Decomposition of vector fields in $\mathbb{R}^d$

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A basic PDEs

In many systems of PDEs one of the equations is the *continuity equation*

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{b}) = \partial_t \rho + \sum_{i=1}^{d} \partial_{x_i}(\rho b_i) = 0.$$

This equation means that the quantity ρ is conserved: in every regular region Ω

$$\frac{d}{dt}\int_\Omega \rho = \int_{\partial\Omega} \rho \mathbf{b} \cdot n,$$

or equivalently in weak form

$$\int \rho(\partial_t \psi + \mathbf{b} \cdot \nabla \psi)\, dxdt + \int \rho\psi(t=0)dx = 0$$

for every smooth test function ψ.

There is a clear relation with the ODE

$$\frac{d}{dt}\mathtt{X}(t,y) = \mathbf{b}(t,\mathtt{X}(t,y)), \qquad \mathtt{X}(0,y) = y.$$

Indeed the function given by

$$\int_{\Omega} \rho(t,x) = \int_{\mathtt{X}^{-1}(\Omega)} \rho_0(y)dy$$

is a solution to the continuity equation: just observe that if

$$\Omega(t) = \{X(t,y), y \in \Omega(0)\}$$

then the lateral flow in 0.

Classical PDEs

One is usually interested in solving the ODE

$$\frac{d}{dt}\mathtt{X}(t,y) = \mathbf{b}(t,\mathtt{X}(t,y)), \qquad \mathtt{X}(0,y) = y.$$

for every initial point y.
If $\mathbf{b}$ is continuous, then Peano's theorem yields an existence result: there exists at least one solution for every initial datum.
If we ask more regularity w.r.t. x, e.g.

$$\big|\mathbf{b}(t,x) - \mathbf{b}(t,x')\big| \leq C|x - x'|,$$

then one has also uniqueness.

For such sufficiently regular vector fields, one has thus existence and (in case) uniqueness for the solution of the continuity equation: indeed one can rewrite the PDE as

$$\partial_t \big(\rho(t, \mathtt{X}(t,y)) \det \nabla_y \mathtt{X}(t,y)\big) = 0.$$

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On the other hand, a weak solution to

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{b}) = 0,$$

requires only $\rho, \rho\mathbf{b}$ to be locally integrable: for ψ smooth

$$\int \rho \big(\partial_t \psi + \mathbf{b} \cdot \nabla \psi\big) dxdt + \int \rho\psi(t=0)dx = 0.$$

The vector field **b** is determined by the solution to a system of PDEs, but in general it is not smooth; in the following we will unlink this dependence of **b** and look for an almost everywhere well posedness of the Cauchy problem for the ODE.

Lagrangian representation

Consider the PDE for $\rho \geq 0$

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{b}) = \mu^+ - \mu^-, \quad \mu^\pm \text{ locally bounded measures.}$$

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Theorem (Smirnov, Ambrosio)

There exists a measure η on the space
$\Gamma = \left\{\gamma : [t_\gamma^-, t_\gamma^+] \to \mathbb{R}^d, \frac{d\gamma}{dt} = \mathbf{b}(t, \gamma)\right\}$
such that

$$\int \psi \rho (1, \mathbf{b}) dxdt = \int \left\{ \int_{t_\gamma^-}^{t_\gamma^+} \psi(t, \gamma(t)) \left(1, \frac{d\gamma}{dt}\right) dt \right\} \eta(d\gamma),$$

$$\int \phi \mu^\pm (dtdx) = \int \psi\big(t_\gamma^\mp, \gamma(t_\gamma^\mp)\big) \eta(d\gamma).$$

Some remark:

- the existence of such a measure can be interpreted as the fact that if we have a solution to the PDE then there are enough solutions to the ODE to represent it;
- it holds also for $\rho, \rho\mathbf{b}$ measures;
- the uniqueness of η is lost for two reasons:
  1. the trajectories cross each other,
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The nonuniqueness is thus related to the first situation: we say that η is *untangled* if it is concentrated on a set of trajectories Υ such that

$$\gamma, \gamma' \in \Upsilon \text{ intersecting} \quad \Longleftrightarrow \quad \gamma \cup \gamma' \text{ still a trajectory.}$$

