TOPOLOGICAL VECTOR SPACES

1. Topological vector spaces and local base

Definition 1.1. A *topological vector space* is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a topology τ such that

- every point is closed;
- the vector space operations are continuous.

This means that $X \setminus \{x\}$ is open and the operations

$$X \times X \ni (x, y) \mapsto x + y \in X, \qquad F \times X \ni (k, x) \mapsto kx \in X \tag{1.1}$$

are continuous, where $F = \mathbb{R}$ or $F = \mathbb{C}$ and $X \times X$ and $F \times X$ have the product topology.

Since the operations

$$T_a x = x + a \qquad M_\lambda x = \lambda x \tag{1.2}$$

are homeomorphism of X, then the topology is *translation invariant*. This means that a set E is open if and only if $E + a$ is open.

As a consequence, the topology τ is completely determined by the neighborhoods of 0. We define a *local base* $\mathcal{B}$ of 0 by requiring that $B \in \mathcal{B}$ are open and every neighborhood of 0 contains a set in $\mathcal{B}$. Clearly the topology τ is generated by the sets

$$E = \bigcup_{\alpha,\beta} (x_\alpha + B_\beta), \quad x_\alpha \in X, B_\beta \in \mathcal{B},$$

i.e. by unions of translated sets belonging to the local base at 0.

In the following by *local base* $\mathcal{B}$ we denote the local base at 0.

We say that a set E in a topological vector space is *bounded* if for every neighborhood V of 0 there corresponds a $s \in F$ such that $E \subset tV$ for all $t > s$.

2. Separation properties

Aim of this section is to prove that the compatibility of the topology with the vector operation implies some good properties on the structure of the local base $\mathcal{B}$.

We say:

- X is *locally convex* if there is a local base $\mathcal{B}$ whose members are convex, i.e.

$$aB + (1 - a)B \subset B \quad \forall B \in \mathcal{B}; \tag{2.1}$$

- X is *locally bounded* if 0 has a bounded neighborhood, i.e. there is $A \in \mathcal{B}$ such that

$$\forall B \in \mathcal{B}\ \exists s \in F\ :\ \forall t > s,\ A \subset tB. \tag{2.2}$$

  One can check that this implies that the local base $\mathcal{B}$ can be chosen to contain only bounded neighborhood of 0;
- X is *locally compact* if 0 has a neighborhood with compact closure. As before, this implies that $\mathcal{B}$ can be generated by relatively compact open neighborhood of 0.

The first result is that the topological vector spaces are "normal" topological spaces.

Proposition 2.1. *If K is compact, C closed, $K \cap C = \emptyset$ then there is V neighborhood of* 0 *such that*

$$(K + V) \cap (C + V) = \emptyset.$$

Proof. If $K = \emptyset$, there nothing to prove. Otherwise, let $x \in K$: by the invariance with translation, we can assume $x = 0$.

Then $X \setminus C$ is an open set of 0. Since addition is continuous, by $0 = 0 + 0 + 0$, there is a neighborhood U such that

$$3U \subset X \setminus C.$$

By defining

$$\tilde{U} = U \cap (-U) \subset U,$$

we have that $\tilde{U}$ is open, symmetric and $3\tilde{U} \subset X \setminus C$. This means that

$$\emptyset = \Big\{3x, x \in \tilde{U}\Big\} \cap C = \Big\{2x\Big\} \cap \Big\{y - x, y \in C, x \in \tilde{U}\Big\} \supset \tilde{U} \cap (C + \tilde{U}).$$

This concludes the proof for a single point.

Since K is compact, then repeating the above argument for all $x \in K$, we obtain symmetric open sets V_x such that

$$(x + 2V_x) \cap (C + V_x) = \emptyset.$$

The sets $\{V_x\}_{x \in K}$ are a covering of K: hence there is a finite number of points $x_i \in K$, $i = 1, \dots, n$, such that

$$K \subset \bigcup_{i=1}^{n} (x_i + V_x).$$

Define the neighborhood V of 0 by

$$V = \bigcap_{i=1}^{n} V_x,$$

Then

$$(K + V) \cap (C + V) \subset \bigg(\bigcup_{i=1}^{n} (x + V_x + V)\bigg) \cap (C + V) \subset \bigcup_{i=1}^{n} \Big((x + 2V_x) \cap (C + V_x)\Big) = \emptyset.$$

This is the statement of the proposition. □

In particular, the closure of $K + V$ does not intersect C. As corollary we have that every member of $\mathcal{B}$ contains the closure of some other member of $\mathcal{B}$.

By the assumption that every $x \in X$ is closed, we conclude that

Theorem 2.2. *A topological vector space is a Hausdorff space.*

We next show that the local base $\mathcal{B}$ at 0 can be chosen balanced, i.e. the sets $B \in \mathcal{B}$ are balanced. If we assume that X is locally convex, then it can be chosen balanced and convex. Note that from the proof of Proposition 2.1 one can deduce that the local base is at least symmetric.

