TOPOLOGICAL SPACES

1. Topological spaces

A *topological space* is a set S with a collection of subset $\tau \subset P(S)$ (called *open sets*) such that

(1) $S, \emptyset \in \tau$;
(2) if E_1, E_2 belong to τ, then $E_1 \cap E_2 \in \tau$;
(3) if $E_\alpha \in \tau$ for all $\alpha \in A$, then $\cup_\alpha E_\alpha \in \tau$.

We recall below come definitions in topological spaces:

- A collection $\tau' \subset \tau \subset P(S)$ of subset of S is a *base* for the topology τ is every $E \in \tau$ is the union of members in τ'.
- A set E is *closed* if $S \setminus E$ is an open set.
- The *closure* $\bar{E}$ of a set E is the smallest close set containing E, while its *interior* $\overset{\circ}{E}$ is the largest open set contained in E.
- A *neighborhood* of a point $p \in S$ is any open set containing p.
- The pair (S, τ) is an *Hausdorff space* (or *separable space*) if distinct points have disjoint neighborhood, i.e. for all x_1, x_2 in S with $x_1 \neq x_2$ there exists E_1, E_2 in τ so that

$$x_1 \in E_1,\ x_2 \in E_2, \quad E_1 \cap E_2 = \emptyset. \tag{1.1}$$

- A map $f : S \mapsto T$, (T, τ), (S, σ) topological spaces, is *continuous* if

$$f^{-1}(E) \in \sigma \quad \forall E \in \tau. \tag{1.2}$$

- A family of open sets $E_\alpha \in \tau$, $\alpha \in A$, is a *covering* of the set E if $\cup_\alpha E_\alpha \supset E$.
- E is called *compact* if every covering $\{E_\alpha\}_{\alpha \in A}$ contains a finite covering $\{E_{\alpha_i}\}_i$, $i = 1, \dots, n$..
- A subset is *relatively compact* if its closure is compact.
- A topological space is *locally compact* if each point has a relatively compact neighborhood.
- A family of sets $\{C_\alpha\}_{\alpha \in A}$ has the finite intersection property if for all finite subsets $\{\alpha_i\}_i \subset A$

$$\bigcap_i C_{\alpha_i} \neq \emptyset. \tag{1.3}$$

If S is compact and the C_α are closed with the finite intersection property, then from (1.3) it follows that

$$\bigcap_{\alpha \in A} C_\alpha \neq \emptyset. \tag{1.4}$$

Conversely, if for all families of closed sets $\{C_\alpha\}_\alpha \subset \mathcal{P}(S)$ with the finite intersection property (1.4) holds, then S is compact.

Proposition 1.1. *In a Hausdorff space, every compact set K is closed.*

Proof. For any $x \in K$, $y \notin K$ there are disjoint neighborhood $E_y(x)$ of x, $E_x(y)$ of y. Since when x varies in K the open sets $E_y(x)$ are a covering of K, then we can extract a finite covering $E_y(x_i)$, with $i = 1, \dots, n$. We thus have that the neighborhood

$$E(y) = \bigcap_{i=1}^n E_{x_i}(y)$$

is disjoint from K. The conclusion follows because

$$X \setminus K = \bigcup_y E(y).$$

□

2. Cartesian product of topological spaces

The Cartesian product $\prod_\alpha X_\alpha$ of topological spaces (X_α, τ_α), $\alpha \in A$, is the set of function $f : A \to \cup_\alpha X_\alpha$, with $f(\alpha) \in X_\alpha$,

$$\prod_{\alpha \in A} X_\alpha = \Big\{ f : A \mapsto \bigcup_{\alpha \in A} X_\alpha, f(\alpha) \in X_\alpha \ \forall \alpha \in A \Big\}. \tag{2.1}$$

Define the projections $P_\beta : \prod_\alpha X_\alpha \mapsto X_\beta$ by

$$P_\beta(f) = f(\beta). \tag{2.2}$$

We define a set $E \in \prod_\alpha X_\alpha$ *open* if there is a finite number of indexes $\alpha_1, \dots, \alpha_n$, and open sets $E_{\alpha_i} \in \tau_{\alpha_i}$, $i = 1, \dots, n$, so that

$$E = \Big\{ f : A \mapsto \bigcup_{\alpha \in A} X_\alpha \colon f(\alpha_i) \in E_{\alpha_i} \in \tau_{\alpha_i}, i = 1, \dots, n \Big\}. \tag{2.3}$$

One can verify that these sets define a base of a topology, which is the weakest topology which makes continuous all maps P_β.

