WEAK TOPOLOGIES

1. The weak topology of a topological vector space

Let X be a topological vector space over the field F, $F = \mathbb{R}$ or $F = \mathbb{C}$. For definiteness we assume $F = \mathbb{C}$.

We recall that the *dual space* X^* of X is

$$X^* = \Big\{ \ell : X \mapsto \mathbb{C}, \ell \text{ continuous} \Big\}. \tag{1.1}$$

The function

$$p_\ell(x) = |\ell x| \tag{1.2}$$

with ℓ continuous is a *continuous seminorm* on X. If we assume that the family of seminorms

$$\mathcal{P} = \Big\{ p_\ell : \ell \in X^* \Big\} \tag{1.3}$$

is *separating*, then the topology generated by $\mathcal{P}$ makes X into a locally convex topological vector space. We denote the topology generated by $\mathcal{P}$ by $\sigma(X, X^*)$.

Remark 1.1. If X is locally convex, for example a Fréchet space, normed or Banach space, then $\mathcal{P}$ is separating by the Hahn-Banach theorem.

Lemma 1.2. *The weak topology $\sigma(X, X^*)$ is the weakest topology such that each map*

$$\ell : X \mapsto \mathbb{C}, \quad \ell \in X^*, \tag{1.4}$$

is continuous.

A sequence x_n converges to x in $\sigma(X, X^*)$ if and only if for all $\ell \in X^*$

$$\lim_{n\to\infty} \ell x_n = \ell x. \tag{1.5}$$

A set E is bounded w.r.t. $\sigma(X, X^*)$ if and only if $\ell(E)$ is bounded in $\mathbb{C}$.

If X is infinite dimensional, then each element of a local base $\mathcal{B}$ contains an infinite dimensional subspace, hence $\sigma(X, X^*)$ is not locally bounded.

The proof is left as an exercise.

Let τ be the original vector topology of X. Clearly $\sigma(X, X^*) \subset \tau$. We thus say that

- the sequence x_n *converges strongly* to x and we write

$$x_n \to x \quad \text{if } x_n \text{ converges to } x \text{ in the original topology } \tau. \tag{1.6}$$

- the sequence x_n *converges weakly* to x and we write

$$x_n \rightharpoonup x \quad \text{if } x_n \text{ converges to } x \text{ in the topology } \sigma(X, X^*). \tag{1.7}$$

Similarly we will speak about strong neighborhood, strongly closed, strongly bounded..., and weak neighborhood, weakly closed, weakly bounded....

A simple consequence of the fact that $\sigma(X, X^*) \subset \tau$ is that

$$x_n \to x \quad \Longrightarrow \quad x_n \rightharpoonup x, \tag{1.8}$$

i.e. *every strongly convergent sequence is weakly convergent.*

Moreover, if $x_n \rightharpoonup x$, then the orbits

$$\Gamma(\ell) = \big\{ \ell x_n \big\}_{n\in\mathbb{N}}$$

are bounded. It follows from uniform boundedness principle that

Proposition 1.3. *If X is a normed space, so that X^* is Banach, then*

$$x_n \rightharpoonup x \quad \Longrightarrow \quad \|x_n\| \text{ bounded.} \tag{1.9}$$

Moreover

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|. \tag{1.10}$$

The same proof shows that if E is weakly bounded and X is a normed space, then E is strongly bounded.

Proof. In fact, from Banach Steinhaus theorem we have that the sequence $\|x_n\|$ is uniformly bounded. Moreover,

$$|\ell x_n| \le \|\ell\|_{X^*} \|x_n\|,$$

and since $\ell x_n \to \ell x$,

$$|\ell x| \le \|\ell\|_{\ell^*} \liminf_{n\to\infty} \|x_n\|.$$

It follows (1.10) by Hahn Banach. □

For infinite dimensional normed spaces, the weak topology $\sigma(X, X^*)$ is always weaker than the strong topology generated by the norm.

Example 1.4. Let us prove that the weak closure of $S(0,1) = \{\|x\| = 1\}$ is $\overline{B_X(0,1)} = \{\|x\| \le 1\}$. If V is a weak neighborhood of 0, then

$$V = \Big\{x : |\ell_i x| < \epsilon, i = 1, \dots, n\Big\}, \quad \ell_i \in X^*.$$

Since X is infinite dimensional, then there is

$$0 \ne y \in \bigcap_{i=1}^{n} N_{\ell_i},$$

so that

$$x + ty \in x + V \quad \forall t \in \mathbb{C}.$$

Since for some $g(t) = \|x + ty\|$ is strongly continuous and

$$g(x) < 1, \quad \lim_{t\to\infty} g(t) = +\infty,$$

then there is $\bar{t}$ such that

$$g(\bar{t}) = 1 \quad \Longrightarrow \quad x + \bar{t}y \in S(0,1).$$

Thus the weak closure of $S(0,1)$ contains $\overline{B_X(0,1)}$. By means of the Hahn Banach separation theorem, if $x \notin \overline{B_X(0,1)}$, then there is a linear functional ℓ separating x and $\overline{B_X(0,1)}$:

$$\|\ell\|_{X^*} = 1, \quad \Re(\ell x) > 1.$$

The weakly open set $\{x : \Re(\ell x) > 1\}$ is thus an open neighborhood of x with empty intersetion with $\overline{B_X(0,1)}$. Thus the weak closure of $S(0,1)$ is contained in $\overline{B_X(0,1)}$. Hence the conclusion follows.