Theorem 2.3. *If X is a vector space, then there is balanced local base. If X is locally convex, then there is a balanced convex local base*

Proof. The first part follows from the continuity of scalar multiplication. In fact, since

$$0 \cdot 0 = 0,$$

so that for all $V \in \mathcal{B}$, there is $V_1 \in \mathcal{B}$, $\delta > 0$ such that

$$x \in V_1,\ |\alpha| < \delta \quad \Longrightarrow \quad \alpha V_1 \subset V.$$

Take now

$$W = \bigcup_{|\alpha| < \delta} \alpha V_1.$$

This is clearly open, contained in V and balanced by construction.

Assume now that X is locally convex, and let U be a convex neighborhood of 0. We first take the balanced convex set

$$A = \bigcap_{|\alpha| = 1} \alpha U.$$

This is clearly convex and not empty because $0 \in A$.

Let now W be constructed as in the first part of the proof: W is a balanced neighborhood of 0 contained in U. Then for $|\alpha| = 1$

$$\alpha^{-1} W \subset W \quad \Longrightarrow \quad W \subset \alpha W \subset \alpha U,$$

This shows that $W \subset A$, i.e. $W \subset \overset{\circ}{A}$, the interior of A.

We prove now that $\overset{\circ}{A}$ is convex. In fact, for all $t \in [0,1]$, the convexity of A and the fact that $\overset{\circ}{A} \subset A$ implies

$$t\overset{\circ}{A} + (1-t)\overset{\circ}{A} \subset A.$$

Since $\overset{\circ}{A}$ is the largest open set in A, and $t\overset{\circ}{A} + (1-t)\overset{\circ}{A}$ is open then we conclude

$$t\overset{\circ}{A} + (1-t)\overset{\circ}{A} \subset \overset{\circ}{A},$$

i.e. $\overset{\circ}{A}$ is convex.

To end the proof, we have to show that $\overset{\circ}{A}$ is balanced. By construction we have that for $|\beta| \leq 1$

$$\beta A = \bigcap_{|\alpha|=1} \beta\alpha U = \bigcap_{|\alpha|=1} \alpha(|\beta|U).$$

Since U is convex and contain 0, then $|\beta|U \subset U$, i.e. for all $|\beta| \leq 1$

$$\beta A \subset A.$$

It follows that for $|\beta| \leq 1$

$$\beta\overset{\circ}{A} \subset \overset{\circ}{A}.$$

The conclusion follows. □

We now show another consequence of the continuity of the linear operations.

Theorem 2.4. *Let X be a topological vector space.*

(1) *If V is a neighborhood of 0, then*

$$X = \bigcup_{n=1}^{+\infty} nV; \tag{2.3}$$

(2) *every compact set is bounded;*

(3) *If V is bounded, i.e. X is locally bounded, then*

$$\left\{\frac{1}{n}V\right\}_{n\in\mathbb{N}} \tag{2.4}$$

is a local base at 0.

The last point says that if X is locally bounded, then it has a countable base.

Proof. By continuity of the scalar product, for x fixed we have

$$\lim_{n\to\infty} \frac{1}{n}x = 0,$$

so that for every V there is m, such that $x/n \in V$ for $n \geq m$. This proves (2.3).

The second statement is a consequence of finite covering property of compact sets. In fact, we can use a balanced open neighborhood W of 0, so that from (2.3)

$$K \subset \bigcup_{n=1}^{\infty} nW \quad \Longrightarrow \quad K \subset \bigcup_{i=1}^{N} n_i W = \max_i \{n_i\} W,$$

where is the last equality we used the fact that W is balanced. This proves the second point.

Finally, if V is bounded, then for all $V' \in \mathcal{B}$ there is $m \in \mathbb{N}$ such that

$$V \subset mV',$$

and hence $m^{-1}V \subset V'$. □

To end this section, we show how some linear operations on sets behave under the topology.

Proposition 2.5. *If X is a topological vector space, then*

(1) *for every $A \in X$, $\bar{A} = \cap(A + V)$, $V \in \mathcal{B}$;*
(2) $\bar{A} + \bar{B} \subset \overline{A+B}$;
(3) *if Y subspace, then also $\bar{Y}$ subspace;*

(4) *if C convex/balanced/bounded, so is $\bar{C}$;*
(5) *if C convex, so is $\overset{\circ}{C}$;*
(6) *if C balanced and $0 \in \overset{\circ}{C}$, so is $\overset{\circ}{C}$.*

We prove only the first, the proof of the other points is left as exercise.

Proof. If $x \notin \bar{A}$, then there is $V \in \mathcal{B}$ such that $x + V \notin A$, so that

$$x \notin \bigcap_{V \in \mathcal{B}} (A + V).$$

Conversely, if $x \in \bar{A}$, then $x + V \cap A \neq \emptyset$, so that

$$x \in \bigcap_{V \in \mathcal{B}} (A + V).$$

□

3. Linear mappings

We now consider the linear maps $\mathbf{M} : X \mapsto Y$, where X, Y are topological vector spaces.

We say that $\mathbf{M}$ is *bounded* if for some V neighborhood of 0 the set $\mathbf{M}V$ is bounded in Y.

If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, we call $\mathbf{M}$ *linear functional*.

