BANACH STEINHAUS, OPEN MAPPING AND CLOSED GRAPH THEOREMS

In this lecture we study which consequences follows from the completeness of a metrizable vector space. The basic tool is Baire's lemma:

Lemma 0.1. *If X is a complete metric space, X_n closed sets with empty interior, then $\cup_n X_n$ has empty interior.*

1. The Banach Steinhaus theorem

If X, Y are topological vector spaces, $\{\mathbf{M}_\nu\}_\nu$ a family of linear operators,

$$\mathbf{M}_\nu : X \mapsto Y.$$

We say that $\{\mathbf{M}_\nu\}_\nu$ is *equicontinuous* if

$$\forall W \subset Y \text{ neighborhood of } 0 \ \exists V \subset X \text{ neighborhood of } 0 \in X \ : \mathbf{M}_\nu V \subset W \ \forall \nu. \tag{1.1}$$

We recall that if X is metrizable, then $\mathbf{M} : X \mapsto Y$ bounded is equivalent to $\mathbf{M}$ continuous.

Theorem 1.1. *If the family $\{\mathbf{M}_\nu\}_\nu$ from X to Y is equicontinuous then it is equibounded, i.e. for all $E \subset X$ bounded there is $F \subset Y$ bounded such that*

$$\mathbf{M}_\nu E \subset F \quad \forall \nu. \tag{1.2}$$

Proof. For $W \subset Y$ open neighborhood, let V as in the assumptions of equicontinuity. If E is bounded, there is t such that $E \subset tV$. Hence

$$\mathbf{M}_\nu E \subset \mathbf{M}_\nu (tV) \subset tW,$$

i.e. $\mathbf{M}_\nu E$ is bounded by tW independently on ν. Thus

$$F = \bigcup_\nu \mathbf{M}_\nu E$$

is bounded. □

In particular the orbits

$$\Gamma(x) = \big\{\mathbf{M}_\nu x\big\}_\nu \tag{1.3}$$

are bounded sets in Y for all $x \in X$ if $\{\mathbf{M}_\nu\}_\nu$ is equicontinuous. If X, Y are normed spaces, then

$$\{\mathbf{M}_\nu\}_\nu \text{ equibounded} \quad \Longleftrightarrow \quad \exists C \colon \|\mathbf{M}_\nu x\|_Y \leq C \|x\|_X \ \forall x \in X. \tag{1.4}$$

If X is a complete metrizable topological vector space, i.e. an F-space, then the boundedness of the sets $\Gamma(x)$ implies that the family $\{\mathbf{M}_\nu\}_\nu$ is equicontinuous, hence equibounded. This is the Banach Steinhaus theorem, also known as the *uniform boundedness principle*: in fact from a local boundedness we recover a uniform estimate.

Theorem 1.2 (Banach-Steinhaus)**.** *If $\{\mathbf{M}_\nu\}_\nu$ is a family of continuous linear operator from the F-space X to the topological vector space Y such that*

$$\Gamma(x) = \big\{\mathbf{M}_\nu x\big\}_\nu \tag{1.5}$$

is bounded in Y for all $x \in X$, then $\{\mathbf{M}_\nu\}_\nu$ is equicontinuous, hence equibounded.

If X, Y are normed spaces, and X Banach, then the condition is that

$$\forall x \in X \ \exists C_x \colon \|\mathbf{M}_\nu x\|_Y \leq C_x \|x\| \ \forall \nu. \tag{1.6}$$

while the condition (1.4) is that C_x can be chosen independently on $x \in X$: the family $\{\mathbf{M}_\nu\}_\nu$ is equicontinuous.

Proof. The proof is based on the principle of uniform boundedness in complete metric spaces.

Take balanced neighborhoods W, W_1 of 0 in Y such that $\bar{W}_1 + \bar{W}_1 \subset W$, and define the sets

$$A_n = \Big\{x \in X : \mathbf{M}_\nu x \in n\bar{W}_1 \ \forall \nu\Big\} = \bigcap_\nu n\mathbf{M}_\nu^{-1}(\bar{W}_1).$$