Similarly, one can show that $B_X(0,1) = \{\|x\| < 1\}$ has empty interior for $\sigma(X, X^*)$. In particular it is not open.

Despite these facts, there are sets whose weak closure is equivalent to strong closure. In some sense, the next result is the formalization of the above example.

Theorem 1.5. *If $K \subset X$ is convex and X locally convex, then it is weakly closed if and only if is it strongly closed.*

In Example, in fact, 1.4 the weak closure of $\overline{B_X(0,1)}$ is again $\overline{B_X(0,1)}$.

Proof. Since $\sigma(X, X^*) \subset \tau$, then if K is weakly closed it is strongly closed.

Conversely, if K is strongly closed and convex, let $x_0 \in X \setminus K$. Then by the Hahn-Banach theorem (for complex vector spaces) there is some $\ell \in X^*$ such that

$$\sup_{x \in K} \Re(\ell x) \leq \gamma_1 < \gamma_2 \leq \Re(\ell x_0).$$

Hence the neighborhood of x_0

$$x_0 + V = x_0 + \Big\{x : |\ell x| \leq \Re(\ell x_0) - \gamma_2\Big\}$$

has empty intersection with K. □

In particular, in a topological vector space the closure of convex sets is convex. Hence we have

Corollary 1.6. *If X is a metrizable locally convex topological vector space, and $x_n \rightharpoonup x$, then there exists a sequence of convex combinations*

$$y_i = \sum_{finite} \alpha_{i,n} x_n, \quad \alpha_{i,n} \geq 0, \sum_n \alpha_{i,n} = 1$$

which converges to x strongly.

Proof. Let H be the convex hull of $\{x_n\}_n$, and K its weak closure. Then we have that $x \in K$, and by theorem 1.5 it follows that x is also in the strong closure of H. Since X is metrizable, this means that there is some sequence y_i converging to x. □

2. The weak* topology

On the dual space X^*, the family of seminorms

$$\mathcal{P}^* = \Big\{p_x : x \in X\Big\} \tag{2.1}$$

is separating by definition, hence generating a topology which makes X^* a locally convex topological vector space. We denote this topology by $\sigma(X^*, X)$.

We have the "dual" result of Lemma 1.2:

Lemma 2.1. *The weak topology $\sigma(X^*, X)$ is the weakest topology such that each map*

$$x : X^* \mapsto \mathbb{C}, \quad x \in X, \tag{2.2}$$

is continuous.

A sequence ℓ_n converges to ℓ in $\sigma(X^*, X)$ if and only if for all $x \in X$

$$\lim_{n \to \infty} \ell_n x = \ell x. \tag{2.3}$$

A set $E \subset X^*$ is bounded w.r.t. $\sigma(X^*, X)$ if and only if

$$\big\{\ell x, \ell \in E\big\}$$

is bounded in $\mathbb{C}$.

If X^* is infinite dimensional, then each element of a local base $\mathcal{B}$ contains an infinite dimensional subspace, hence $\sigma(X^*, X)$ is not locally bounded.

A priori, one can look at the second dual Y of the locally convex topological vector space $(X, \sigma(X^*, X))$, i.e.

$$Y = \Big\{\lambda : X^* \mapsto \mathbb{C}, \ell \text{ continuous w.r.t. } \sigma(X^*, X)\Big\}. \tag{2.4}$$

By construction, it follows that

$$X \subset Y,$$

i.e. X can be embedded into Y.

It turns out that $X = Y$, i.e. the dual of $(X^*, \sigma(X^*, X))$ can be identified with X.

Theorem 2.2. *If $\lambda : X^* \mapsto \mathbb{C}$ is linear and continuous w.r.t. $\sigma(X^*, X)$, then there exists $x \in X$ such that*

$$\lambda(\ell) = \ell x \quad \forall \ell \in X^*. \tag{2.5}$$

Proof. By definition of continuity w.r.t. $\sigma(X^*, X)$, for all $\epsilon > 0$ there are $\delta > 0$ and $x_1, \dots, x_n$ such that

$$\lambda\Big\{\ell : |\ell x_i| \leq \delta, i = 1, \dots, n\Big\} \subset (-\epsilon, \epsilon).$$

In particular, if ℓ is such that $\ell x_i = 0$ for all i, then $\lambda\ell = 0$. This show that

$$N_\lambda \supset \bigcap_{i=1}^{n} N_{x_i}.$$

Consider the linear mapping $T : X^* \mapsto \mathbb{C}^{n+1}$ defined by

$$T(\ell) = \Big(\begin{array}{cccc} \lambda\ell & \ell x_1 & \dots & \ell x_n \end{array} \Big).$$