Proposition 3.1. *The map $\mathbf{M} : X \to Y$ is continuous iff it is continuous in 0. Moreover, if $\mathbf{M}$ is a linear functional, then each of the following is equivalent:*
(1) *$\mathbf{M} : X \mapsto F$ is continuous;*
(2) *the null space of $\mathbf{M}$ is closed;*
(3) *$N_{\mathbf{M}} = X$ or it is not dense in X;*
(4) *$\mathbf{M}$ is bounded in some neighborhood of 0.*

Proof. The first part follows directly from

$$\mathbf{M}^{-1}(y + U) = \mathbf{M}^{-1}y + \mathbf{M}^{-1}U \supset \mathbf{M}^{-1}y + V,$$

where V is the neighborhood of $0 \in X$ such that for U neighborhood of $0 \in Y$

$$\mathbf{M}(V) \subset U.$$

Assume that $\mathbf{M}$ is continuous from X into Y. Since

$$N_{\mathbf{M}} = \mathbf{M}^{-1}(\{0\}),$$

it follows that $1 \Rightarrow 2$.

If the null space is closed, then either is X or is not dense in X, so that $2 \Rightarrow 3$.

In the step $3 \Rightarrow 4$ we use the fact that $\mathbf{M}$ is a linear functional. If $N_{\mathbf{M}} = X$ then clearly $\mathbf{M}x = 0$ for all x so that $\mathbf{M}$ is bounded. We thus assume that $N_{\mathbf{M}}$ is not dense. Then its complement $X \setminus N_{\mathbf{M}}$ is not empty and has not empty interior. Thus for some $x \in X$ there exists V neighborhood of 0 such that

$$(x + V) \cap N_{\mathbf{M}} = \emptyset.$$

Thus since V can be taken balanced, $\mathbf{M}V$ can be either bounded or

$$\forall n, \epsilon \in \mathbb{N} \quad \left| \frac{1}{n} \mathbf{M}V \right| = \left| \mathbf{M}(n^{-1}V) \right| \geq \epsilon.$$

Since V is balanced, so is $\mathbf{M}V$, and then $\mathbf{M}V = F$, where $F = \mathbb{R}$ of $F = \mathbb{C}$. In this case, there is a $y \in V$ such that $\mathbf{M}y + \mathbf{M}x = 0$, but this contradict the assumption. This proves that $3 \Rightarrow 4$.

Finally, if $\mathbf{M}$ is bounded in some neighborhood V of 0, then for all open neighborhood $U \subset Y$ of 0 there is $n \in N$ such that

$$U \supset \frac{1}{n} \mathbf{M}V = \mathbf{M}(n^{-1}V).$$

Since $n^{-1}V$ is open, then $\mathbf{M}$ is continuous at 0, i.e. $4 \Rightarrow 1$. □

Note that in the proof of the theorem we used the assumption of $\mathbf{M}$ being a linear functional only when proving $3 \Rightarrow 4$. Clearly this implication is not true ever for maps from $\mathbb{R}^2$ into itself.

However the essential part of the above theorem can be extended to maps from X into $\mathbb{R}^n$ or $\mathbb{C}^n$, by noticing that each component $e_i \cdot \mathbf{M} : X \mapsto F$ is a linear functional.

We observe that a continuous functional $\mathbf{M} : X \mapsto Y$ is uniformly continuous in the following sense:

for all U neighborhood of $0 \in Y$ there is a V neighborhood of $0 \in X$ such that

$$\mathbf{M}(x + V) \subset \mathbf{M}x + U. \tag{3.1}$$

This depends on the translation invariance of the local base $\mathcal{B}$.

4. Finite dimensional spaces

We see here that the topology of $\mathbb{C}^n$ (or $\mathbb{R}^n$) is the only topology compatible with the linear operations on finite dimensional spaces.

Lemma 4.1. *If X is a linear space over $\mathbb{R}$ or $\mathbb{C}$ and has linear dimension n, then every invertible linear map $\mathbf{M}$ from $\mathbb{R}^n$ or $\mathbb{C}^n$ into X is an homeomorphism.*

Proof. Denote with e_n the standard base in $\mathbb{R}^n$ or $\mathbb{C}^n$. Clearly from the linearity and invertibility of $\mathbf{M}$

$$X = \operatorname{span}\{x_1, \ldots, x_n\} = \operatorname{span}\Big\{\mathbf{M}e_1, \ldots, \mathbf{M}e_n\Big\}.$$

Moreover $\mathbf{M}$ is continuous because every component

$$\mathbf{M}_i : (\alpha_1, \ldots, \alpha_n) \mapsto x = \alpha_i x_i$$

is continuous.

