HILBERT SPACES

1. Scalar product

If X is a vector space over $\mathbb{C}$, a function $(\cdot,\cdot) : X \times X \mapsto \mathbb{C}$ is a *scalar product* over X if it is

(1) *sesquilinear*: linear w.r.t. the first component and skewlinear w.r.t. the second,

$$(ax_1 + bx_2, y) = a(x_1, y) + b(x_2, y), \quad (x, ay_1 + by_2) = \bar{a}(x, y_1) + \bar{b}(x, y_2), \quad a, b \in \mathbb{C}; \tag{1.1}$$

(2) *skew symmetric*,

$$(x, y) = \overline{(y, x)}; \tag{1.2}$$

(3) *positive*,

$$(x, x) > 0 \quad \text{if} \quad x \neq 0. \tag{1.3}$$

Given the scalar product $(\cdot,\cdot)$, we define

$$\|x\| = (x, x)^{1/2}. \tag{1.4}$$

Proposition 1.1 (Schwarz inequality)**.** *For all $x, y \in X$ we have*

$$|(x, y)| \leq \|x\| \|y\|, \tag{1.5}$$

and the equality holds only for x, y linearly dependent.

Proof. Since the inequality is true for $y = 0$, we can assume $y \neq 0$. Consider the function

$$\mathbb{C} \ni t \mapsto \|x + ty\|^2 = \|x\|^2 + |t|^2 \|y\|^2 + 2\Re\{t(x, y)\} \geq 0.$$

The function is strictly positive unless $x = ty$ for some t. Choosing

$$t = -\frac{\overline{(x, y)}}{\|y\|^2},$$

where for $\alpha \in \mathbb{C}$, $\bar{\alpha}$ is the complex conjugate, we obtain that

$$0 \leq \|x\|^2 + \frac{|(x, y)|^2}{\|y\|} - 2\frac{|(x, y)|^2}{\|y\|^2} = \|x\|^2 - \frac{|(x, y)|^2}{\|y\|^2}.$$

Note that the t we choose is the minimum of the function $\|x + ty\|^2$. □

Note that for $t = \pm 1$ we obtain the *parallelogram identity*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.6}$$

Corollary 1.2. *The function* (1.4) *is a norm on X.*

The proof is left as an exercise: note that the triangle inequality follows from the Schwartz inequality.

With the topology generated by the norm $\|\cdot\|$ of (1.4), we can speak about continuity, in particular completeness:

A linear space H is a *Hilbert space* if it has a scalar product and it is complete w.r.t. the norm generated by the scalar product.

The most important example of Hilbert space is the space L^2 of square integrable functions on some measure space Ω, with the scalar product

$$(u, v) = \int_\Omega u(x) v(x) d\mu.$$

2. Closest point in a closed convex subset

Theorem 2.1. *Let K a non empty closed convex subset of a Hilbert space H. Then, for all $x \in X$, there is a unique point $y \in K$ such that*

$$\|x - y\| = \inf_{z \in K} \|x - z\|. \tag{2.1}$$

Proof. Let z_n be a minimizing sequence of the left hand side of (2.1). We can use the parallelogram identity (1.6) to obtain

$$\big\|x - (y_n + y_m)/2\big\|^2 + \frac{1}{4}\|y_n - y_m\|^2 = \frac{1}{2}\big(\|x - y_n\|^2 + \|x - y_m\|^2\big).$$

Since K is convex, then $(y_n + y_m)/2 \in K$, so that

$$\|y_n - y_m\|^2 \leq 2\big(\|x - y_n\|^2 + \|x - y_m\|^2\big) - 4 \inf_{z \in K} \|x - z\|^2.$$

It follows that y_n is a Cauchy sequence, and from completeness of H and closeness of K, $y_n \to y \in K$. This is a minimizer, and using again the parallelogram identity one sees that it is unique. □

This result holds also on uniformly convex Banach spaces.

