EXAMINATION 4

(1) Let $K \subset H$ closed, convex and non empty, $P_K : H \mapsto H$ be the projection operator on K.
- Show that $\forall z \in K$
$$(x - P_K x, z - P_K x) \leq 0.$$
- Prove that if $y \in K$ and
$$(x - y, z - y) \leq 0 \quad \forall z \in K,$$
then $y = P_K x$.
- Deduce that for all $x, w \in H$
$$\|P_K x - P_K w\| \leq \|x - w\|.$$
- Fix $H = \ell^2$ and
$$K = \Big\{ u \in \ell^2, u_{2n-1} = 2^n u_{2n} \Big\}.$$
Prove that K is closed, convex, find $K^\perp$ and write the projector P_K. Is P_K linear?
- Show that for ℓ^p, $p \neq 2$, it can happen that $\|P_K x - P_K w\|_{\ell^p} > \|x - y\|_{\ell^p}$.

(2) Consider the space ℓ^2 and the space
$$h^1 = \Big\{ u : \mathbb{N} \mapsto \mathbb{C} : \{nu(n)\} \in \ell^2 \Big\},$$
with the scalar products
$$(u, v)_{\ell^2} = \sum_n u_n \bar{v}_n, \quad (u, v)_{h^1} = \sum_n n^2 u_n \bar{v}_n.$$
- Show that h^1 is an Hilbert space, which can be embedded into ℓ^2.
- Prove that $h^1 \hookrightarrow\hookrightarrow \ell^2$, i,e. the embedding is compact.
- Let $v \in \ell^2$, and consider the linear functional on h^1 defined by
$$(u, v) = \sum_n u(n) \bar{v}(n).$$
Find the Riesz-Fréchet representation v_{h_1} of v in h^1. Show that the above representation of linear functionals is meaningful if $u \in h^1$ and $\{v_n/n\}_n \in \ell^2$. Thus $(\ell^2)^* h^1$.
- Prove conversely that every $w \in h^1$ can be extended to a linear functional over ℓ^2 by
$$wu = \sum_n u(n) w(n),$$
i.e. $(h^1)^* \subset \ell^2$.

(3) Let H be a separable Hilbert space with base $\{e_n\}_{n \in \mathbb{N}}$. Let $\mathbf{M} : H \mapsto H$ be such that
$$\sum_n \|\mathbf{M} e_n\|^2 < \infty.$$
- Show that $\mathbf{M}$ is compact.
- Prove that the quantity
$$\|\mathbf{M}\|_{\mathrm{HS}} = \bigg(\sum_n \|T e_n\|^2 \bigg)^{1/2}$$
is independent on the choice of the base $\{e_n\}_{n \in \mathbb{N}}$.
- Show that the subspace of $\mathcal{L}(H)$
$$\mathcal{H} = \Big\{ \mathbf{M} : H \mapsto H \colon \|\mathbf{M}\|_{\mathrm{HS}} < \infty \Big\}$$
is a closed subspace w.r.t. $\| \cdot \|_{\mathrm{HS}}$.

(4) Let $\mathbf{M} \in \mathcal{L}(X, Y)$, X, Y Banach, with finite index. We will denote this set by $\mathcal{F}(X, Y)$.

- Show that from the fact that $R_{\mathbf{M}}$ has finite codimension, then $R_{\mathbf{M}}$ is closed.
- Show that the pseudo inverse belongs to $\mathcal{L}(Y, X)$.
- Assume that there is $\mathbf{L}$ such that
$$\mathbf{M} \circ \mathbf{L} - \mathbf{I}_Y \in \mathcal{K}(Y), \quad \mathbf{L} \circ \mathbf{M} - \mathbf{I}_X \in \mathcal{K}(X).$$
Prove that $\mathbf{M} \in \mathcal{F}(X, Y)$.
- Deduce that $\mathcal{F}(X, Y)$ is open w.r.t. the operator norm.
- Using the stability of the index, prove that $\mathrm{Ind}\mathbf{M}$ is constant w.r.t. the operator norm, hence is constant on every connected component of $\mathcal{F}(X, Y)$.

(5) Consider the linear operator
$$\mathbf{M} : C([0,1]; \mathbb{C}) \mapsto C([0,1]; \mathbb{C}), \quad \mathbf{M}(u)(t) = \int_0^t h(s)u(s)ds,$$
with $h \in C([0,1], \mathbb{C})$.
- Show that $\mathbf{M}$ is compact, and find its spectrum.
- Is $\mathbf{M} : L^2(0,1) \mapsto L^2(0,1)$ self adjoint?
- Assume that $h(s) = 1$. Find explicitly the solutions to
$$u - \mathbf{M}u = f$$
for $f \in C([0,1], \mathbb{C})$.