By the assumption, $T(X^*)$ is a subspace of $\mathbb{C}^{n+1}$ and the point $(1, 0, \dots, 0)$ is not in $T(X^*)$. Then there are $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{C}^{n+1}$ such that

$$\alpha \cdot T(X^*) = \Big\{\alpha_1 \lambda\ell + \sum_{i=2}^{n+1} \alpha_i \ell x_{i-1}, \ell \in X^*\Big\} = 0 < \Re\alpha_1.$$

It follows that $\alpha_1 \neq 0$ and

$$\lambda\ell = \sum_{i=1}^{n} \frac{\alpha_{i+1}}{\alpha_1} \ell x_i.$$

□

If X is in particular a normed space, then we know that X^* is a Banach space. Hence, if τ is the vector topology of X^* generated by the norm $\|\cdot\|_{X^*}$, $\sigma(X^*, X) \subset \tau$. We thus say that

- the sequence ℓ_n *converges strongly* to ℓ and we write

(2.6) $$\ell_n \to \ell \quad \text{if} \quad \|\ell_n - \ell\|_{X^*} \to 0.$$

- the sequence ℓ_n *converges weakly star* to ℓ and we write

(2.7) $$\ell_n \rightharpoonup^* \ell \quad \text{if } \ell_n \text{ converges to } \ell \text{ in the topology } \sigma(X^*, X).$$

Similarly we will speak about strong neighborhood, strongly closed, strongly bounded... in $(X^*, \|\cdot\|_{X^*})$, and weak* neighborhood, weakly* closed, weakly* bounded in $(X^*, \sigma(X^*, X))$....

We observe also that we have the weak topology $\sigma(X^*, X^{**})$.

It follows from Uniform boundedness principle that

Proposition 2.3. *If X is a Banach space, then*

(2.8) $$\ell_n \rightharpoonup^* \ell \quad \Longrightarrow \quad \|\ell_n\| \text{ bounded.}$$

Moreover

(2.9) $$\|\ell\| \leq \liminf_{n\to\infty} \|\ell_n\|.$$

Finally, if $E \subset X^*$ is bounded w.r.t. $\sigma(X^*, X)$, then E is strongly bounded.

3. The Banach-Alaoglu theorem

Let V be a neighborhoof of $0 \in X$. Define the polar of V as

(3.1) $$K = \Big\{\ell \in X^* : |\ell x| \leq 1 \ \forall x \in V\Big\}.$$

We have the following fundamental result:

Theorem 3.1 (Banach-Alaoglu)**.** *The polar K of any neighborhood V of 0 is compact in the weak* topology $\sigma(X^*, X)$.*

Proof. Since each V local neighborhood is absorbing, then there is a $\gamma(x) \in \mathbb{C}$ such that

$$x \in \gamma(x)V.$$

Hence it follows that

$$|\ell x| \leq \gamma(x) \quad x \in X, \ell \in K.$$

Consider the topological space

$$P = \Pi_{x\in X}\big\{\alpha \in \mathbb{C}\colon |\alpha| \leq \gamma(x)\big\}, \tag{3.2}$$

with the product topology σ. By Tychonoff's theorem (P, σ) is compact.

By construction, the elements of P are functions $f : X \mapsto \mathbb{C}$ (not necessarily linear) such that

$$|f(x)| \leq \gamma(x).$$

In particular, the set K is the subset of P made of the linear functions.

We first show that K is a closed subset of P w.r.t. the topology σ. This follows from the fact that if f_0 is in the σ closure $\bar{K}$ of K, then for scalars α, β and point $x, y \in X$ one has that

$$\Big\{|f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon, |f(x) - f_0(x)| < \epsilon, |f(y) - f_0(y)| < \epsilon\Big\} \cap K \neq \emptyset.$$

Take thus ℓ in the intersection, so that

$$\begin{aligned}\Big|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)\Big| &= \Big|\Big(f_0(\alpha x + \beta y) - f(\alpha x + \beta y)\Big) + \alpha\big(f(x) - f_0(x)\big) + \big(f(y) - f_0(y)\big)\Big| \\ &< (1 + |\alpha| + |\beta|)\epsilon.\end{aligned}$$

Since ϵ is arbitrary, f_0 is linear. Moreover, since $|f_0(x)| \leq \gamma(x)$, then for $x \in V$

$$|f_0(x)| \leq 1.$$

It follows that we have two topologies on K:

- the weak* topology $\sigma(X^*, X)$ inherited by X^*;
- the product topology σ inherited by P. Since K is closed in (P, σ), then (K, σ) is compact.

