EXAMINATION 6

(1) On the complete metric space (X,d) consider the set of maps

$$\mathbf{T}_\lambda : X \mapsto X, \quad d(\mathbf{T}_\lambda(x), \mathbf{T}_\lambda(y)) \leq \lambda d(x,y),$$

with $\lambda \in [0,1)$. Denote with x_λ the fixed point of $\mathbf{T}_\lambda$.

- Show that if $\lambda \mapsto \mathbf{T}_\lambda(x)$ is pointwise continuous, then $\lambda \mapsto x_\lambda$ is continuous.
- Give an example for which $\lambda \to 1$ but x_λ does not converge.
- Show that the set of contractions is not open in the uniform topology

$$D(\mathbf{T}, \mathbf{S}) = \sup_{x \in X} d(\mathbf{T}(x), \mathbf{S}(x)).$$

In the case X Banach, suggest a suitable topology such that the set of contractions is open and prove it.

(2) Consider the space of simple integrable functions

$$X = \left\{ u : [0,1] \mapsto \mathbb{C} : \; u = \sum_i \alpha_i \chi\{x : a_i < x < b_i\}, \; \alpha_i \in \mathbb{C}, \; a_i, b_i \in \mathbb{Q} \cap [0,1] \right\},$$

with the norm

$$\|u\|_{L^1} = \int_0^1 |u(t)| dt.$$

- Which is the closure of X? Is it separable?
- Prove that $\sigma(X^*, X)$ is metrizable and write an explicit metric.
- Construct a family of linear functionals which are pointwise bounded over X but not uniformly bounded.

(3) In the Hilbert space $L^2((-1,1), \mathbb{C})$ consider the set

$$K = \left\{ u \in L^2 : \; |u(t)| \leq 1 \text{ a.e.} \right\}.$$

- Study if K is convex, closed, compact and absorbing.
- Show that

$$(\mathbf{P}_K u)(t) = \begin{cases} u(t)/|u(t)| & |u(t)| > 1 \\ u(t) & |u(t)| \leq 1 \end{cases}$$

is the projection operator over K.

- Prove that if $K \subset E$, with E closed subspace of H Hilbert, then

$$\mathbf{P}_K = \mathbf{P}_K \circ \mathbf{P}_E.$$

- Write the projection operator $\mathbf{P}$ over $K \cap E$, where $E = \{u : u(t) = u(-t) \text{ a.e.}\}$.

(4) Let $A \subset \mathbb{C}$ be a compact set, and consider the subset of L^p, $p \in [1, \infty)$,

$$K_A = \left\{ u \in L^p((0,1), \mathbb{C}) : u(t) \in A \text{ a.e.} \right\}.$$

- Show that K_A is strongly closed in L^p.
- Prove that if A is convex, then K_A is weakly closed L^p.
- K_A satisfies the assumptions of Krein Milman theorem? Write its extremal points.

(5) Let $a_i \in \mathbb{R}$ be a sequence of real numbers, and $1 < q < p < \infty$. Assume that

$$au = \{a_1 u_1, a_2 u_2, \dots\} \in \ell^q \qquad \forall\; u = \{u_1, u_2, \dots\} \in \ell^p.$$

Show that $a \in \ell^r$ with $r = pq/(p-q)$. What happens for $p = +\infty$?

(6) Consider the sequence of functions

$$u_n(x) = \chi_{(n,n+1)}(x) \in L^1(\mathbf{R}).$$

Show that there are no converging subsequences $u_{n_k}(x)$ for the topology $\sigma(L^1, L^\infty)$.