Thus

$$\Big\{x = \sum_{i=1}^n \alpha_i x_i : \sum_{i=1}^n |\alpha_i|^2 = 1\Big\} = \mathbf{M}(S(0,1)) = \mathbf{M}\Big\{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^n |\alpha_i|^2 = 1\Big\}$$

is compact. Since $0 \notin \mathbf{M}(S(0,1))$, then there is a balanced neighborhood V of 0 such that $V \cap \mathbf{M}(S(0,1)) = \emptyset$. In particular $\mathbf{M}^{-1}(V)$ is balanced, hence connected, thus

$$\mathbf{M}^{-1}(V) \subset B(0,1) = \Big\{(\alpha_1, \ldots, \alpha_n) : \sum |\alpha_n|^2 < 1\Big\}.$$

This implies that each component $\mathbf{M}_i^{-1}$ of $\mathbf{M}^{-1}$ is bounded, hence by Proposition 3.1 it is continuous, so that also $\mathbf{M}^{-1}$ is continuous. □

Proposition 4.2. *If $Y \subset X$ is a finite dimensional linear subspace of X, then Y is closed.*

Proof. If $\{x_1, \ldots, x_n\}$ is a vector base for Y, we can introduce the homeomorphism from $\mathbb{R}^n$ or $\mathbb{C}^n$ to Y with the induced topology

$$\mathbf{M}(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n \alpha_i x_i.$$

Clearly $\mathbf{M}$ is continuous also into X.

Let now $p \in \bar{Y}$, and consider a neighborhood V of 0 disjoint from $\mathbf{M}(S(0,1))$. For some $t > 0$, p belongs to the closure of

$$Y \cap (tV) \subset \mathbf{M}(tB) \subset \mathbf{M}(t\bar{B}).$$

Since $t\bar{B}$ is compact, then it is compact and then closed in X. Thus

$$p \in \overline{\mathbf{M}(t\bar{B})} = \mathbf{M}(t\bar{B}) \subset Y.$$

□

This shows that there only one topology in $\mathbb{R}^n$ or $\mathbb{C}^n$ compatible with the linear operations: it is the standard topology generated by the balls

$$B(x,r) = \Big\{x \in \mathbb{R}^n \text{ or } \mathbb{C}^n, \sum_{i=1}^n |x_i|^2 < r^2\Big\}.$$

Theorem 4.3. *Every locally compact topological vector space X has finite dimension.*

Proof. If X has a neighborhood V which is compact, then by the finite covering property

$$V \subset (x_1 + V/2) \cup \cdots \cup (x_n + V/2).$$

Let Y be the vector space generated by $(x_1, \dots, x_n)$. The previous formula reads as

$$V \subset Y + V/2.$$

Moreover Y is closed. By linearity, we have also

$$V/2 \subset Y + V/4,$$

so that it follows

$$V \subset Y + V/4.$$

Thus proceding we have

$$V \subset \left(\bigcap_{i=1}^{\infty} (Y + 2^{-n} V) \right) = \bar{Y} = Y,$$

because $V/2^n$ is a local base (V being compact is bounded), Y closed and we have used Proposition 2.5. Since $X = \cup_n nV \subset Y$, X is finite dimensional. □

5. Metrization

The topological vector space X is *metrizable* if its topology is generated by some metric d. In this case we say that:

- X is an *F-space* if it is generated by a complete invariant metric d. This means that

$$d(x + z, y + z) = d(x, y);$$

- X is a *Fréchet space* if it is a locally convex F-space;

A *norm* $\| \cdot \| : X \mapsto \mathbb{R}$ is defined by

(1) $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$;
(2) $\|\lambda x\| = |\lambda| \|x\|$;
(3) $\|x + y\| \leq \|x\| + \|y\|$

It is easy to see that $d(x, y) = \|x - y\|$ is a translation invariant distance, and that the base

$$B(0, n^{-1}) = \left\{ x : \|x\| \leq n^{-1} \right\}$$

is a convex balanced base. We thus can consider two more topological vector spaces:

- X is *normable* if there is a norm which generate a metric compatible with the topology;
- X is *Banach* if it is normable and the metric generated by the norm is complete.

A Banach space X is then a Fréchet space.

There are several conditions on general topological spaces which state that the topology is generated by a metric $d : X \times X \mapsto \mathbb{R}^+$. One of the basic requirement is that *every point x has a countable base of neighborhoods*, because the sets

$$B(x, n^{-1}) = \left\{ y : d(x, y) \leq n^{-1} \right\}$$

is a local base for the topology generated by the metric.

While the above conditions is in general only necessary for general topological spaces, for topological vector spaces it is also sufficient.

Theorem 5.1. *If X is a topological vector space with a countable base, then there is a metric d such that*

(1) *d is compatible with τ;*
(2) *the open balls of 0 are balanced;*
(3) *d is translation invariant, i.e. $d(x + z, y + z) = d(x, y)$.*

If X is locally convex, then d can be chosen to have the open balls convex.

The technical part in the proof is to construct a metric which satisfies the triangle inequality.

Proof. By assumption there is a numerable base V_n. By using the continuity of the vector operations, we can assume that the base is balanced and

$$4V_n \subset V_{n-1}.$$

If the space is locally convex, then this base is convex, as shown in Theorem 2.3.

Let D be the set of rational numbers with

$$r = \sum_{n=1}^{\infty} c_n(r)2^{-n}, \qquad c_n(r) \in \{0,1\},\ 0 \le r < 1,$$

with $c_n \neq 0$ only for a finite number of n.