To conclude, we need only to show that the two topologies coincide. This follows because the bases of the two topologies are generated by the sets

$$V_{\sigma(X^*,X)} = \Big\{\big|\ell x_i - \ell_0 x_i\big| < \epsilon, i = 1, \dots, n\Big\},$$

$$V_\sigma = \Big\{\big|f(x_i) - f_0(x_i)\big| < \epsilon, i = 1, \dots, n\Big\}.$$

There is thus a one to one correspondence among local bases, hence the two topologies coincide. □

The application of Theorem 3.1 to normed space gives

Corollary 3.2. *If X is a normed space, then the unit ball in X^**

$$\Big\{\ell \in X^*, \|\ell\|_{X^*} \leq 1\Big\}$$

is compact in the weak topology $\sigma(X^*, X)$.*

We recall that in general in a compact sets there are sequences without converging subsequences. It is thus important the next result.

We say that X is *separable* if there is a countable dense subset of X.

Theorem 3.3. *If X is separable, and $K \subset X^*$ is weakly* compact, then K is metrizable in the weak* topology.*

As a consequence, each sequence in $K \subset X^*$ has a converging subsequence.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of X, and define the distance

$$d(\ell,\ell') = \sum_{n=1}^{+\infty} 2^{-n} \min\big\{1, |\ell x_n - \ell' x_n|\big\}, \quad \ell,\ell' \in X^*. \tag{3.3}$$

One can check that since $\ell,\ell' \in X^*$, this is actually a distance.

We first show that the d-topology is contained in the $\sigma(X^*,X)$ topology. In fact, if we consider the neighborhood

$$V = \Big\{|(\ell-\ell')x_n| < \epsilon, n = 1,\dots,k\Big\},$$

then

$$d(\ell,\ell') \leq \epsilon + 2^{-k+1}.$$

Since ϵ is arbitrarily small and k arbitrarily large, then the topology generated by d is weaker than $\sigma(X^*,X)$.

To prove the opposite, we need to use that K is weak* compact. We first show some uniform continuity of $\ell \in K$: if $y_i \in X$ converges to 0, then for all ϵ there is N such that

$$|\ell y_i| < \epsilon, \quad \forall i > N,\ \ell \in K.$$

In fact, if this does not hold, then the family of open sets

$$V_i = \Big\{\ell : |\ell y_i| < \epsilon\Big\}$$

covers X^*, hence covers K, but it is not possible to extract a finite covering.

As a consequence, we have that if $\ell_i \to \ell$ in the d topology, i.e.

$$\ell_i x_n \to \ell x_n \quad \forall x_n,$$

then

$$\big|\ell_i x - \ell_j x\big| \leq \big|\ell_i(x-x_n)\big| + \big|\ell_j(x-x_n)\big| + \big|\ell_i x_n - \ell_j x_n\big| \leq \epsilon,$$

if we choose x_n sufficiently close to x (use uniform continuity) and then $i,j \gg 1$ (use d convergence). Thus $\tilde{\ell}$ can be extended by continuity to all $x \in X$, and from the definition of weak convergence it follows $\tilde{\ell} \in K$. Hence K is d closed. Since the d-topology is weaker than the $\sigma(X^*,X)$ topology, K is d-compact.

From the form of the local base of $\sigma(X^*,X)$, it follows that we have to prove that for all $x \in X$ fixed

$$\big\{\ell' \in K : d(\ell',\ell) < r\big\} \subset K \cap \big\{\big|(\ell'-\ell)x\big| < \epsilon\big\}.$$

if $r > 0$ is sufficiently small. We prove it by contradiction.

If such an $r > 0$ does not exists, then for all $m \in \mathbb{N}$ there is $\ell_m \in K$ such that

$$\big|(\ell_m - \ell)x_n\big| < \frac{1}{m}, \quad n = 1,\dots,m,$$

but

$$\big|(\ell_m - \ell)x\big| \geq \epsilon.$$

It follows that ℓ_m has a converging subsequence in the d-topology ((K,d) is a metric space), so that we can assume that there is $\{\ell_i\}_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \ell_i x_n = \ell x_n \quad \forall x_n.$$

Since ℓ is continuous and in K, we can write

$$\big|\ell x - \ell_m x\big| \leq \big|\ell x - \ell x_n\big| + \big|\ell_m x - \ell_m x_n\big| + \big|\ell x_n - \ell_m x_n\big| < 3\epsilon',$$

with ϵ' arbitrary (use uniform continuity), and we have a contradiction. □

We remark that $(X^*,\sigma(X^*,X))$ is never metrizable, unless X has a countable vector base.

4. The Krein-Milman theorem

If K is convex set in a linear space X, we recall that E is an extreme subset of K if E is convex and

$$x, y \in K \text{ such that } \frac{x+y}{2} \in E \quad \Longrightarrow \quad x, y \in E.$$

In particular, an extreme point is an extreme set consisting of only one point.

In $\mathbb{R}^n$ we have the following theorem:

Theorem 4.1 (Carathéodory). *Every compact subset K of $\mathbb{R}^n$ has extreme points, and every point of K can be written as a convex combination of $n+1$ extreme points.*

For locally convex topological vector spaces on $\mathbb{R}$, one can prove a generalization of the above result:

Theorem 4.2 (Krein-Milman). *If $K \subset X$ is convex and compact, X locally convex topological vector space over $\mathbb{R}$, then*

1. *K has at least one extreme point;*
2. *K is the closure of the convex hull of its extreme points,*

$$K = \overline{\bigg\{ \sum_{\textit{finite}} t_i x_{\alpha_i} : t_i \geq 0, \sum_i t_i = 1, x_{\alpha_i} \text{ extreme point of } K \bigg\}}. \tag{4.1}$$

Proof. We first prove that every non empty closed extreme subset of K has an extreme point.

