EXAMINATION 5

(1) Let H be a separable Hilbert space, let $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal and consider the space

$$X = \left\{ \sum_{i \text{ finite}} \alpha_i e_i, \alpha_i \in \mathbb{C} \right\} = \mathrm{span}\{e_n, n \in \mathbb{N}\},$$

with the norm induced by H.

- Show that X is of first category in H, and find an explicit representation of X^*. Is X^* reflexive?
- Find a sequence $\{\ell_n\}_n \subset X^*$ converging to 0 in the weak* topology $\sigma(X^*, X)$ such that $\|\ell_n\|_{X^*} \to \infty$.
- Prove that if $\|\ell_n\|_{X^*} \leq C < \infty$ for all n, then $\ell_n \rightharpoonup \ell$ in $\sigma(X^*, X)$ implies that

$$\ell_n u \to \ell u, \quad \forall u \in H.$$

(2) In the Banach space $C([0,1], \mathbb{C})$ consider the operator

$$(\mathbf{M}u)(t) = \int_0^1 g(t,s)u(s)ds, \quad g(t,s) \in C^1(\mathbb{R}^2, \mathbb{C}).$$

- Show that $\mathbf{M}$ is compact.
- In the special case

$$g(t,s) = g(t)h(s),$$

compute the spectrum of $\mathbf{M}$. Is $\lambda = 0$ an eigenvalue of $\mathbf{M}$?
- Which conditions should $g(t,x)$ satisfy if we require $\mathbf{M}$ to be self-adjoint on the Hilbert space $L^2((0,1), \mathbb{C})$?

(3) Consider the Banach space ℓ^∞.

- Show that $\ell^\infty = (\ell^1)^*$.
- Deduce the existence of extremal point of ℓ^∞ and write explicitly all the extremal points.
- Consider the set of extremal points

$$\ell^\infty \ni \{u_k\}_{k\in\mathbb{N}}, \quad u_k(n) = \begin{cases} 0 & n \neq k \\ 1 & n = k. \end{cases}$$

Show that $\overline{B_{\ell^\infty}(0,1)}$ is the weak* closure of $\{u_k\}_k$ in the topology $\sigma(\ell^\infty, \ell^1)$, but not in the topology $\sigma(\ell^\infty, (\ell^\infty)^*)$.

(4) Consider the Banach space $C([0,1], \mathbb{C})$ and the operator

$$\mathbf{M} : C([0,1], \mathbb{C}) \mapsto C([0,1], \mathbb{C}), \quad (\mathbf{M}_{\bar{s}}u)(t) = u(t) - u(\bar{s}), \quad \bar{s} \in [0,1].$$

- Compute $N_{\mathbf{M}_{\bar{s}}}$, $R_{\mathbf{M}_{\bar{s}}}$.
- Find the index of $\mathbf{M}_{\bar{s}}$. Is $\mathbf{M}_{\bar{s}}$ compact?
- Find the conditions such that

$$\mathbf{M}_{\bar{s}}u = v,$$

admits at least one solution, and in that case find all solutions.

(5) *i)* Let f_n be a sequence of $L^2(0,1)$ such that

$$f_n \rightharpoonup f \quad \text{and} \quad ||f_n||_2 \to ||f||_2.$$

Show that $f_n \to f$ strongly in L^2.

ii) Construct a sequence $f_n \in L^1(0,1)$, $f_n \geq 0$, such that $f_n \rightharpoonup f$ in $\sigma(L^1, L^\infty)$, $||f_n||_1 \to ||f||_1$ but f_n does not converge in norm to f in L^1.