EXAMINATION 7

(1) Let X be an infinite dimensional Banach space and consider the duality application:
$$X \ni u \mapsto \Big\{ \ell \in X^* : \|\ell\|_{X^*} = 1, \ell u = \|u\| \Big\}.$$
- Construct two sequences $\{u_n\}_{n\in\mathbb{N}} \subset X$, $\{\ell_n\}_{n\in\mathbb{N}} \subset X^*$ such that
$$\ell_n u_m = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad \text{e} \quad \|u_n\|_X = 1 \ \forall n \in \mathbb{N}.$$
- Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers, and consider the operator
$$\mathbf{M} : X \mapsto X, \quad \mathbf{M}u = \sum_{n=1}^{\infty} \alpha_n (\ell_n u) u_n.$$
Give conditions on $\{\alpha_n\}_n$ so that $\mathbf{M}$ is well defined.
- Is $\mathbf{M}$ compact? Compute its spectrum.
- Prove that if $X = \ell^1$, $u_n = e_{n+1}$, $\ell_n = e_1 + e_{n+1}$, with
$$e_n : \mathbb{N} \mapsto \mathbb{C}, \quad e_n(i) = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$$
then the assumption $|\alpha_n| \leq C < \infty$ does not imply that $\mathbf{M}$ is well defined.
- Give a necessary and sufficient condition on $\{\alpha_n\}_n$ if $X = H$ Hilbert.

(2) Let A be a compact set in $\mathbb{C}$ and
$$K_A = \Big\{ u \in \ell^p, p = 1, \infty : u(t) \in A \Big\}.$$
- Prove that K_A is weakly closed in ℓ^1 and weakly* closed in ℓ^∞.
- Write the projection operator for ℓ^1. Give an example for which there is not a unique projection for $u \in \ell^\infty$.
- Consider the subspace of ℓ^1
$$K = \Big\{ u \in \ell^1 : \sum_n u(n) = 0 \Big\}.$$
Show that K is not empty, closed and convex, but the projection of an element $u \notin K$ on K is not unique.

(3) Consider $L^\infty((0,1),\mathbb{C})$ with the sup norm.
- Show that $C([0,1],\mathbb{C})$ is a closed subspace of L^∞.
- Show that if $u \in L^\infty$ satisfies
$$\int_0^1 u(t) f(t) = 0 \quad \forall f \in L^1((0,1),\mathbb{C}),$$
then $u \equiv 0$.
- Let Z be a closed linear complement of $C([0,1])$ in L^∞. Show that
$$L^\infty / Z = C([0,1]), \quad (L^\infty/Z)^* = \Big\{ \ell \in (L^\infty)^* : \ell z = 0 \ \forall z \in Z \Big\},$$
and deduce that $Z = \{0\}$.

(4) Consider the space $C((-1,1),\mathbb{C})$ of continuous functions in $(-1,1)$ with the family of seminorms
$$p_n(u) = \max_{|x|\leq 1-1/n} \big|u(x)\big| = \|u\|_{C(|x|\leq 1-1/n)}.$$
- Show that $C((-1,1))$ is a Fréchet space.

- Let C^* be the dual space of $C((-1,1)$. Show that
$$P_n(\ell) = \sup\Big\{\ell u : p_n(u) \leq 1\Big\}$$
makes C^* a Fréchet space. The topology generated by P_n is the *strong topology.*
- Find a sequence ℓ_n weakly* convergent, but not strongly convergent.

(5) Consider the space of bounded Radon measures on $\mathbb{R}$, considered as the dual space of
$$C_0(\mathbb{R};\mathbb{C}) = \Big\{u \text{ continue}, \lim_{|x|\to\infty} u(x) = 0\Big\}.$$

- Construct a sequence μ_n with norm 1 but weakly* convergent to 0. Describe the set
$$\Big\{\mu(\mathbb{R}), \mu = \lim_{n\to\infty} \mu_n \text{ weakly* }\Big\}.$$
- Show that is the support of all μ_n, μ_n positive measures with norm 1, is contained in a compact set, then every weak* limit has norm 1.

(6) Consider the Banach space $C([0,1],\mathbb{C})$ with the sup norm, and two sequences of functions
$$\{g_1,\dots,g_n\}, \quad \{h_1,\dots,h_n\},$$
Moreover define the operator
$$(\mathbf{M}u)(t) = \sum_{i=1}^{n} g_i(t) \int_0^1 u(s)h_i(s)ds.$$

- Show that $\mathbf{M}$ is compact.
- Compute eigenvalues and eigenvectors.
- Give conditions such that
$$u - \mathbf{M}u = v$$
has a solution.