In fact, fixed F, let $\{F_\alpha\}_{\alpha \in A}$ be the set of closed nonempty extreme subset of F. We partially order it by inclusion

$$F_\alpha \leq F_{\alpha'} \quad \text{if} \quad F_\alpha \supset F_{\alpha'}.$$

Since F is a closed subset of K, then it is compact, so that the finite intersection property holds:

$$\bigcap_\beta F_{\alpha_\beta} = \emptyset \quad \Longrightarrow \quad \exists (\alpha_1, \dots, \alpha_n) \text{ finite: } \bigcap_i F_{\alpha_i} = \emptyset.$$

In particular, it follows that every sequence of totally ordered F_α has a upper bound, which is the intersection of all F_α. This is not empty because of the finite intersection property, and one can check that it is an extreme subset of F.

By Zorn's lemma there is at least a maximal element $\bar{F}$.

Assume that $\bar{F}$ consists of more that 1 element. Since X is locally convex, its dual separates points, so that there is ℓ such that

$$\min\big\{\ell x, x \in F\big\} < \max\big\{\ell x, x \in F\big\} = \gamma.$$

Thus the set $\ell^{-1}(\gamma) \cap \bar{F}$ is an extremal subset of $\bar{F}$ and it is strictly contained in $\bar{F}$. Hence $\bar{F}$ consists of a single point.

We now prove that the convex hull of the extremal points of K (which is not empty because of the first part of the proof) is dense in K. We assume by contradiction that there is $x \in K$ such that

$$x \notin \overline{\bigg\{ \sum_{\text{finite}} t_i x_{\alpha_i} : t_i \geq 0, \sum_i t_i = 1, x_{\alpha_i} \text{ extreme point of } K \bigg\}} = \overline{K_e}.$$

By the Hahn-Banach theorem there is thus a linear continuous functional which separates x and K_e strictly, i.e.

$$\ell x \leq \gamma_1 < \gamma_2 = \min\big\{\ell y : y \in K_e\big\}.$$

Then the set

$$\Big\{x \in K : \ell x = \min\big\{\ell x : x \in K\big\}\Big\}$$

is a closed non empty extreme subset of K. By the previous part it has an extreme point, but this yields a contradiction. □

5. Dual of Banach spaces and reflexive spaces

A particular case is when X is normed: in this case X^* is a Banach space with norm

$$\|\ell\|_{X^*} = \sup_{\|x\|=1} |\ell x|. \tag{5.1}$$

One can introduce the *second dual* of X, i.e. the dual of X^*, denoted by X^{**}.

Clearly, there is a canonical immersion $\mathbf{J}$ of X into X^{**}, defined by

$$\mathbf{J} : X \mapsto X^{**}, \quad (\mathbf{J}x)\ell = \ell x, \ \|\mathbf{J}x\|_{X^{**}} = \|x\|_X. \tag{5.2}$$

Since $\mathbf{J} : X \mapsto X^{**}$ is continuous, it follows that $\mathbf{J}(X)$ is a closed subspace of X^{**}. In particular, either $\mathbf{J}(X) = X^{**}$ or it is not dense.

For the weak* topology $\sigma(X^{**}, X^*)$ we have that the image of the unitary ball

$$\mathbf{J}(\overline{B_X(0,1)}) = \mathbf{J}\big\{x : \|x\|_{\leq} 1\big\}$$

is dense in $\overline{B_{X^{**}}(0,1)}$. We will use the following lemma:

Lemma 5.1 (Helly)**.** *Let X be a Banach space, $\ell_i \in X^*$, $i = 1, \dots, n$, n linear functionals in X^*, and $\alpha_i \in \mathbb{C}$, $i = 1, \dots, n$, n scalars. Then the following properties are equivalent:*

(1) *for all $\epsilon > 0$ there is x_ϵ, $\|x_\epsilon\| < 1$ such that*

$$\big|\ell_i x_\epsilon - \alpha_i\big| \leq \epsilon \quad i = 1, \dots, n; \tag{5.3}$$

(2) *for all $\beta_1, \dots, \beta_n \in \mathbb{C}$*

$$\bigg|\sum_{i=1}^n \beta_i \alpha_i\bigg| \leq \bigg\|\sum_{i=1}^n \beta_i \ell_i\bigg\|_{X^*}. \tag{5.4}$$

Proof. The first implication follows by

$$\begin{aligned}\bigg|\sum_{i=1}^n \beta_i \alpha_i\bigg| &= \bigg|\sum_{i=1}^n \beta_i \big(\alpha_i - \ell_i x_\epsilon\big)\bigg| + \bigg|\sum_{i=1}^n \beta_i \ell_i x_\epsilon\bigg| \\ &\leq \epsilon \sum_{i=1}^n |\beta_i| + \bigg\|\sum_{i=1}^n \beta_i \ell_i\bigg\|_{X^*},\end{aligned}$$

since $\|x_\epsilon\|_X \leq 1$.

