\mathrm{Tr}}(t) = \sum_{i<j} \int_{x<y} |w_i(t,y)||w_j(t,x)| dxdy$$

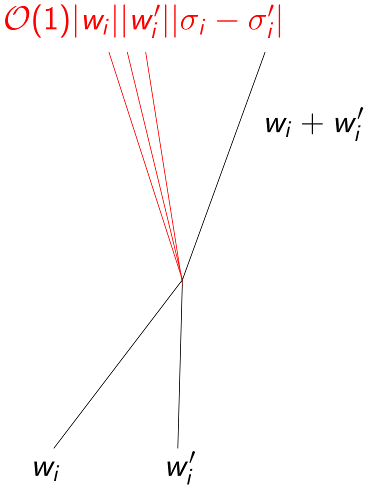
decreases of

$$\frac{d}{dt} Q^{\mathrm{Tr}}(t) = \int |w_i(t,x)||w_j(t,x)| dx. \tag{2}$$

The new wavefronts generated by the nonlinearity are at most

$$\mathrm{Tot.Var.}(u(t)) - \mathrm{Tot.Var.}(u(t-)) \leq \mathcal{O}(1) \int |w_i(t,x)||w_j(t,x)| dx. \tag{3}$$

Non-transversal interactions. This term is purely *nonlinear*.



Non-transversal interactions. This term is purely *nonlinear*.

The wavefronts generated by this kind of interaction are

$$\begin{aligned} & \mathrm{Tot.Var.}(u(t)) - \mathrm{Tot.Var.}(u(t-)) \\ & \lessapprox \mathcal{O}(1) \sum_i \int |v_i(t,x)||v_i'(t,x)|\big|\sigma_i(t,x) - \sigma_i'(t,x)\big| dx, \end{aligned} \tag{4}$$

where $\sigma_i(t,x)$ is the *speed* of the wavefront $w_i(t,x)$ (and $'$ the wavefront coming from right).

The *cubic* functional

$$Q^{\mathrm{NTr}}(t) = \sum_i \int_{x<y} |w_i(t,y)||w_i(t,x)|\big|\sigma_i(t,x) - \sigma_i(t,y)\big| dxdy$$

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$$\frac{d}{dt} Q^{\mathrm{Tr}}(t) = \int |w_i(t,x)||w_i'(t,x)|\big|\sigma_i(t,x) - \sigma_i'(t,x)\big| dxdx. \tag{5}$$

Hence one introduces the *Glimm* functional

$$\Gamma(t) := \mathrm{Tot.Var.}(u(t)) + C\big(Q^{Tr} + Q^{NTr}\big)$$

and deduce from (2-5) that for $C \gg 1$

$$\frac{d\Gamma}{dt} \leq 0.$$

In particular

$$\mathrm{Tot.Var.}(u(t)) \leq \mathrm{Tot.Var.}(u_0)\big(1 + C\mathrm{Tot.Var.}(u_0)\big),$$

and by compactness one concludes. □

The validity of the Lipschitz stability in L^1 can be understood by considering the equation for the perturbation $u + \epsilon h$, $\epsilon \searrow 0$,

$$h_t + (Df(u)f)_x = 0,$$

which is the same as the equation of u_x, and thus obtaining

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Remark. If f is genuinely nonlinear, i.e. $D\lambda_i r_i \neq 0$ (if $N = 1$ this means that f is convex/concave) then the *quadratic* functional

$$Q^{\mathrm{Gl}}(t) = \sum_i \int_{x<y} |w_i(t,x)||w_i(t,y)| dxdy$$

is decreasing and controls the non-transversal interactions:

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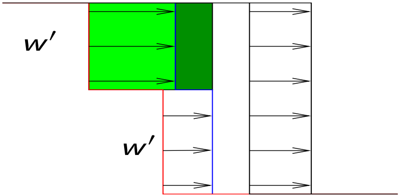
is decreasing and controls the non-transversal interactions:

i-wavefronts which interact never split.