A "classical" approach

($\mu^{\pm} = 0$ for simplicity.) One can give a meaning to

$$\partial_t u + \mathbf{b} \cdot \nabla u = 0, \quad u(t = 0) = u_0, \tag{1}$$

by requiring that

$$\partial_t (\rho u) + \operatorname{div}\big(\rho u \mathbf{b}\big) = 0, \quad \rho u(t = 0) = \rho_0 u_0. \tag{2}$$

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The chain rule in the classical sense given that if u is a solution then also $\beta(u)$ is a solution to (1) for β smooth: indeed $u(t, \gamma(t))$ is constant.

In the literature such a property

$$\forall u \in L^\infty(\rho)\Big(\rho u \text{ solution to (2)} \implies \rho\beta(u) \text{ is a solution to (2)}\Big)$$

is called *renormalization property (of $\rho(1, \mathbf{b})$)*.

If every solution ρ enjoys the renormalization property, then uniqueness for every initial data:

- if for some initial point x_0 one has two solutions $\gamma_1 \neq \gamma_2$, $\gamma_1(0) = \gamma_2(0) = x_0$, then take

$$\rho(t) = \frac{1}{2}\big(\delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}\big), \quad \rho u(t) = \frac{1}{2}\big(\delta_{\gamma_1(t)} - \delta_{\gamma_2(t)}\big)$$

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- if uniqueness for every initial data, then

$$\rho(t) = \int \delta_{\gamma(t)} \rho_0(d\gamma(0)), \quad \rho u(t) = \int \delta_{\gamma(t)} \rho_0 u_0(d\gamma(0)),$$

  so that u is constant along the trajectories used by ρ.

In general one relaxes this condition requiring ρ to belong to some class, e.g.

$$0 \leq \rho \leq C.$$

In this case one obtains that

renormalization property for all $\rho \leq C$

$$\Updownarrow$$

uniqueness among *regular Lagrangian flows*.

Regular Lagrangian flows $\mathtt{X}(t,x)$ are such that

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Remark. We are not saying that for almost every initial data x_0 there exists a unique solution to the ODE (OPEN).

Key computation [Diperna-Lions]: ($\rho = 1$ and $\operatorname{div} \mathbf{b} = 0$) if $u^\epsilon = \phi^\epsilon * u$ is a smoothing of u, then one computes

$$\partial_t \beta(u^\epsilon) + \mathbf{b} \cdot \nabla \beta(u^\epsilon) = \beta'(u^\epsilon)\big(\mathbf{b} \cdot \nabla u^\epsilon - (\mathbf{b} \cdot \nabla u)^\epsilon\big),$$

and thus the problem reduces in showing that as $\epsilon \to 0$ the commutator

$$\mathbf{b} \cdot \nabla u^\epsilon - (\mathbf{b} \cdot \nabla u)^\epsilon = \int u(x + \epsilon y) \frac{\mathbf{b}(x) - \mathbf{b}(x + \epsilon y)}{\epsilon} \cdot \nabla \phi(y) dy \to 0.$$

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If $\nabla b \in L^1$ then this can be done, otherwise the convergence of the integrand is not strong enough to pass to the limit.
Remark. It seems that **b** should have some sort of weak derivative...

Keyfitz-Kranzer system

The system of conservation laws

$$\partial_t u + \sum_i \partial_{x_i}\big(f_i(|u|)u\big) = 0, \quad u \in \mathbb{R}^m,$$

can be written as

$$\partial_t |u| + \operatorname{div}\big(\mathbf{f}(|u|)|u|\big) = 0, \quad \partial_t \theta + \mathbf{f}(|u|) \cdot \nabla\theta = 0,$$

where $\theta = u/|u|$.
The theory of Kruzkhov yields for scalar conservation laws that $\nabla|u|$ is a bounded measure, and thus one reduces to a transport equation with vector fields whose derivative is a measure.