Define the function $A : D \cup [1, +\infty) \mapsto P(X)$ as

$$A(r) = \begin{cases} \sum_{n=0}^{\infty} c_n(r)V_n & r \in D \\ X & r \ge 1 \end{cases}$$

Since the sets V_n are balanced and open, then $A(r)$ is open and balanced. If the space is locally convex, then $A(r)$ is convex.

We have that

$$A(r) + A(s) = \sum_{n=0}^{\infty} (c_n(r) + c_n(s))V_n = \sum_{n=0}^{\infty} c_n(r+s)V_n$$

if $r + s < 1$ and $c_n(r + s) = c_n(r) + c_n(s)$, and

$$A(r) + A(s) \subset A(r + x) = X$$

in the case $r + s \ge 1$.

If for some $n > 1$ $c_n(r + s) \neq c_n(r) + c_n(s)$, then it is easy to verify that at the smallest n one has $c_n(r + s) = 1$, $c_n(r) = c_n(s) = 0$. Thus in this case we have

$$A(r) \subset \sum_{m=1}^{n-1} c_m(r)V_m + 2V_{n+1}, \quad A(s) \subset \sum_{m=1}^{n-1} c_m(s)V_m + 2V_{n+1},$$

so that

$$A(r) + A(s) \subset \sum_{m=1}^{n-1} c_m(r + s)V_m + 4V_{n+1} \subset A(r + s),$$

because of the assumptions on V_n.

Let now

$$f(x) = \inf\{r : x \in A(r)\},$$

and define the distance by

$$d(x, y) = f(x - y).$$

Then clearly by construction $d \ge 0$ and $d(x, y) = 0$ iff $x = y$. Moreover d is invariant and symmetric (the set are balanced). If the space is locally convex, the open balls are convex. The triangle inequality follows by proving that

$$f(x + y) \le f(x) + f(y),$$

only in the case the right hand side is less than 1. Let r, s be such that

$$f(x) < r,\ f(y) < s,\ r + s < f(x) + f(y) + \epsilon$$

Then $x \in A(r)$, $y \in A(s)$ ans hence $x + y \in A(r + s)$, so that

$$f(x + y) \le r + s \le f(x) + f(y) + \epsilon.$$

Since ϵ is arbitrary, the conclusion follows. □

In metric spaces we can speak about Cauchy sequences. However, even if X is only a topological vector space with local base $\mathcal{B}$, we can define $\{x_n\}_{n\in\mathbb{N}}$ a *Cauchy sequence* if

$$\forall V \in \mathcal{B},\ \exists N \text{ such that } n, m > N \quad \Longrightarrow \quad x_n - x_m \in V. \tag{5.1}$$

Clearly different local bases generate the same Cauchy sequences.

In particular, if X is metrizable with invariant metric d, then its "metric" Cauchy sequences coincides with the "topological" Cauchy sequences. Thus if two invariant metrics d_1, d_2 on X induce the same topology, then they have the same Cauchy sequences. The invariance of the metric is fundamental, otherwise one can generates diverging sequences which are Cauchy for some non translation invariant metric d.

In particular the translation invariant metric d_1 is complete if and only if the translation invariant metric d_2 is complete.

For the translation invariant metric constructed in the theorem, if $n \leq k < n+1$, then

$$d(nx,0) \leq d(kx,0) \leq (n+1)d(x,0). \tag{5.2}$$

The first inequality follows from the fact that the balls $B(x,r) = \{x : d(x,0) < r\}$ are balanced, while the second from the translation invariance.

If the metric is a norm, we recall that

$$\|\alpha x\| = |\alpha| \|x\|.$$

6. Boundedness and continuity

We recall an equivalent definition of bounded sets $E \subset X$:

the set $E \subset X$ is bounded if and only if for all $\alpha_n \to 0$ in F, and every sequence $x_n \in E$,

$$\lim_{n\to\infty} \alpha_n x_n = 0. \tag{6.1}$$

We leave the proof as an exercise.

We say that $\mathbf{M} : X \mapsto Y$ is *bounded* if it maps bounded sets in bounded sets.

Proposition 6.1. *We have the following implications:*

1. *If $\mathbf{M}$ is continuous, then is bounded.*
2. *If $\mathbf{M}$ is bounded, then for all sequences $x_n \to 0$ the set $\{\mathbf{M}x_n\}_{n\in\mathbb{N}}$ is bounded.*
3. *If X is metrizable and for all $x_n \to 0$ the set $\{\mathbf{M}x_n\}_{n\in\mathbb{N}}$ is bounded, then $\mathbf{M}x_n \to 0$.*
4. *If X is metrizable and $\mathbf{M}x_n \to 0$ for all $x_n \to 0$, then $\mathbf{M}$ is continuous.*

Proof. The first implication follows from continuity and the previous characterization of boundedness: if fact, if $E \subset X$ is bounded, and $U \subset Y$ is a neighborhood of 0, then $\mathbf{M}^{-1}(U) = V$ is a neighborhood of $0 \in X$. Hence for some s, $t \geq s$, $E \subset tV$, and

$$\mathbf{M}(E) \subset \mathbf{M}(tV) = tU.$$

The second implication from the fact that $\{x_n\} \to 0$ is bounded. In fact, $x_n \to 0$ implies that for all V there is N such that for $n > N$ we have $x_n \in V$. Then it remains to find α such that the finite set $\alpha\{x_1, \dots, x_N\} \subset V$. This follows from continuity of the scalar product.