Stability

The only proof of stability using hyperbolic technique is based on the quadratic Q^{Gl}:



Measure area with weight

$\mathtt{W}(z) = 1 + |w'|\chi_{s \in w}$

$\mathtt{d}(u, u') = \int \left(\int_u^{u'} \mathtt{W}(s) ds \right) dx$

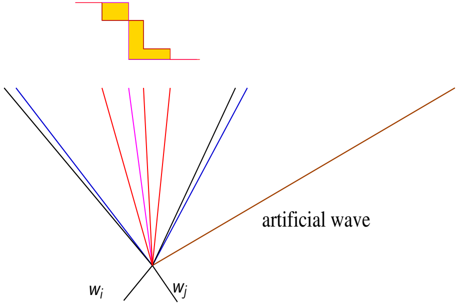
By differentiating in time

$\frac{d}{dt}\mathtt{d}(u, u') = |w||w'||\sigma - \sigma'|$

For the general case the use of the cubic Q^{NTr} would give a forth order decreasing term, not sufficient for stability.

Convergence of approximate solutions

The main perturbation when constructing approximate solutions is that we change the speed of the wavefronts:

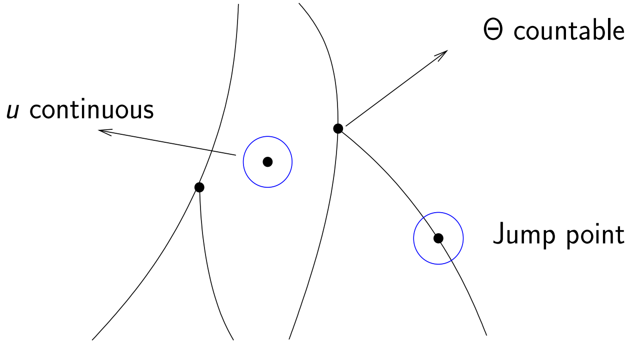


The *error* in $L^1(\mathbb{R})$ at time t can be thus estimated as

$$\sum_i \int |w_i(t,x)| \big|\sigma_i(t,x) - \tilde{\sigma}_i(t,x)\big| dx \approx \mathrm{Tot.Var.}(u(t))^2.$$

Structure

The control of the change in speed which is given by Q^{Gl} yields that u enjoys more regularity that just being $BV_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$:



Quadratic potential

There are several papers addressing this issue: the main idea is to study the quadratic functional

$$Q^{AM}(u(t)) := \sum_i \int_{x<y} \underbrace{\frac{|\sigma_i(t,x) - \sigma_i(t,y)|}{\mathrm{Tot.Var.}(u(t),[x,y])}}_{\mathcal{O}(1)} |w_i(t,x)||w_i(t,y)| dxdy.$$

The fact that $Q^{AM}(u(t))$ is *not* decreasing in t forces to study the solution from $(t, +\infty)$, and add a term

$$\mathcal{G}(t) = \int_t^{+\infty} \Big\{ \text{unwanted oscillations of } Q^{AM} \Big\} ds.$$

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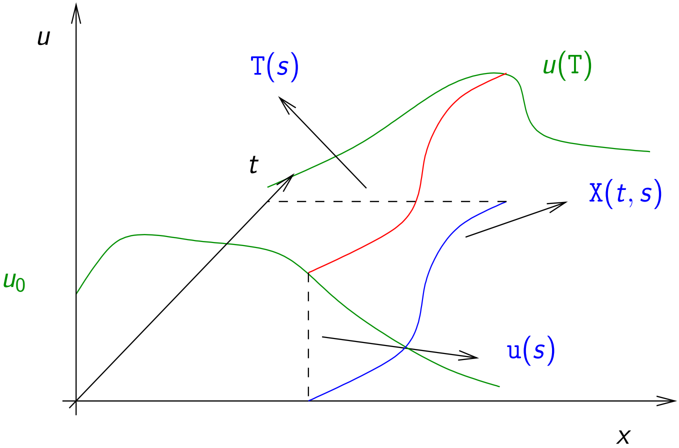
(However some gaps in the literature...)