Let $Y \subset H$ be a linear subspace of H. We define the *orthogonal complement* $Y^\perp$ as

$$Y^\perp = \Big\{x \in H : (x, y) = 0 \ \forall y \in Y\Big\}. \tag{2.2}$$

Theorem 2.2. *Let $Y \subset H$ be a closed subspace, $Y^\perp$ its orthogonal complement. Then*

1. *$Y^\perp$ is a closed subspace of H;*
2. *$H = Y + Y^\perp$, $Y \cap Y^\perp = \{0\}$;*
3. *$(Y^\perp)^\perp = Y$.*

Proof. From the linearity of the scalar product, it follows that $Y^\perp$ is a subspace. Since $(\cdot,\cdot)$ is also continuous w.r.t. $\|\cdot\|$, then $Y^\perp$ is closed. This concludes (1).

Clearly if $x \in Y \cap Y^\perp$, then $(x, x) = 0$, so that $Y \cap Y^\perp = \{0\}$. On the other hand, for any $x \in H$, by Theorem 2.1 there exists a unique $y \in Y$ closest to x. If we denote by $v = x - y$, then the minimum is

$$\|v\|^2 \leq \|v + ty\|^2 = \|v\|^2 + 2\Re(t(v, y)) + |t|^2\|y\|^2, \quad \forall t \in \mathbb{C}, \ y \in Y.$$

Hence it follows $(v, y) = 0$, so that $v \in Y^\perp$.

The last part follows from (2). □

Remark 2.3. It follows that every closed subspace has a closed linear complement. This is not true for Banach spaces: in fact, if a Banach space has the property that every closed linear subspace has a closed linear complement, then there is a scalar product which generates a norm, equivalent to the initial norm.

3. Linear functionals

For any $y \in H$, we can construct a linear continuous functional $\ell_y \in H^*$ defined by

$$\ell_y(x) = (x, y). \tag{3.1}$$

One checks that Schwartz inequality gives

$$\|\ell_y\|_{H^*} = \|y\|_H.$$

Thus a Hilbert space has a continuous embedding of H into H^*. It follows that this is an isometry: every continuous functional can be represented as a scalar product with some fixed element.

Theorem 3.1 (Riesz Fréchet representation)**.** *Let $\ell \in H^*$ be a continuous linear functional on the Hilbert space H,*

$$|\ell x| \leq C\|x\|.$$

Then there is some $y \in H$ such that

$$\ell x = (x, y).$$

Proof. We can assume that $\ell \neq 0$. Consider the closed subspace

$$N_\ell = \{x : \ell x = 0\}.$$

Since $\ell \neq 0$, there is $z \in H$ such that $\ell z \neq 0$. Let $y \in N_\ell$ be its projection on N_ℓ, define $v = z - y$, so that $\ell v \neq 0$ and

$$N_\ell^\perp = \Big\{x \in H : (x, y) = 0 \ \forall y \in N_\ell\Big\} = \mathbb{C}\{v\}.$$

The fact that $N_\ell^\perp$ is one dimensional is a consequence of the fact that ℓ is a linear functional.

The linear functional

$$\ell_v x = \frac{\ell v}{\|v\|^2}(x, v)$$

is equal to ℓ on $N_\ell^\perp$, hence in the whole H. □

We can generalize the representation to maps $B : H \mapsto H \in \mathbb{C}$ which are

- *sesquilinear*: linear w.r.t. the first component and skewlinear w.r.t. the second,

$$B(ax_1 + bx_2, y) = aB(x_1, y) + bB(x_2, y), \quad B(x, ay_1 + by_2) = \bar{a}B(x, y_1) + \bar{b}B(x, y_2), \quad a, b \in \mathbb{C}; \tag{3.2}$$

- *bounded*: there exists C such that

$$|B(x, y)| \leq C\|x\|\|y\|; \tag{3.3}$$

- *positive*: there is a positive constant $b > 0$ such that

$$|B(x, x)| \geq b\|x\|^2. \tag{3.4}$$

Theorem 3.2 (Lax-Milgram)**.** *Every linear functional $\ell \in H^*$ can be written uniquely as*

$$\ell x = B(x, y), \quad y \in H \tag{3.5}$$

Proof. For any y fixed, the map

$$x \mapsto B(x, y)$$

is a linear functional over H, hence there is some $\ell_y = B(\cdot, y)$. By Theorem 3.1, we can construct a map

$$\mathbf{T} : H \mapsto H, \quad B(x, y) = (x, Ty).$$

This map is skewlinear. Moreover we have by the properties of B that

$$b\|y\|_H \leq \|\mathbf{T}y\|_{H^*} \leq C\|y\|_H.$$

One sees easily that from the above relation it follows that $\mathbf{T}H$ is a closed subspace of H.