The third one from the fact that if $x_n \to 0$, then for the invariant metric d

$$d_n = d(x_n, 0) \to 0.$$

and thus also $[1/\sqrt{d_n}]x_n \to 0$, where $[\cdot]$ is the integer part. In fact, by using the translation invariance of d and (5.2),

$$d\big([1/\sqrt{d_n}]x_n, 0) \leq [1/\sqrt{d_n}]d(x_n,0) \to 0.$$

We conclude by

$$\mathbf{M}x_n = \frac{1}{[1/\sqrt{d_n}]}\mathbf{M}\big([1/\sqrt{d_n}]x_n]\big) \to 0.$$

because of boundedness of $\{[1/\sqrt{d_n}]x_n\}_{n\in\mathbb{N}}$.

Finally, if $\mathbf{M}$ not continuous, then there is W neighborhood of $0 \in Y$ such that $\mathbf{M}^{-1}W$ does not contains any neighborhood U of $0 \in X$. Since X has a local countable base $\{V_n\}_{n\in\mathbb{N}}$, it follows that for any V_n there is an $x_n \notin \mathbf{M}^{-1}W$. By construction $x_n \to 0$, but $\mathbf{M}x_n \not\to 0$. ☐

7. An example: $L^p(0,1)$ with $p \in (0,1)$

The elements of $L^p(0,1)$, $p \in (0,1)$ are the measurable functions such that

$$\Delta_p(f) = \int_0^1 |f(t)|^p dt < \infty.$$

Since we have the inequality

$$(a+b)^p \le a^p + b^p, \quad \forall p \in (0,1),$$

we have

$$\Delta(f+g) \le \Delta(f) + \Delta(g).$$

Hence

$$d(f,g) = \Delta(f-g)$$

is an invariant distance. By repeating the argument used to prove completeness in the case $p \ge 1$, one can prove that (L^p, d) is a complete metric space.

We thus have that (L^p, d) is an F-space.

We now prove that in (L^p, d) there are no open convex sets other than $\emptyset$, X.

In fact, if $V \neq \emptyset$ is open and convex, with $0 \in V$, then it contains some open balls $B(0,r)$. Let now $f \in L^p$, and divide the interval $(0,1)$ into n segments (x_i, x_{i+1}), $i = 0, \dots, n-1$, with $x_0 = 0$, $x_n = 1$, such that

$$\int_{x_i}^{x_{i+1}} |f(y)|^p dy = \frac{1}{n} \int_0^1 |f(y)|^p dy = \frac{1}{n} \Delta(f).$$

Define now

$$g_i(x) = n f_i(x) \chi\{x \in (x_i, x_{i+1})\}, \ i = 0, \dots, n-1, \quad \Delta(g_i) = n^{p-1} \Delta(f).$$

so that

$$f = \frac{1}{n} \sum_{i=0}^{n-1} g_i.$$

Since for n sufficiently large $g \in B(0,r)$, then $f \in V$, i.e. $V = L^p$.

Assume now that $\mathbf{M} : L^p \mapsto Y$ is continuous and Y is locally convex. Then for U convex neighborhood of $0 \in Y$

$$\mathbf{M}^{-1}(U) \text{ convex and open} \quad \Longrightarrow \quad \mathbf{M}^{-1}(U) = L^p,$$

or equivalently $\mathbf{M} = 0$. In particular the only continuous functional on L^p is 0.

This example shows that the lack of local convexity can imply that the set of the continuous linear functional is extremely small.

8. Seminorm and local convexity

A *seminorm* on a vector space X is a real valued function $p : X \mapsto \mathbb{R}$ such that

1. $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$;
2. $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$, $\alpha \in F$.

We note first that

$$p(0) = p(0x) = 0, \quad 0 = p(x-x) \le 2p(x),$$

i.e. the seminorm is positive. Moreover

$$p(x) \le p(x-y) + p(y) \quad \Longrightarrow \quad |p(x) - p(y)| \le p(x-y).$$

If $p(x) = 0$ implies $x = 0$, then the seminorm is actually a *norm*.

A family of seminorms is said to be *separating* if for each $x \neq 0$ there is p such that $p(x) \neq 0$.

There is a standard way to construct seminorms on a vector space X.

Let now A be a balanced convex set. We say that A is *absorbing* if

$$\forall x \in X \ \exists t > 0 \text{ such that } x \in tA, \tag{8.1}$$

i.e. $x \in tA$ for t sufficiently large. Define the *Minkowski functional* $\mu_A(x)$ as

$$\mu_A(x) = \inf\{t : x \in tA\}. \tag{8.2}$$

Using the fact that A is balanced and convex, it follows that μ_A is a seminorm, and conversely if p is a seminorm, $\{x : p(x) < 1\}$ is balanced convex absorbing and generates the seminorm. There is thus a correspondence among balanced convex absorbing sets and seminorms.