Conversely, if (1) does not hold, then this means that the closure of the set

$$(\ell_1, \dots, \ell_n)\big\{x : \|x\| \leq 1\big\} \in \mathbb{C}^n$$

does not contains $(\alpha_1, \dots, \alpha_n)$. Thus there is $(\beta_1, \dots, \beta_n) \in \mathbb{C}^n$ such that

$$\max \Re\bigg\{\sum_{i=1}^n \beta_i \ell_i x, \|x\| \leq 1\bigg\} < \Re\bigg\{\sum_{i=1}^n \beta_i \alpha_i\bigg\} \leq \bigg|\sum_{i=1}^n \beta_i \alpha_i\bigg|.$$

Since $\{\|x\| \leq 1\}$ is balanced, it follows that (2) is false. □

We can now prove

Proposition 5.2 (Goldstine)**.** *If X is a Banach space, then $J(\overline{B_X(0,1)})$ is dense in $\overline{B_{X^{**}}(0,1)}$ for the weak* topology.*

Proof. If $\xi \in X^{**}$, take a neighborhood of the form

$$V = \Big\{\eta \in X^* \colon \big|(\eta - \xi)\ell_i\big| < \epsilon, \ell_i \in X^*, i = 1, \dots, n\Big\}.$$

We need only to find $x \in X$ such that

$$\big|\ell_i x - \xi \ell_i\big| < \epsilon.$$

Since $\|\xi\|_{X^{**}} \leq 1$, then

$$\bigg|\sum_{i=1}^n \beta_i \big(\xi \ell_i\big)\bigg| \leq \bigg\|\sum_{i=1}^n \beta_i \ell_i\bigg\|_{X^*},$$

so that for Lemma 5.1 it follows that there is an $x_\epsilon \in X$ which belongs to V. □

A Banach space X is *reflexive* if $J(X) = X^{**}$.

It is important to observe that in the previous definition the canonical immersion $\mathbf{J}$ is used: even for particular non reflexive spaces one can find continuous linear surjection form X into X^{**}.

We prove the following main result:

Theorem 5.3 (Kakutani)**.** *The Banach space X is reflexive if and only if*

$$\overline{B_X(0,1)} = \Big\{ x \in X : \|x\|_X \leq 1 \Big\}$$

is compact for the weak topology $\sigma(X, X^*)$.

Proof. If X is reflexive, then $\mathbf{J} : X \mapsto X^{**}$ is continuous, injective and surjective. Hence $\mathbf{J}^{-1}$ is linear and continuous w.r.t. the strong topologies of X and X^{**}. Actually both $\mathbf{J}$, $\mathbf{J}^{-1}$ are isometries.

It is clear that

$$\mathbf{J}\Big\{ x : |\ell x| < \epsilon \Big\} = \Big\{ \eta : |\eta \ell| < \epsilon \Big\},$$

so that the topology $\mathbf{J}^{-1}(\sigma(X^{**}, X^*))$ coincides with the topology $\sigma(X, X^*)$. Since $\overline{B_{X^{**}}(0,1)}$ is weak* compact, so is $\overline{B_X(0,1)}$.

Conversely, if $\overline{B_X(0,1)}$ is compact, then $\mathbf{J}(\overline{B_X(0,1)})$ is closed, and by Proposition 5.2 it coincide with the whole $\overline{B_{X^{**}}(0,1)}$. □

In general, if X is Banach separable, the dual space X^* is not. However the converse is true.

Theorem 5.4. *If X is a Banach space and X^* is separable, then X is separable.*

Proof. Let $\{\ell_n\}_{n \in \mathbb{N}}$ be a dense countable set in X^*. Let $x_n \in X$, $\|x_n\|_X \leq 1$, be a point where

$$|\ell_n x_n| \geq \frac{1}{2} \|\ell_n\|_{X^*},$$

and consider the countable set

$$Q = \bigg\{ \sum_{\text{finite}} \alpha_i x_i : \alpha_i \text{ belongs to a countable dense subset of } \mathbb{C} \bigg\}.$$

Clearly Q is countable and dense in the vector space L generated by $\{x_n\}_{n \in \mathbb{N}}$, so that it remains to prove that L is dense in X.