Theorem 2.1 (Tychonoff)**.** *The Cartesian product $X = \Pi_\alpha X_\alpha$ of compact spaces X_α is compact.*

Proof. Let $\{M_\beta\}_{\beta \in B}$ be a covering of X, and assume that there is not a finite subcovering $\{M_{\beta_i}\}_{i=1}^N$. Consider the closed sets

$$C_\beta = \big(\Pi_\alpha X_\alpha\big) \setminus M_\beta.$$

By assumption, every finite intersection of C_β is not empty, but $\cap_\beta C_\beta = \emptyset$.

Let now $\mathcal{F}$ be the collection of all families $K = \{D_\gamma\}_\gamma$ of closed sets with the finite intersection property and containing $\{C_\beta\}_\beta$, ordered by inclusion. If $\{K_\eta\}_\eta$ is a totally ordered sequence in $\mathcal{F}$, then the upper bound is given by their union $\cup_\eta K_\eta$. Thus by Zorn's lemma there is a maximal collection of closed sets

$$\{D_\gamma\}_{\gamma \in \Gamma} \supset \{C_\beta\}_{\beta \in B}$$

so that if $D \in \Pi_\alpha X_\alpha$ is closed and $D \notin \{D_\gamma\}_\gamma$, then $\{D, \{D_\gamma\}_\gamma\}$ does not have the finite intersection property.

Consider now the sets

$$D_{\alpha,\gamma} = P_\alpha D_\gamma = \Big\{ f(\alpha) : f \in D_\gamma \Big\}.$$

Since X_α is compact for all $\alpha \in A$ and the family $\{\overline{D_{\alpha,\gamma}}\}_\gamma$ has the finite intersection property, then

$$\bigcap_{\gamma \in \Gamma} \overline{D_{\alpha,\gamma}} \neq \emptyset.$$

Choose $x_\alpha \in \cap_\gamma \overline{D_{\alpha,\gamma}}$, define $\bar f$ by

$$\bar f(\alpha) = x_\alpha,$$

and let $E_{\bar f}$ be an open neighborhood of $\bar f$ of the form

$$E_{\bar f} = \Big\{ f : A \mapsto \bigcup_{\alpha \in A} X_\alpha \colon f(\alpha_i) \in E_{\bar f, \alpha_i} \in \tau_{\alpha_i} \text{ with } \bar f(\alpha_i) \in E_{\bar f, \alpha_i}, i = 1, \dots, n \Big\}.$$

If $E_{\bar f} \cap D_{\bar \gamma} = \emptyset$ for some $\bar\gamma$, then

$$D_{\bar f} = \Pi_\alpha X_\alpha \setminus E_{\bar f} \supset D_{\bar\gamma}$$

is a closed set such that $\{D_{\bar f}, \{D_\gamma\}_\gamma\}$ has the finite intersection property:

$$D_{\bar f} \cap D_{\gamma_1} \cap \dots \cap D_{\gamma_n} \supset D_{\bar\gamma} \cap D_{\gamma_1} \cap \dots \cap D_{\gamma_n} \neq \emptyset.$$

Thus $D_{\bar f} \in \{D_\gamma\}_\gamma$.

Consider then the sets

$$D_{\bar f, \alpha_i} = \Big\{ f \in \Pi_\alpha X_\alpha : f(\alpha_i) \in X_{\alpha_i} \setminus E_{\bar f, \alpha_i} \Big\}.$$

If none of the $D_{\bar f, \alpha_i}$ belongs in the family $\{D_\gamma\}_\gamma$, then there exists closed sets $D_1, \dots, D_n \in \{D_\gamma\}_\gamma$ so that

$$D_{\bar f, \alpha_i} \cap D_i = \emptyset.$$

But since $D_{\bar f} = \cup_i D_{\bar f, \alpha_i}$ it follows that

$$D_{\bar f} \cap D_1 \cap \cdots \cap D_n = \emptyset,$$

contradicting the fact that $\{D_\gamma\}_\gamma$ has the finite intersection property. Thus some $D_{\alpha_i} \in \{D_\gamma\}_\gamma$, but this contradicts the construction of $\bar f$.