By construction the sets A_n are closed, and due to pointwise boundedness

$$\bigcup_{n\in\mathbb{N}} A_n = X.$$

Hence, since X is metric complete, some $A_{\bar{n}}$ has not empty interior, i.e. there is $V \subset X$ open neighborhood of 0 and $\bar{x} \in X$ such that

$$\bar{x} + V \subset A_{\bar{n}}, \quad \mathbf{M}_\nu V \subset \bar{n}\bar{W}_1 - \mathbf{M}_\nu \bar{x} \subset \bar{n}(\bar{W}_1 + \bar{W}_1) \subset \bar{n}W \quad \forall \nu.$$

It follows that $\mathbf{M}_\nu(V/\bar{n}) \subset W$. □

Let X, Y be two normed spaces, and $\mathcal{L}(X,Y)$ be the vector space of the continuous linear operator $\mathbf{M} : X \mapsto Y$. We can make it into a normed space by defining

$$\|\mathbf{M}\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X \le 1} \|Tx\|_Y. \tag{1.7}$$

This is the strong topology. One can check that if Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space even if $X$is not. In particular the dual space X^* is Banach.

The topology of pointwise convergence on $\mathcal{L}(X,Y)$ is the weakest topology such that all applications

$$x \mapsto \mathbf{M}x \tag{1.8}$$

are continuous.

If X is Banach, then as a corollary of the uniform boundedness principle we have

Corollary 1.3. *If $\{\mathbf{M}_n\}_{n\in\mathbb{N}}$ is a family of continuous operators for X Banach to Y normed, such that $\mathbf{M}_n x$ converges to $\mathbf{M}x$ for all $x \in X$, then $\mathbf{M}$ is a linear bounded operator and*

$$\|\mathbf{M}\|_{\mathcal{L}(X,Y)} \le \liminf_{n\to\infty} \|\mathbf{M}_n\|_{\mathcal{L}(X,Y)}. \tag{1.9}$$

Another important consequence is that

Corollary 1.4. *Let X be a normed space, and $B \subset X$ be such that*

$$\forall \ell \in X^* \quad \ell B \subset \mathbb{C} \text{ is bounded.} \tag{1.10}$$

Then B is bounded.

Conversely, if X is Banach and $B' \subset X^*$ is such that

$$\forall x \in X \quad \big\{\ell x, \ell \in B'\big\} \subset \mathbb{C} \text{ is bounded,} \tag{1.11}$$

then B' is bounded.

Proof. Since every application

$$\Big\{T_x : X^* \mapsto \mathbb{C} \colon T_x(\ell) = \ell x, x \in B\Big\} \subset \mathcal{L}(X^*, \mathbb{C})$$

is pointwise bounded and X^* is Banach, then there is C such that

$$|\ell x| \le C\|\ell\|_{X^*} \quad \forall x \in B.$$

By the Hahn-Banach theorem we have that for all $x \in X$ there is a linear functional ℓ such that

$$\ell x = \|x\|,$$

so that we conclude $\|x\| \le C$ for all $x \in B$.

For the second part, we use the family

$$\Big\{T_\ell : X \mapsto \mathbb{C} \colon T_\ell(x) = \ell x, \ell \in B'\Big\} \subset \mathcal{L}(X, \mathbb{C}).$$

As before there is some constant C such that

$$|\ell x| \le C\|x\|_X \quad \forall \ell \in B'.$$

By definition of norm $\|\cdot\|_{X^*}$, it follows that

$$\sup_{\ell\in B'}\|\ell\|_{X^*}\le C.$$

□

2. The open mapping theorem

Suppose $f : X \mapsto Y$, X, Y topological spaces.

We say that f is *open* at $x \in X$ if $f(V)$ contains an open neighborhood of $f(x) \in Y$ for all V neighborhood of x. The map f is an open mapping if it is open at each $x \in X$, i.e. $f(V)$ is open for all V open.

For linear mapping $\mathbf{M} : X \mapsto Y$, X, Y F-spaces, we have that $\mathbf{M}$ is open if and only if $\mathbf{M}(X) = Y$.