If L is not dense, then there is a non null continuous functional $\bar{\ell}$ such that

$$\bar{\ell} \neq 0 \quad \bar{\ell} x_n = 0 \ \forall n \in \mathbb{N}.$$

Since ℓ_n is dense, there is $\bar{n}$ such that $\|\bar{\ell} - \ell_{\bar{n}}\| < \epsilon$, so that

$$\frac{1}{2} \|\ell_{\bar{n}}\|_{X^*} \leq |\ell_{\bar{n}} x_{\bar{n}}| \leq \big| (\bar{\ell} - \ell_{\bar{n}}) x_{\bar{n}} \big| + |\bar{\ell} x_{\bar{n}}| \leq \epsilon.$$

Thus $\|\ell_{\bar{n}}\| \leq 2\epsilon$, which implies that $\bar{\ell} = 0$. □

We next prove that

Proposition 5.5. *If $M \subset X$ is a closed subspace of a reflexive space, then M is reflexive.*

Proof. The proof follows by proving that the topology $\sigma(M, M^*)$ coincide with the topology $M \cap \sigma(X, X^*)$, and $\overline{B_M(0,1)}$ is closed for $\sigma(X, X^*)$ (closed for strong topology and convex). □

As a corollary, we have that

$$X \text{ separable and reflexive} \quad \Longleftrightarrow \quad X^* \text{ separable and reflexive.}$$

Proof. Clearly if X is reflexive, the unit ball $\overline{B_{X^*}(0,1)}$ is compact for the topology $\sigma(X^*, X^{**})$ because of Banach-Alaoglu theorem 3.1 and the fact that $\sigma(X^*, X^{**}) = \sigma(X^*, X)$. Moreover if X is reflexive and separable, then X^{**} is separable, hence by Theorem 5.4 X^* is separable.

Conversely, if X^* is reflexive, then X^{**} is reflexive, so that $\mathbf{M}(X)$ is reflexive by Proposition 5.5, hence X is reflexive. Moreover, we know from Theorem 5.4 that X is separable, if X^* is separable. □

For these spaces,

$$\{x_n\}_{n \in \mathbb{N}} \text{ bounded} \quad \Longrightarrow \quad \exists \{x_{n_i}\}_{i \in \mathbb{N}} \text{ convergent},$$

with $x_n \in X$ or X^*.

5.1. **Uniformly convex Banach spaces.** We say that X Banach space is *uniformly convex* if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x\|_X, \|x'\|_X \leq 1, \left\|\frac{x+y}{2}\right\| \geq 1-\delta \quad \Longrightarrow \quad \|x-x'\| < \epsilon. \tag{5.5}$$

For these spaces we have:

Theorem 5.6 (Milman)**.** *If X is a uniformly convex Banach space, then X is reflexive.*

Proof. Let $\xi \in X^{**}$, $\|\xi\|_{X^{**}} = 1$. We want to prove that for all $\epsilon > 0$ there is $x \in X$, $\|x\| \leq 1$ such that

$$\big\|\xi - \mathbf{J}x\big\|_{X^{**}} < \epsilon.$$

Since $\mathbf{J}(X)$ is strongly closed ($\mathbf{J}$ is an isometry), then $\mathbf{J}$ is surjective.

Let $\ell \in X^*$ be such that

$$\|\ell\|_{X^*} = 1, \quad \xi\ell > 1-\delta,$$

where δ is the constant chosen by the uniform convexity estimate corresponding to ϵ, and consider the neighborhood of ξ of the form

$$V = \Big\{\eta \in X^{**} : \big|(\xi-\eta)\ell\big| < \delta/2\Big\}.$$

By Proposition 5.2, it follows that there is some $x \in B_X$ such that $\mathbf{J}x \in V$.

Assume that $\xi \notin \mathbf{J}x + \epsilon B_{X^{**}}$. Then we obtain a new neighborhood of ξ for the weak* topology which does not contains x. With the same procedure, we can find a new $\hat{x}$ in this new neighborhood.

Thus we have

$$\big|\ell x - \xi\ell\big| \leq \frac{\delta}{2}, \quad \big|\ell\hat{x} - \xi\ell\big| \leq \frac{\delta}{2}.$$

Adding we obtain

$$2|\xi\ell| \leq |\ell(x+\hat{x})| + \delta \leq \|x+\hat{x}\| + \delta.$$

Then $\|(x+\hat{x})/2\| \geq (1-\delta)$, so that $\|x+\hat{x}\| < \epsilon$, which is a contradiction. □

6. Exercises

(1) Let $E \subset X$, X Banach space, be a compact subset for the weak topology $\sigma(X, X^*)$. Show that E is bounded for the strong topology.
(2) Let u, v be two continuous functions from T topological space into X Banach space with the weak topology. Prove that
   - the map $u+v$ is continuous;
   - if $a : T \mapsto \mathbb{C}$ is continuous, then $a(x)u(x)$ is continuous.
(3) Prove that the topology $\sigma(X, X^*)$ is not metrizable if X is an infinite dimensional normed space (show that $\sigma(X, X^*)$ does not have any countable local base, because X^* is metric complete and thus its vector base is not countable).
(4) Let X be a Banach space, $M \subset X$ be a subspace and $\ell_0 \in X^*$. Show that there is m_0,

$$m_0 \in M^\perp = \Big\{\ell \in X^* \colon \ell x = 0 \ \forall x \in M\Big\}$$

