## EXAMINATION 3

(1) Consider the topological vector space

$$X = \Big\{u : [0, 1] \mapsto \mathbb{C}, u(x) \neq 0 \text{ only in a finite number of points}\Big\},$$

with the topology inherited by the product space $\mathbb{C}^{\mathbb{N}}$.

- Write explicitly a local base $\mathcal{B}$ of $X$.
- A sequence $\{u_n\}$ is a Cauchy sequence if

$$\forall V \in \mathcal{B} \text{ there is } N \text{ such that } n, m > N \quad \Longrightarrow \quad x_n - x_m \in V.$$

  Show that $X$ can be continuously embedded into a "complete" topological vector space $\tilde{X}$ (i.e. a topological vector space in which every Cauchy sequence has a limit). Find an explicit representation of $\tilde{X}$.
- Prove that every continuous functional on $X$ is of the form

$$f(u) = \sum_{i=1}^{n} a_i u(x_i).$$

(2) Consider the Banach space

$$\ell^\infty = \Big\{u : \mathbb{N} \mapsto \mathbb{C}, \|u\|_{\ell^\infty} = \sup_n |u_n| < \infty\Big\}.$$

- Show that the family of functional

$$\ell^1 = \Big\{v : \mathbb{N} \mapsto \mathbb{C}, \|v\|_{\ell^1} = \sum_{n=1}^{\infty} |v_n| < \infty\Big\}.$$

  is separating on $X$, and thus generates a vector topology $\tau$.
- By means of the Hahn-Banach theorem, extend to $\ell^\infty$ the functional

$$\ell u = \lim_{n \to +\infty} u_n,$$

  defined on the sequences $u$ which has limit. Thus there are linear continuous functionals on $\ell^\infty$ which are not in $\ell^1$. Denote this extended functional by LIM.
- Consider the set

$$K = \Big\{u : \big|\text{LIM}u\big| \leq 1\Big\}.$$

  Show that $K$ is absorbing, however its interior is $\emptyset$ w.r.t. to the vector topology $\tau$.

(3) For $p \in (0, \infty)$, let $\ell^p$ be the space

$$\ell^p = \Big\{u : \mathbb{N} \mapsto \mathbb{C} : \sum_{n=1}^{+\infty} |u(n)|^p < \infty\Big\}.$$

If $p \geq 1$, define the norm

$$\|u\|_p = \bigg(\sum_{n=1}^{\infty} |u(n)|^p\bigg)^{1/p}.$$

For $p = \infty$, define

$$\ell^p = \Big\{u : \mathbb{N} \mapsto \mathbb{C} : \|u\|_\infty = \sup_n |u(n)| < \infty\Big\}.$$

- Prove that $\ell^p$, $p \in [1, \infty]$ is a Banach space, and $\ell^p$, $p \in (0, 1)$ i an $F$ space with the metric

$$\Delta(u, v) = \sum_{i=1}^{+\infty} |u(n) - v(n)|.$$

- Prove that for $p \in [1,\infty)$, the dual space of $\ell^p$ is $\ell^q$, with
$$\frac{1}{p} + \frac{1}{q} = 1.$$
In particular $\ell^p$, $p \in (1,\infty)$, is reflexive.
- Show by the Hahn Banach theorem that the dual of $\ell^\infty$ is strictly larger than $\ell^1$.
- Prove that for $p \in (1+\infty)$ there are sequences which converges weakly but not strongly.
- Show that in $\ell^1$ every weak convergent sequence is strongly convergent.
- Prove that $\ell^p$, $p \in (0,1)$ is not locally convex, but $(\ell^p)^* = \ell^\infty$ separates points on $\ell^p$. In particular the set
$$\left\{ \sum_{i=1}^{+\infty} |u(n)| < 1 \right\}$$
is not bounded for the original topology, but it is weakly bounded.

(4) Let $X$ be a Fréchet space with invariant metric $d$, $X^*$ its dual. Define the family of seminorms on $X^*$ by
$$p_n(\ell) = \sup\Big\{ |\ell x| : d(x,0) < n^{-1} \Big\}.$$
- Show that the family of seminorms is separating, hence generates a metrizable topology, which makes $X^*$ a Fréchet space. This is the *strong topology* of $X^*$.
- Prove the the strong topology is stronger than the weak* topology.
- Prove that weakly bounded sets are strongly bounded in $X$.
- Prove that weakly* bounded sets are strongly bounded in $X^*$.
- Show that if $X$ is normed, then the strong topology coincides with the norm topology of $X^*$, and show with an example that the second dual $X^{**}$ of the Fréchet space $X$ may not coincide with $X$.

(5) Let $(X, (\cdot,\cdot))$ be a separable Hilbert space with a complete orthonormal base $\{e_n\}_n$. Define
$$\|x\|^2 = (x,x), \quad |x| = \sum_n \frac{1}{n} |(x, e_n)|.$$
- Prove that $|\cdot|$ is a norm on $X$, but $X$ is not complete.
- Show that $\{x_k\}_k$ converges weakly in $H$ if and only if $\|x_k\|$ is bounded and $|x_k - x| \to 0$.
- Prove that $\{\|x\| < 1\}$ is compact for $(X, |\cdot|)$.