Let X be a vector space, and assume there is a family $\mathcal{P}$ of separating seminorms. We can introduce an invariant topology τ on X by defining the local base $\mathcal{B}$ at 0:

(8.3)

$$\forall \text{ finite collection on seminorms } p_1, \dots, p_m \in \mathcal{P}, \forall n \in \mathbb{N} \quad \Longrightarrow \quad B(\{p_i\}, n) = \bigcap_{i=1}^{m} \left\{ x : p_i(x) < \frac{1}{n} \right\} \in \mathcal{B}.$$

Theorem 8.1. *The topology generated by the family of separating seminorms $\mathcal{P}$ makes X into a locally convex topological vector space.*

Proof. Clearly the sets B given by (8.3) are convex and balanced, so that it remains to show that every point is closed and the linear operations are continuous.

Since $\mathcal{P}$ is separating, then for all $x \neq 0$ there is p_x such that $p_x(x) = \delta > 0$. Thus the neighborhood of x

$$x + p_x^{-1}([0, \delta/2))$$

does not contains 0. Then $X \setminus \{0\}$ is open.

To prove that addition is continuous, if $V = B(\{p_i\}, n)$ is a neighborhood of $x + y$ of the form (8.3), then from the subadditivity of the seminorms it follows that

$$B(\{p_i\}, 2n) + B(\{p_i\}, 2n) \subset B(\{p_i\}, n)$$

so that

$$(x + B(\{p_i\}, 2n)) + (y + B(\{p_i\}, 2n)) \subset x + y + B(\{p_i\}, n).$$

Similarly, for the point αx one has

$$(|\alpha| + \epsilon) B(\{p_i\}, n(|\alpha| + \epsilon)) \subset B(\{p_i\}, n),$$

so that

$$\big(\alpha + (-\epsilon, \epsilon)\big) \Big(x + B(\{p_i\}, n(|\alpha| + \epsilon)) \Big) \subset \alpha x + B(\{p_i\}, n).$$

□

With the above topology, it is easy to verify the following fact:

(1) every seminorm $p \in \mathcal{P}$ is continuous;
(2) a set $E \in X$ is bounded if and only if $p(E)$ is bounded for all $p \in \mathcal{P}$.

Note that there can be different families of separating seminorms, generating different topologies.

We now show in some sense the converse of the above result: if we have a locally convex topology, then there exists a family of continuous seminorms which generates the topology by (8.3).

Theorem 8.2. *Let X be a locally convex topological space, with balanced convex base $\mathcal{B}$. For all $V \in \mathcal{B}$ consider the Minkowski functional μ_V. Then*

(1) *for all $V \in \mathcal{B}$*

$$V = \Big\{ x : \mu_V(x) < 1 \Big\};$$

(2) *$\{\mu_V : V \in \mathcal{B}\}$ is a separating family of continuous seminorms on X.*

Proof. If $\mathcal{B}$ is a local balanced convex base, then let $\mathcal{P}$ be the family of seminorms generated by μ_V, $V \in \mathcal{B}$.

Clearly since $x \in V$ implies $x/t \in V$ for some $t > 1$, so that $\mu_V(x) < 1$. Conversely, if $x \notin V$, then $x/t \in V$ implies $t \geq 1$, so that $\mu_v(x) \geq 1$. This proves 1.

If $x \neq 0$, then there is V such that $x \cap V = \emptyset$, so that $\mu_V(x) \geq 1$.

By contruction these seminorms are continuous, and the topology generated by the local base given in (8.3) is the same topology generated by $\mathcal{B}$ □

One can easily check that the topology generated by (8.3) is the smaller topology such that each p is continuous.

If the local base is countable, the family of seminorm can be chosen countable, $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$. In this case one can write an equivalent metric using the family $\{p_n\}_{n\in\mathbb{N}}$, namely

$$d(x,0) = \max_i \left\{ \frac{c_i p_i(x)}{1 + p_i(x)} \right\}, \quad c_i \to 0. \tag{8.4}$$

We finally give the necessary and sufficient condition such that topology of X is generated by a norm

Theorem 8.3. *The topological vector space X is normable if and only if there is a bounded convex neighborhood of* 0*.*

Proof. If the norm $\|\cdot\|$ exists, then $\|x\| < 1$ is bounded.

On the other hand, if V is convex and bounded, its Minkowski functional μ_V is equal to 0 only for $x = 0$, hence it is a norm. □

9. Two examples

9.1. **The space $C(\Omega)$, Ω open in $\mathbb{R}^n$.** Let Ω open in $\mathbb{R}^n$. Then Ω is the union of countable many compact sets $K_n \subset K_{n+1}$. We define $C(\Omega)$ as the vector space of complex valued functions on Ω, with the topology generated by the family of seminorms

$$p_n(f) = \sup\big\{|f(x)|, x \in K_n\big\}. \tag{9.1}$$

Since the family is countable, then $C(\Omega)$ is metrizable by

$$d(f,g) = \max_n \left\{ \frac{2^{-n}\|f-g\|_{C(K_n)}}{1 + \|f-g\|_{C(K_n)}} \right\}. \tag{9.2}$$

Repeating the argument of completes on every compact set K_n, one can prove that $C(\Omega)$ is complete, hence it is a Fréchet space.