such that

$$\inf_{\ell \in M^\perp} \|\ell_0 - \ell\|_{X^*} = \|\ell_0 - m_0\|.$$

(5) Prove that if x_n converges to x strongly in X Banach, ℓ_n converges to ℓ weakly in X^*, then $\ell_n x_n \to \ell x$. Show with an example that if ℓ_n converges only weakly, then $\ell_n e_n$ may not coincide.
(6) Show that in finite dimension the weak topology end the norm topology coincide.
(7) If X is a reflexive separable space, show that there are sequences converging weakly but not converging strongly (use the result that if $M \subsetneq X$ is a closed subspace, then there is a point x, $\|x\| = 1$ and $\|x-y\| \geq 1-\delta$ for all $\delta > 0$ and $y \in M$).
(8) Let X be a Banach space, and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X.
   - Assume that $x_n \rightharpoonup x$, then show that

$$\bigcap_{n=1}^{+\infty} \overline{\mathrm{conv}\big\{x_n, x_{n+1}, \dots\big\}} = \{x\}.$$

- Assume X reflexive, and $\{x_n\}_{n\in\mathbb{N}}$ bounded. Prove that if

$$\bigcap_{n=1}^{+\infty} \overline{\mathrm{conv}\{x_n, x_{n+1}, \dots\}} = \{x\},$$

  then $x_n \rightharpoonup x$.

(9) Let X be a Banach space.
- Let $\{\ell_n\}_{n\in\mathbb{N}}$ be a sequence in X^*, and assume that $\ell_n x$ has limit for all $x \in X$. Prove that there is a weak* limit ℓ: $\ell_n \rightharpoonup^* \ell$.
- Assume X reflexive, $\{x_n\}_{n\in\mathbb{N}}$ sequence in X such that for all $\ell \in X^*$ the sequence ℓx_n has limit. Show that x_n converges weakly to some $x \in X$.
- Consider c_0 and the sequence

$$u_n = \Big(\underbrace{1, \dots, 1}_{n \text{ numbers}}, 0, \dots \Big).$$

  Knowing that the dual space of c_0 is ℓ^1, show that the conclusion of the second point is false. Hence c_0 is not reflexive.

(10) Consider the space ℓ^1 with norm

$$\|u\|_{\ell^1} = \sum_{n=1}^{\infty} |u(n)|.$$

- Prove the the dual space of ℓ^1 is ℓ^∞.
- Show that ℓ^1 is separable, but ℓ^∞ is not.
- Construct in ℓ^∞ a bounded sequence which does not have any weakly convergent subsequence.

(11) Let X be a uniformly convex Banach space, and $E \subset X$ a convex closed subset.
- Prove that for all $x \in X$ there exists a unique point x_0 such that

$$\|x - x_0\| = \inf_{y\in E} \|x - y\|.$$

  Denote this projection by $P_E x$.
- Show that $P_E x$ is strongly continuous.

(12) Let $\mathbf{M} : X \to Y$ be a strongly continuous operator, X, Y Banach. Show that $\mathbf{M}$ is also continuous from the weak topology of X into the weak topology of Y, and that the opposite holds.

(13) Let C be a closed subset of $\mathbb{R}$, and consider the subset of ℓ^2 defined by

$$X_C = \bigg\{ u : \mathbb{N} \mapsto \mathbb{R}, \|u\|_{\ell^2} = \bigg(\sum_{n=1}^{\infty} |u(n)|^2 \bigg)^{1/2}, u(n) \in C \bigg\}.$$

- Show that X_C is strongly closed.
- Show that if C is convex, then it is also weakly closed.
- Find an example of non weakly closed X_C.

(14) Consider the space $X = C([0,1];\mathbb{R})$ with the sup norm, and the sets

$$\overline{B(0,1)} = \Big\{ u : [0,1] \mapsto \mathbb{R}, \|u\| \le 1 \Big\}, \quad B^1(0,1) = \Big\{ u : [0,1] \mapsto \mathbb{R}, \|u\| \le 1, \|u'\| \le 1 \Big\}.$$

- Is $B^1(0,1)$ a closed subset of $\overline{B(0,1)}$?
- Has $B^1(0,1)$ compact closure in $\overline{B(0,1)}$?
- Is $\overline{B(0,1)}$ closed and compact in X?

(15) Consider the normed space

$$X = \Big\{ u \in C^1([-1,1];\mathbb{R}), \|u\| = \sup_t |u(t)| \Big\},$$

and the family of linear functionals

$$\ell_n u = \frac{1}{n^2} \int_{-1/n}^{1/n} \mathrm{sgn}(t) u(t) dt.$$

- Show that for all $u \in X$ the set $\{\ell_n u\}_{n\in\mathbb{N}}$ is bounded.

- Prove that however ℓ_n is not bounded, i.e. there are $u_n \in X$, $\|u_n\| = 1$ and
$$T_n u_n \to \infty.$$
- Show that
$$K = \bigcap_{n=1}^{\infty} \ell_n^{-1}(-1, 1)$$
is convex, balanced and absorbing, but it has empty interior.