The "only if" part follows form the observation that if a linear subspace M of Y contains an open set, then $M = Y$. The other part is stated in the next theorem:

Theorem 2.1 (open mapping)**.** *Let $\mathbf{M} : X \mapsto Y$ be a linear continuous operator, where X, Y are F-spaces, and assume that $\mathbf{M}(X) = Y$. Then $\mathbf{M}$ is an open mapping.*

Proof. Since X, Y are F-spaces, then we can choose the local bases

$$V_{r,n}=\big\{x: d_X(x,0)<2^{-n}r\big\},\ r>0,\quad W_m=\big\{y: d_Y(y,0)<2^{-m}\big\}.$$

Since $X=\cup_i iV_{r,n}$, then

$$Y=\mathbf{M}(X)=\bigcup_{i=1}^{\infty}\overline{\mathbf{M}(iV_{r,n})}.$$

Hence some

$$\overline{\mathbf{M}(iV_{r,n})}=i\overline{\mathbf{M}(V_{r,n})}$$

has not empty interior, in particular $\overline{\mathbf{M}(V_{r,n})}$ has not empty interior for all $n\in\mathbb{N}$:

$$\exists m(n)\colon W_{m(n)}\subset\overline{\mathbf{M}(V_{r,n})}.$$

Since $\mathbf{M}$ is continuous, $m(n)\to\infty$ as $n\to 0$.

Since X, Y are F-spaces, then for all $y\in W_{m(1)}\subset\overline{\mathbf{M}(V_{r,1})}$ there is a point x_1 in V_1 such that

$$d_Y(\mathbf{M}x_1,y)\le m(2).$$

Since we have $W_{m(2)}\subset\overline{\mathbf{M}(V_{r,2})}$, then we can find a point x_2 such that

$$d_Y(\mathbf{M}(x_1+x_2),y)\le m(3).$$

Proceding, at step $\bar n$ we find points $x_1\in V_{r,1}$, $x_2\in V_{r,1}$,...,$x_{\bar n}\in V_{\bar n}$ such that

$$d_Y\bigg(\mathbf{M}\bigg(\sum_{i=1}^{\bar n}x_i\bigg),y\bigg)\le m(\bar n+1).$$

Due to the choice of x_i, the sequence

$$\sum_{i=1}^{\bar n}x_i\to\bar x,$$

and since $\mathbf{M}$ is continuous $\mathbf{M}\bar x=y$. Moreover

$$d(\bar x,0)=d\bigg(\sum_{i=1}^{+\infty}x_i,0\bigg)\le\sum_{i=1}^{+\infty}d(x_i,0)<r,$$

so that $\bar x\in V_{0,r}$. □

As an application, we have

Corollary 2.2. *If $\mathbf{M} : X \mapsto Y$ is continuous and one to one, then $\mathbf{M}^{-1}$ is continuous. If X, Y are Banach spaces, this means that there are constant c, C such that*

$$c\|x\|_X\le\|\mathbf{M}x\|_Y\le C\|x\|_X,\quad c,C>0. \tag{2.1}$$

In particular, if X has two norms $\|\cdot\|_1$, $\|\cdot\|_2$ such that both $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ are Banach spaces, then the two norms are equivalent, i.e.

$$c\|x\|_X \leq \|\mathbf{M}x\|_Y \leq C\|x\|_X, \quad c, C > 0. \tag{2.2}$$

3. Closed graph theorem

If $f : X \mapsto Y$, then the *graph* of f is the set

$$\mathcal{G}(f) = \Big\{(x, y) \in X \times Y : y = f(x)\Big\}.$$

Proposition 3.1. *If X is a topological space and Y is topological Hausdorff separable, $f : X \mapsto Y$ continuous, then $\mathcal{G}(f)$ is closed.*

Proof. Let $(x_0, y_0) \in (X \times Y) \setminus \mathcal{G}(f)$. Then y_0 and $f(x_0)$ have disjoint neighborhood W_0, W_1. Since f is continuous, then there is an open neighborhood V_0 of x_0 such that

$$f(V_0) \subset W_0.$$

Hence for all $x \in V_0$ $f(x) \notin W_1$, so that $(V_0 \times W_1) \cap \mathcal{G}(f) = \emptyset$. □

As we know, topological vector spaces are separable, so that the graph of continuous linear mappings is closed. If the spaces are metric complete, then we can prove the converse.

Theorem 3.2 (closed graph)**.** *If $\mathbf{M} : X \mapsto Y$ is a linear mapping between F-spaces, and $\mathcal{G}(\mathbf{M})$ is closed, then $\mathbf{M}$ is continuous.*

Proof. The space $X \times Y$ is a vector space, which becomes an F space with the distance

$$d_{X\times Y}\Big((x_0, y_0), (x_1, y_1)\Big) = d_X(x_0, x_1) + d_Y(y_0, y_1).$$

Since $\mathbf{M}$ is linear, then $\mathcal{G}(\mathbf{M})$ is a linear space, and from the closure is an F space with the same distance.