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2. in the jump part of ∇u: even if the commutator estimate cannot be passed to the limit, then one can use the traces [Ambrosio-DeLellis-Maly] to show that the chain rule holds

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3. in the so called Cantor part, no clear choice of ϕ and no trace theory.

A regularity approach

In the smooth case one has

$$\frac{d}{dt}\nabla \mathtt{X} = \nabla \mathbf{b}(t, \mathtt{X})\nabla \mathtt{X},$$

so that

$$\frac{d}{dt}\text{"}\log \nabla X\text{"} = \nabla \mathbf{b}, \quad \text{"}\nabla X = \exp\left\{\int \nabla b dt\right\}\text{"}.$$

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Unfortunately log is sublinear, so that no compactness estimate can be obtained, but one can hope to have an estimate for

$$A_r(t,x) = \frac{1}{r^d} \int_{|y|<r} \log \left(1 + \frac{|\mathtt{X}(t,x+y) - \mathtt{X}(t,x)|}{r} \right) dy.$$

[DeLellis-Crippa] Indeed

$$\frac{d}{dt}A_r \leq \frac{1}{r^d}\int_{|y|<r} \frac{|\mathbf{b}(t,x+y)-\mathbf{b}(t,x)|}{|y|}dy$$
$$\leq M_{\nabla \mathbf{b}}(x) + M_{M_{\nabla \mathbf{b}}}(x),$$

being the maximal function M

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If $\nabla \mathbf{b} \in L^p$, $p > 1$, then

$$\int \sup_r A_r(t,x)dx \leq C_0 + C_1\int_0^t \|\nabla \mathbf{b}\|_{L^p}dt.$$

In particular, if at time T we have a mixing of order δ, i.e. it means that a constant fraction of trajectories starting from T at a distance $< \delta$ are then separated by 1, so that (inverting time)

$$\int \sup_r A_r(0,x)dx \simeq \log(1+1/\delta) \leq C_0 + C_1 \int_0^t \|\nabla \mathbf{b}\|_{L^p} dt.$$

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Bressan Mixing Conjecture: if the vector fields $\mathbf{b}$ mixes $\rho \in [1/C, C]$ to order δ at time $T = 1$, then

$$\int_0^1 |\nabla \mathbf{b}| dxdt \geq C \log \delta^{-1}.$$

This is an explicit estimate on compactness.

A way of restating uniqueness

The classical uniqueness/continuity can be written as

$$\forall R > 0 \; \exists r \; \Big(|\gamma'(0) - \gamma(0)| < r \implies |\gamma'(t) - \gamma(t)| \leq R \Big).$$

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In the language of Lagrangian representations η we could write: for all $R, \varpi > 0$ there exists r such that

$$\int \frac{\eta(\{|\gamma'(0) - \gamma(0)| < r, \sup_t |\gamma'(t) - \gamma(t)| > R\})}{\eta(\{|\gamma'(0) - \gamma(0)| < r\})} \eta(d\gamma) < \varpi.$$

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Theorem

The above condition implies untangling of trajectories, hence uniqueness.

In Bressan's case, i.e. when $\nabla \mathbf{b}$ is a measure, one considers the PDE

$$\partial_t 1 + \mathrm{div}\big(1 \cdot \mathbf{b}\big) = \mathrm{div}\mathbf{b}.$$

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One can prove an even stronger condition: there exists "sufficiently smooth" cylinders which approximate the cylinders of $\mathtt{X}$, hence deducing the approximate differentiability of the flow.
A consequence of this fact that that the Lagrangian representation η of $(1, \mathbf{b})$ is unique up to the degeneracy of the initial/final points of the curves.

Sorry for the people not cited here and

THANK YOU!