Remark. From now on only the scalar case $N = 1$.

Wave representation

There exists three functions $\mathtt{T}(s)$, $\mathtt{X}(t,s)$, $\mathtt{u}(s)$



Wave representation

There exists three functions

Time $\mathtt{T} : (0, \mathrm{Tot.Var.}(u_0)] \to \mathbb{R}^+$ Borel

Position $\mathtt{X} : \big\{x \in (0, \mathrm{Tot.Var.}(u_0)], 0 \leq t < \mathtt{T}(s)\big\} \to \mathbb{R}$
Lipschitz in t and increasing in y

Value $\mathtt{u} : (0, \mathrm{Tot.Var.}(u_0)] \to \mathbb{R}$ 1-Lipschitz

such that

$$D_x u(t) = \mathtt{X}_\sharp\big(D_s \mathtt{u}\, \mathtt{T}^{-1}(\{t \leq \mathtt{T}(s)\})\, \mathcal{L}^1\big)$$
$$D_t u(t) = \mathtt{X}_\sharp\big(- \sigma(t)\, D_s \mathtt{u}\, \mathtt{T}^{-1}(\{t \leq \mathtt{T}(s)\})\, \mathcal{L}^1\big)$$

where

Speed $\sigma(t, s) := \frac{d}{dt}\mathtt{X}(t, s)$ is the speed of the wave s.

Quadratic estimate

Theorem (Modena-B.)

If $u(t)$ is the entropic solution, then

$$\int \mathrm{Tot.Var.}\big(\sigma(s),[0,\mathtt{T}(s))\big)\,ds \leq \mathcal{O}(\|D^2 f\|_\infty)\mathrm{Tot.Var.}(u_0)^2.$$

The proof is based on the fact that we can distinguish between

1. waves which have *already* interacted

$$\mathcal{I}(t) = \Big\{ s < s' : \exists \tau \leq t \big(\mathtt{X}(\tau,s) = \mathtt{X}(\tau,s')\big) \Big\},$$

2. waves which have *never* interacted

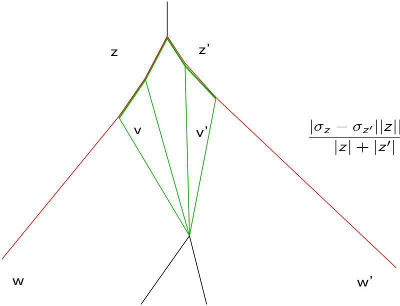
$$\mathcal{N}(t) = \Big\{ s < s' : \forall \tau \leq t \big(\mathtt{X}(\tau,s) < \mathtt{X}(\tau,s')\big) \Big\}.$$

Proof.

One can prove that the original Glimm functional

$$\mathfrak{Q}(t) := \mathcal{L}^2\big(\mathcal{N}(t)\big)$$

is sufficient, because if two waves split, in order to make them interact again one needs to use waves which have never interacted. □



$$\frac{|\sigma_z - \sigma_{z'}||z||z'|}{|z| + |z'|} \leq \mathcal{O}(1)|w + v'||w' + v|$$

Regularity

The results on the regularity are very similar to the genuinely nonlinear case:

1. the countable set is determined by the set where

$$\mu^a := \left[\mathtt{X}_\sharp \left(\int |D_t \sigma(t)| dt \right) \right]^{\mathrm{atomic}}$$

   is concentrated, i.e. where a positive set of waves s has a jump in the speed,

2. on the jump set is only 1-rectifiable, because it can open and close on a Cantor like set in t (there is not a strong stability as in the genuinely nonlinear case).

Perspectives

Extend to systems. In preparation....

Lagrangian (wave) representation. The map $(\mathtt{X}, \mathtt{T}, \mathtt{u})$ are "compact" even when $\mathrm{Tot.Var.}(u_0) \to \infty$.
Is this a *Lagrangian representation* of L^∞ solutions? Ok for continuous...
For scalar multi-d?

Quadratic estimate for singular approximations. Is it possible to prove some quadratic interaction for viscous conservation laws?