Thus $\bar f$ is in the closure of all D_γ, and since these sets are closed it follows $\bar f \in D_\gamma$ for all γ. In particular

$$\bigcap_{\beta \in B} C_\beta \supset \bigcap_{\gamma \in \Gamma} D_\gamma \supset \{\bar f\}.$$

Thus $\Pi_\alpha X_\alpha$ is compact. □

We say that a topological space is *normal* if for any two disjoint closed sets F_1, F_2 there exists disjoint neighborhood G_1, G_2. Clearly a normal topological space in which each point is closed is Hausdorff separable.

Theorem 2.2 (Urysohn's Lemma)**.** *Let A, B disjoint closed sets in a normal space X. Then there exists a real valued continuous function $f(t)$ on X such that*

$$0 \leq f \leq 1, \quad f|_A \equiv 1, \ f|_B \equiv 0.$$

Proof. By assumption there are $\mathring{A}$, $\mathring{B}$ disjoint neighborhood of A, B, and define

$$A_{1/2} = \overline{\mathring{A}}, \quad B_{1/2} = \overline{\mathring{B}}$$

Then $B_{1/2}$ is disjoint from A, because it is contained in $X \setminus \mathring{A}_{1/2}$. Similarly $A_{1/2}$ is contained in $X \setminus \mathring{B}_{1/2}$. We can thus write

$$A \subset \mathring{A}_{1/2} \subset A_{1/2} \subset X \setminus \mathring{B}_{1/2} \subset X \setminus \mathring{B}, \ B \subset \mathring{B}_{1/2} \subset X \setminus \mathring{A}_{1/2} \subset X \setminus \mathring{A}.$$

We can repeat the procedure with the sets A, $X \setminus \mathring{A}$ obtaining $A_{3/4}$ and $B_{3/4}$, and also with B, $X \setminus \mathring{B}$, obtaining $A_{1/4}$, $B_{1/4}$. Note that by construction we have

$$X \setminus \mathring{B}_{3/4} \subset A_{1/2} \subset X \setminus \mathring{B}_{1/2}, \quad X \setminus \mathring{A}_{1/2} \subset B_{3/4} \subset X \setminus \mathring{A}_{3/4}.$$

Proceding with this construction, for all number of the form $x = 2^{-n}k$, with $k \in \{1, 3, 2^n - 1\}$, there are closed sets $A_{k2^{-n}}$, $B_{k2^{-n}}$ such that for $k2^{-n} < k'2^{-n'}$

$$A = A_1, \quad B = B_0, \quad A_{k'2^{-n'}} \subset A_{k2^{-n}}, \ B_{k,2^{-n}} \subset B_{k'2^{-n'}},$$

and

$$X \setminus \mathring{B}_{k'2^{-n'}} \subset A_{k2^{-n}} \subset X \setminus \mathring{B}_{k2^{-n}}, \quad X \setminus \mathring{A}_{k2^{-n}} \subset B_{k'2^{-n'}} \subset X \setminus \mathring{A}_{k'2^{-n'}}.$$

Define now

$$\phi(x) = \sup\Big\{k2^{-n} \colon x \in A_{k2^{-n}}\Big\}, \quad \psi(x) = \inf\Big\{k2^{-n} \colon x \in B_{k2^{-n}}\Big\}.$$

Clearly if $x \notin A_{k2^{-n}}$, then $x \in B_{k'2^{-n'}}$ for $k2^{-n} < k'2^{-n'}$, so that one see that if $\phi(x) < k2^{-n}$ then $\psi(x) \leq k2^{-n}$. Similarly, if $x \notin B_{k'2^{-n'}}$, then $x \in A_{k2^{-n}}$ for $k2^{-n} < k'2^{-n'}$, so that $\psi(x) > k'2^{-n'}$ implies $\phi(x) \geq k'2^{-n'}$. Then these functions coincides.

By construction

$$\phi^{-1}([b,1]) = \bigcap_{k2^{-n} \leq b} A_{k2^{-n}}, \quad \phi^{-1}([0,a]) = \bigcap_{k2^{-n} \geq a} B_{k2^{-n}},$$

are closed sets. Hence for $0 < a < b < 1$,

$$\phi^{-1}((a,b)) = X \setminus \Big(\phi^{-1}([0,a]) \cup \phi^{-1}([b,1])\Big)$$

is open. Similarly one can treat the cases $a = 0$ or $b = 1$. □

The converse of the above theorem is straightforward.