Since every neighborhood of 0 generated by p_n contains functions which are arbitrarily large, then $C(\Omega)$ is not locally bounded, hence not normable.

9.2. **The spaces $C^\infty(\Omega)$ and $\mathcal{D}$.** A complex valued function f is in $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, if $D^\alpha f \in C(\Omega)$ for all $\alpha \in \mathbb{N}^n$. As before $\Omega = \cup_n K_n$, with K_n compact. We can define a countable family of seminorms by

$$p_N(f) = \max\Big\{|D^\alpha f(x)|, x \in K_N, |\alpha| \leq N\Big\}. \tag{9.3}$$

As in the previous example, $C(\Omega)$ is a Fréchet space, and it is not locally bounded, hence not normable.

The space $\mathcal{D}_K$ is the subspace of $C^\infty(\Omega)$ generated by the functions whose support is contained in some compact set K. Since the maps

$$T_x f = f(x)$$

are continuous and

$$\mathcal{D}_K = \bigcap_{x \notin K} T_x^{-1}(0),$$

then $\mathcal{D}_K$ is closed. Hence is a Fréchet space.

As in the previous case, one can show that $\mathcal{D}_K$ is not locally bounded, hence not normable.

10. Exercises

(1) We say that $E \subset X$, X metrizable, is d-bounded if there is $n \in \mathbb{N}$ such that

$$d(x,0) \leq n \quad \forall x \in E.$$

Prove that in the metric introduced in 5.1, X is d-bounded.

(2) If p is a seminorm on X, prove that $p^{-1}(0)$ is a subspace of X.

(3) Prove that if $A \subset X$ is absorbing, then the sets
$$B = \{x : \mu_A(x) < 1\}, \quad C = \{x : \mu_A(x) \leq 1\}$$
satisfy
$$B \subset A \subset C, \quad \mu_A = \mu_B = \mu_C.$$
(4) Let $N \subset X$ be a closed subpace of the topological vector space X. Prove that the topology
$$V \text{ open} \quad \Longleftrightarrow \quad V + N \text{ open}$$
makes X/N a topological vector space.
(5) Let $N \subset X$ be a closed subspace of the topological vector space X. Prove that
- if X is locally convex, so is X/N;
- if X is locally bounded, so is X/N;
- if X is locally metrizable, so is X/N;
- if X is normable, so is X/N;
- if X is complete, so is X/N.

(6) Prove that is p is a continuous seminorm on X and $N = p^{-1}(0)$, then X/N is a normed space.
(7) Let $X = \{f : [0,1] \mapsto \mathbb{R}\}$, with the topology generated by the seminorms
$$p_x(f) = |f(x)|.$$
Show that the convergence of $f_n \to f$ is the pointwise convergence.
(8) Prove that in $L^p([0,1])$, the subspace with finite codimensions are dense in L^p.
(9) If $f : [0,1] \mapsto \mathbb{C}$, then define for $0 < \alpha \leq 1$
$$\|f\|_\alpha = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{\delta^\alpha}, |x - y| \leq \delta, x, y \in [0,1]\right\}.$$
Prove that the space $\mathrm{Lip}_\alpha = \{f : \|f\|_\alpha < \infty\}$ is a Banach space, and that
$$\mathrm{lip}_\alpha = \left\{f \in \mathrm{Lip}_\alpha : \lim_{\delta \to 0} \sup\left\{\frac{|f(x) - f(y)|}{\delta^\alpha}, |x - y| \leq \delta, x, y \in [0,1]\right\} = 0\right\}$$
is a closed subspace of Lip_α.
(10) Let $I = [a,b]$ be a compact interval, $h : I \mapsto \mathbb{R}$ a continuous function and X the set of all continuous functions $f : I \mapsto \mathbb{R}$ such that
$$\|f\|_h = \sup_{x \in I}\left\{e^{\int_a^x h(t)dt}|f(x)|\right\} < \infty.$$
Prove that this is a norm on X equivalent to the usual sup norm.
(11) Let $I = [0,1]$ and X the set of functions $f : I \times \mathbb{R} \mapsto \mathbb{R}$ such that
$$\|f\| = \sup_{(t,x) \in I \times \mathbb{R}}\left\{\frac{|f(t,x)|}{1 + |x|}\right\} < \infty.$$
Prove that X with the above norm is a Banach space.
(12) Let U be open subset of $\mathbb{R}$, and $f_n \in C^\infty(\Omega)$ be such that for all $k \geq 0$
$$\sup_{x \in U}|D^k f_n(x)| \leq C_k \quad \forall n.$$
Prove that the sequence is compact in the Fréchet space $C^\infty(\Omega)$.
(13) Let $Q : C^\infty(\mathbb{R}) \mapsto \mathbb{R}$ be a linear functional such that $Q(f) \geq 0$ for all $f \in C^\infty$, with $f(0) = 0$ and $\{f \geq 0\}$ contains a neighborhood of $x = 0$. Prove that
$$Q(f) = af''(0) + bf'(0) + cf(0),$$
with $a, b, c \in \mathbb{R}$ constant.