EXAMINATION 2

(1) Let X be a Banach space, and Y be a closed subspace.
- Show that X/Y is a Banach space.
- On the dual space X^*, consider the set (annihilator of Y)
$$X^* \supset Y^\perp = \Big\{\ell \in X^* \colon \ell y = 0 \ \forall y \in Y\Big\}.$$
Show that $Y^\perp$ is closed, and $(X/Y)^*$ is isomorphic to $Y^\perp$.
- If
$$X = \ell^\infty = \Big\{u = (u_1, u_2, \dots) \colon \|u\| = \sup_i |u_i| < \infty\Big\},$$
$$Y = c_0 = \Big\{u = (u_1, u_2, \dots) \colon \lim u_n = 0\Big\},$$
show that Y is closed subspace of X. Represent $(\ell^\infty/c_0)^*$ as the continuous functional on ℓ^∞ such that ("supported at $+\infty$")
$$\ell e_n = 0, \quad (e_n)_i = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$$
- Let Z be a linear complement of c_0 in ℓ^∞. Assume Z closed, so that $(\ell^\infty/Z)^* = (c_0)^* = \ell^1$. Deduce that $Z = 0$, so that c_0 does not have any closed linear complement in ℓ^∞.

(2) For $\alpha \in (0,1)$, consider the space
$$X_\alpha = \bigg\{u \in C([0,1];\mathbb{R}) \colon \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty\bigg\},$$
with the norm
$$\|u\|_\alpha = \sup_x |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$
- Show that X_α is a Banach space, and that the unit ball
$$\Big\{u \in C([0,1];\mathbb{R}), \|u\|_\alpha \leq 1\Big\}$$
is compact in $C([0,1];\mathbb{R})$ with the sup norm.
- Prove that the sets
$$O_n = \bigg\{u \in X_\alpha \colon \exists \{I_i\}_i \text{ open covering of } [0,1] \text{ with } |I_i| < 1/n$$
$$\text{such that } \frac{|u(x_{i-}) - u(x_{i+})|}{|x_{i-} - x_{i+}|} > n, \ x_{i-} = \inf I_i, x_{i+} = \sup I_i\bigg\}$$
are open and dense in X_α.
- Deduce the existence of a function $u \in X_\alpha$ not differentiable in every point $x \in [0,1]$.

(3) Consider the space
$$X = \bigg\{u \in C([0,1];\mathbb{R}) : \frac{du}{dt} \in C([0,1],\mathbb{R})\bigg\}$$
with the norm
$$\|u\| = \sup_{t \in [0,1]} |u(t)|.$$
- Show that X is of first category, i.e. countable union of closed set with empty interior.
- Consider the operator
$$\mathbf{A} : X \mapsto C([0,1];\mathbb{R}) \quad \mathbf{A}u = \frac{du}{dt}$$
Show that this operator is closed graph, but it is not continuous.

- Show that the completion of X is $C([0,1];\mathbb{R})$ and that
$$\mathcal{D}(\mathbf{A}) = \Big\{ u \in C([0,1];\mathbb{R}) : \mathbf{A}u \in C([0,1];\mathbb{R}) \text{ is well defined} \Big\},$$
is dense in $C([0,1];\mathbb{R})$ and $\mathbf{A}$ is closed graph in $C([0,1];\mathbb{R}) \times C([0,1];\mathbb{R})$.

(4) Let X be a normed space, $Y \subsetneq X$ a closed subspace.

- Prove that for all $\alpha \in (0,1)$ there is a point $x_\alpha \in X \setminus Y$ such that
$$\|x_\alpha\| = 1 \quad d(x_\alpha, Y) = \inf_{y \in Y} \big\{ \|x_\alpha - y\| \big\} = \alpha.$$
(If $x_0 \notin Y$, then $d(x_0, Y) = c > 0$, and hence there is y_α such that $\|x_0 - y_\alpha\| = c/\alpha$. Consider $(\alpha/c)(x_0 - y_\alpha)$.)
- As a consequence prove that if X has infinite dimension, then
$$B(0,1) = \Big\{ x \in X \colon \|x\| \leq 1 \Big\}$$
is closed but not compact in the topology generated by $\|\cdot\|$.
- Let X be the Fréchet space
$$X = \bigg\{ u = (u_1, u_2, \dots), d(u,0) = \sum_{i=1}^{\infty} 2^{-i} \frac{|u_i|}{1 + |u_i|} \bigg\}.$$
$$Y_{m,n} = \Big\{ u \in X : u_m + u_n = 0 \Big\} \subset X.$$
Prove that $Y_{m,n}$ is a closed subspace of X, and
$$\forall x \in X \quad d(x, Y_{m,n}) \leq 2^{-n} + 2^{-m}.$$

(5) Consider $X = C^1([0,1];\mathbb{R})$, with the norm
$$\|u\|_{L^1} = \int_0^1 |u(t)| dt,$$
and the linear operator $\mathbf{M} : X \mapsto X$ defined by
$$\mathbf{M}u = \frac{du}{dt}.$$

- Show that
$$N_{\mathbf{M}} = \Big\{ u = \text{ constant} \Big\},$$
and thus N_M is closed.
- Prove that $\mathbf{M}$ is unbounded, thus not continuous. Is its graph closed?
- In the completion $\bar{X} = L^1((0,1);\mathbb{R})$ of X, prove that there is a subspace Y (linear complement) such that
$$L^1((0,1);\mathbb{R}) = X + Y, \quad Y \cap X = \{0\}.$$
Using the fact that X is dense, show that the projections P_X, P_Y defined by the decomposition $u = P_X u + P_Y u$ are not continuous.
- Extend $\mathbf{M} : X \mapsto X$ to $\overline{\mathbf{M}} : L^1((0,1);\mathbb{R}) \mapsto X$ by
$$\overline{\mathbf{M}}u = \mathbf{M}(P_X u).$$
Show that $\overline{\mathbf{M}}$ is well defined, $\mathcal{D}(\overline{\mathbf{M}}) = L^1((0,1);\mathbb{R})$, but it is not continuous.
- Define $\widehat{\mathbf{M}} : L^1((0,1);\mathbb{R}) \mapsto (L^1((0,1);\mathbb{R}) \times L^1((0,1);\mathbb{R}))$ by
$$\widehat{\mathbf{M}}u = \big(\mathbf{M}(P_X u), P_Y u \big).$$
Show that $N_{\widehat{\mathbf{M}}}$ is closed, the domain is the whole $L^1((0,1);\mathbb{R})$, but $\widehat{\mathbf{M}}$ is not continuous.