3. Metric spaces

The set X is a *metric space* if there is function distance $d : X \times X \mapsto \mathbb{R}$ such that

(1) $d(x,y) \geq 0$, $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \leq d(x,z) + d(z,y)$.

We say that a topology τ is generated by the metric d if a base of open sets is given by the open balls $B_r(x) = \{y; d(x,y) < r\}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in the metric space (X,d) is *convergent* to $\bar{x} \in X$ if

$$\forall \epsilon > 0 \text{ there is } n(\epsilon) \text{ such that if } n > n(\epsilon) \implies d(x_n, \bar{x}) < \epsilon. \tag{3.1}$$

The limit, if it exists, is unique. We write in this case

$$\lim_{n\to\infty} x_n = \bar{x}. \tag{3.2}$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if

$$\forall \epsilon > 0 \text{ there is } n(\epsilon) \text{ such that if } n, m > n(\epsilon) \implies d(x_n, x_m) < \epsilon. \tag{3.3}$$

We say that metric space is *complete* if every Cauchy sequence is convergent, i.e. if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence there is a $\bar{x}$ such that

$$\lim_{n\to\infty} x_n = \bar{x}.$$

In complete metric spaces we can prove the following lemma:

Lemma 3.1 (Baire)**.** *If X is a complete metric space, X_n closed sets with empty interior, then $\cup_n X_n$ has empty interior.*

We can rewrite the lemma as

If O_n are dense open sets in a complete metric space X, then $\cap_n O_n$ is dense.

Proof. Let $O_n = X \setminus X_n$ open and dense. Take $x \in X$, and its neighborhood $B(x,r)$. By the assumption that the O_n are open and dense, then $O_1 \cap B(x,r) \supset B(x_1,r_1)$ for some x_1, $r_1 > 0$. We can choose $r_1 < r/2$.

Since O_2 is dense, there is a ball $B(x_2,r_2) \subset O_2 \cap B(x_1,r_1)$. We can choose $r_2 \leq r_1/2 < r/4$.

Going on we obtain a sequence of x_n, r_n such that

$$B(x_n,r_n) \subset O_n \cap B(x,r), \quad B(x_n,r_n) \subset O_m, \ \forall m \leq n.$$

and $r_n \leq r_{n-1}/2 < 2^{-n} r$.

The points x_n are a Cauchy sequence: for $m \leq m$

$$d(x_n,x_m) \leq \sum_{i=0}^{n-m-1} d(x_{m+i+1}, x_{m+i}) < \sum_{i=1}^{n-m-1} 2^{-m-i} r = \big(2^{1-m} - 2^{1-n}\big) r.$$

Since X is complete there is a limit $\bar{x}$. By construction $\bar{x}$ is in all O_n and in the neighborhood $B(x,r)$ of x. □

Some terminology: if S is a topological space, a set $A \subset S$ is said to be of *first category* if it is the countable union of nowhere dense closed subset. The other sets are of *second category*. In particular we have that *complete metric spaces are of second category.*

Proposition 3.2 (uniform boundedness principle)**.** *In a complete metric space X, if $f_\nu : X \mapsto \mathbb{R}$, $\nu \in \Upsilon$ are a collection of continuous pointwise bounded functions,*

$$|f_\nu(x)| \leq M(x), \quad \forall \nu \in \Upsilon,$$

then $|f_\nu|$ are uniformly bounded by some constant M in some non empty open set $O \subset X$.

Proof. Define the closed sets
$$C_n = \bigcap_{\nu \in \Upsilon} f_\nu^{-1}([-n, n]).$$
Then we have that $\cup_n C_n = X$, because of local boundedness. Then by Baire's Lemma we conclude that some C_N has non empty interior: there is open set $O \subset C_N$, i.e.
$$|f_\nu(x)| \leq N \quad \forall x \in O, \nu \in \Upsilon.$$
□

4. Contraction principle

Let (X, d) be a complete metric space. A function $T : X \mapsto X$ is said to be a contraction mapping if there is a constant q, $0 \leq q < 1$, such that
$$d(Tx, Ty) \leq q \cdot d(x, y)$$
for all $x, y \in X$.