If $\mathbf{T}H \neq H$, then there is $v \in \mathbf{H}^\perp$, so that

$$B(v, y) = 0 = B(v, v) \geq b\|v\|^2.$$

Hence $v = 0$, so that $\mathbf{T}H = H$.

It follows that $\mathbf{T}^{-1} : H \mapsto H$ exists and it is continuous. The representation theorem of linear functionals yields the statement. □

4. Linear span

Given a set $S \subset H$, we define the *closed linear span* of S as the smallest closed linear subspace containing S.

Theorem 4.1. *The closed linear span of S is*

$$\overline{\mathrm{span}\{S\}} = \big\{y : (s, y) = 0, \ s \in S\big\}^\perp. \tag{4.1}$$

Proof. The right hand side of (4.1) is a closed subspace, and it contains S.

If $x \in \mathrm{span}(S)$, then $(x, y) = 0$ if $(s, y) = 0$ for all $s \in S$. Since the closed linear span is the closure of the linear span, the conclusion follows. □

5. Orthonormal bases

A collection of vectors $\{x_k\}_k \subset H$ is *orthonormal* if

$$(x_k, x'_k) = \begin{cases} 1 & k = k' \\ 0 & k \neq k' \end{cases} \tag{5.1}$$

The family $\{x_k\}_k$ is an *orthonormal base* if moreover

$$\overline{\text{span}\{x_k, k\}} = H. \tag{5.2}$$

For orthonormal vectors, we have a simple way to characterize their closed linear span.

Lemma 5.1. *The closed linear span of the orthonormal family $\{x_k\}_k$ consists of all vectors of the form*

$$x = \sum_k a_k x_k, \quad \sum_k |a_k|^2 < \infty. \tag{5.3}$$

where the convergence is in the sense of norm. Moreover

$$\|x\|^2 = \sum_k |a_k|^2, \quad a_k = (x, x_k), \tag{5.4}$$

We leave the proof as an exercise. One can also verify that only a countable number of a_k is different from 0.

We conclude with the proof that there exists at least an orthonormal base for every Hilbert space H. This is done again using Zorn's lemma.

Theorem 5.2. *Every Hilbert space H has an orthonormal base.*

Proof. Consider the orthonormal sets, partially ordered by inclusion. Since every totally ordered collection of orthonormal sets has an upper bound given by their union, by Zorn's lemma there is a maximal element $\{x_k\}$.

If $H \neq \overline{\text{span}\{x_k, k\}}$, then there is an element y, which is orthogonal to $\overline{\text{span}\{x_k, k\}}$. Normalize it to 1, and add to the sequence $\{x_k\}_k$, contradicting the maximality of $\{x_k\}_k$. □

6. Exercises

(1) Show that a norm satisfying the parallelogram identity comes from a scalar product.
(2) Show that the scalar product is continuous w.r.t. the topology generated by the norm (1.4).
(3) Consider the space ℓ^2

$$\ell^2 = \left\{ u : \mathbb{N} \mapsto \mathbb{C} : (u, v) = \sum_{n=1}^{\infty} u(n)\bar{v}(n) \right\}.$$

Show that this is a Hilbert space.
(4) Prove that a Hilbert space is uniformly convex.
(5) Show that the closed linear span is the closure of the linear span.
(6) Show that if H is separable, then the orthonormal base is countable.
(7) Show that if H is a separable Hilbert space, then it is isomorphic to ℓ^2.
(8) Let H be an infinite separable space. Show that $\{\|x\| = 1\}$ is not compact.
(9) Define

$$L_o = \left\{ \phi \in L^2(-a, a), \phi \text{ odd } \right\}, \quad L_e = \left\{ \phi \in L^2(-a, a), \phi \text{ even } \right\}.$$

Find the distance of $\phi \in L^2(-a, a)$ form L_o and L_e.
(10) Let H be a Hilbert space, $\{e_\alpha\}_\alpha$ be an orthonormal base, and assume that for the bounded sequence $\{x_n\}_n$

$$(u_n, e_\alpha) \to (u, e_\alpha) \quad \forall e_\alpha.$$

Show that $u_n \rightharpoonup u$.