Consider the two projections

$$\begin{array}{cccc} \pi_1 & : & \mathcal{G} & \mapsto & X \\ & & (x, \mathbf{M}x) & \mapsto & x \end{array}, \qquad \begin{array}{cccc} \pi_2 & : & X \times Y & \mapsto & Y \\ & & (x, y) & \mapsto & y \end{array}$$

By constructions both projections are continuous, and π_1 is 1 to 1 and onto. Then by Corollary 2.2 π_1^{-1} is continuous, and since

$$\mathbf{M}x = \big(\pi_2 \circ \pi_1^{-1}\big)x,$$

then $\mathbf{M}$ is continuous. □

A standard way to prove that graph of $\mathbf{M}$ is closed for F spaces is to check that

$$(x_n, \mathbf{M}x_n) \to (x, y) \quad \Longrightarrow \quad \mathbf{M}x = y. \tag{3.1}$$

4. Exercises

(1) Prove that if the family $\mathbf{M}_\nu : X \mapsto Y$ is equibounded and X is metrizable, then it is equicontinuous.

(2) Prove that if X is an F-space, Y topological vector space, and the sequence $\{\mathbf{M}_n\}_{n\in\mathbb{N}}$ of continuous linear operator satisfies

$$\forall x \in X \quad \{\mathbf{M}_n x\}_{n\in M} \text{ converges to } \mathbf{M}x,$$

then $\mathbf{M}$ is continuous.

(3) Let X be the topological vector space generated by a countable Hamel base $\{e_n\}_{n\in\mathbb{N}}$, i.e.

$$x = \sum_{\text{finite}} \alpha_n e_n, \quad \alpha_n \in \mathbb{C}.$$

Show that X is the countable union of closed sets with empty interior. In particular no F-spaces can have countable Hamel base.

(4) Let X, Y, Z Banach spaces, and consider a continuous map
$$B : X \times Y \mapsto Z,$$
such that (bilinearity)
$$B\big(\alpha_0 x_0 + \alpha_1 x_1, y\big) = \alpha_0 B(x_0, y) + \alpha_1 B(x_1, y),$$
$$B\big(x, \beta_0 y_0 + \beta_1 x_1\big) = \beta_0 B(x, y_0) + \beta_1 B(x, y_1).$$
Show that there is $M > 0$ such that
$$\|B(x, y)\|_Z \leq M\|x\|_X\|y\|_Y.$$

(5) Let X be a Banach space, A, B closed vector spaces such that $A \cap B = \{0\}$, $X = A + B$. Prove that every x can be uniquely decomposed as
$$x = P_A x + P_B x, \quad P_A x \in A, P_B x \in B,$$
and that P_A, P_B are continuous.

(6) Consider a sequence of operators $\mathbf{M}_n$ in $\mathcal{L}(X, Y)$, with X, Y Banach, and assume that $\mathbf{M}_n x$ converges to $\mathbf{M}x$ for all $x \in X$. Show that
$$x_n \to x \quad \Longrightarrow \quad \mathbf{M}_n x_n \to \mathbf{M}x.$$

(7) Let X, Y be Banach spaces, and assume that the sequence of operators $\mathbf{M}_n \in \mathcal{L}(X, Y)$ satisfies
$$\forall \ell \in Y^* \quad \ell(\mathbf{M}_n x) \to \ell(\mathbf{M}x).$$
Is $\mathbf{M} \in \mathcal{L}(X, Y)$, i.e. linear and continuous?

(8) Using the closed graph theorem, show that if $\mathbf{M} : X \mapsto X^*$, X Banach, satisfies
$$(\mathbf{M}x)x \geq 0 \quad \forall x \in X,$$
then it is continuous.

(9) Prove that if $\mathbf{M} : X \mapsto X^*$, X Banach, satisfies
$$(\mathbf{M}x)y = (\mathbf{M}y)x \quad \forall x \in X,$$
then it is continuous.

(10) Let X, Y be two Banach spaces, and $\mathbf{M} \in \mathcal{L}(X, Y)$. Assume
$$R_{\mathbf{M}} \text{ closed and } \dim N_{\mathbf{M}} < \infty.$$
Assume that on X we have another norm $|\cdot|$, such that
$$|x| \leq M\|x\|_X \quad \forall x \in X.$$
Show that for some constant C
$$\|x\|_X \leq C\Big(\|Tx\|_Y + |x|\Big).$$