Theorem 4.1. *Every contraction has a unique fixed point.*

The proof, based on the fact that the sequence $\{T^n x\}_{n \in \mathbb{N}}$ is Cauchy, is left as an exercise.

There is an estimate to this fixed point that can be useful in applications. Let T be a contraction mapping on (X, d) with constant q and unique fixed point $x^* \in X$. For any $x_0 \in X$, define recursively the following sequence
$$x_n = T(x_{n-1}).$$
The following inequality holds:
$$d(x^*, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0).$$

5. Exercises

(1) Prove that the topology on the product space $\Pi_\alpha X_\alpha$ introduced in Section 1 is a topology.
(2) Prove that if $f : S \mapsto T$ is continuous, S, T topological spaces, then $f(K)$ is compact if K is compact.
(3) Prove *Alexandroff one point compactification* of a locally compact space, i.e. if X is locally compact, it can be continuously embedded into $\tilde{X} = X \cup \{y_\infty\}$, with $y_\infty \notin X$, such that $\tilde{X}$ is compact.
(Add to the topology the open neighborhood of y_∞ defined as $\{y_\infty\} \cup (X \setminus \bar{O})$, with O relatively compact open subset of X.)
(4) A system of sets has the finite intersection property if every finite collection has a non empty intersection. Prove that a topological space X is compact iff for every system of closed sets M_α, $\alpha \in A$ with the finite intersection property, then
$$\bigcap_{\alpha \in A} M_\alpha \neq \emptyset.$$
(5) Prove that if X is compact and Hausdorff, and $\{C_\alpha\}_\alpha$ is a maximal collection of closed sets with finite intersection property, then
$$x_1, x_2 \in \bigcap_\alpha C_\alpha \quad \Longrightarrow \quad x_1 = x_2.$$
(6) Prove that in a compact metric space every sequence has at least one limit.
(7) Consider the space $A = \{f : [0,1] \mapsto [0,1]\}$ with the topology of the product space
$$[0,1]^{[0,1]} = \Pi_{\alpha \in [0,1]} [0,1] = \big\{ f : [0,1] \mapsto [0,1] \big\}.$$
Find a sequence which is not convergent while A is compact. Deduce that A is not metrizable.
(8) Prove that a compact Hausdorff space X is normal.
(9) Prove that locally compact metric spaces are of second category.
(10) Let X be a metric space and $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of closed subsets of X such that $K_n \subset K_{n-1}$. Prove that

- if X is complete and $\mathrm{diam}(K_n) \to 0$, then $\cap_n K_n$ contains a unique point;
- If one K_n is compact, then $\cap_n K_n \neq \emptyset$.

(11) Let X be a complete metric space and, for $\lambda \in [0,1]$, let $T_\lambda : X \mapsto X$ be such that
$$d(T_\lambda x, T_\lambda y) \leq \frac{1}{2} d(x,y).$$
Denote with x_λ the fixed point of T_λ. Prove that if for every x in a dense set we have
$$\lim_{\lambda \to 0} T_\lambda(x) = T_0(x),$$
then $x_\lambda \to x_0$.

(12) Let X be a complete metric space, and $f : X \mapsto X$. Assume that for some $k < 1$
$$d(f(x), f(y)) \leq k d(x,y) \quad \forall x, y, \in \bar{B}(\bar{x}, r) = \Big\{ x \colon d(x, \bar{x}) \leq r \Big\}.$$
Prove that if $d(\bar{x}, f(\bar{x})) \leq r(1-k)$, then f admits a unique fixed point in $\bar{B}(\bar{x}, r)$.

(13) Prove that there is a unique continuous function $\phi : [0,1] \mapsto [0,1]$ such that
$$\phi(x) = \frac{1}{2} \int_0^1 \sin(x + \phi(y)) dy \quad x \in [0,1].$$

(14) Let $K \in C([0,2]; \mathbb{R})$ be positive and strictly decreasing, with $K(0) = 1$. Prove that for every $h \in C([0,1]; \mathbb{R})$ there is a unique solution $u \in C([0,1]; \mathbb{R})$ to the equation
$$u(x) = h(x) + \int_0^1 K(x+y) u(y) dy \quad x \in [0,1].$$