EXAMINATION 1

(1) Let $\mathbf{A}$ be a linear operator from the vector space X into itself.
- Assume that X is finite dimensional. Using the decomposition of $\mathbf{A}$ in Jordan blocks, prove that
$$\operatorname{codim}(R_{\mathbf{A}}) - \dim(N_{\mathbf{A}}) = 0.$$
Prove also that if $N = \dim X$, then
$$\dim(N_{\mathbf{A}^N}) = \dim(N_{\mathbf{A}^{N+k}}) \quad \forall k \in \mathbb{N}.$$
- Find an example of an infinite dimensional space X and an operator $\mathbf{A} : x \mapsto X$ such that
$$\operatorname{codim}(R_{\mathbf{A}}) - \dim(N_{\mathbf{A}}) \neq 0$$
and
$$\dim(N_{\mathbf{A}^n}) > \dim(N_{\mathbf{A}^m}) \quad \forall n > m.$$

(2) Consider the space
$$X = \Big\{u = (u_1, u_2, u_3, \dots), \sharp\{n : u_n \neq 0\} < \infty\Big\},$$
i.e. the set of sequences which has a finite number of elements different form 0, with the topology inherited from $\mathbb{R}^{\mathbb{N}}$.
- Find the bounded sets of X and prove that they are compact.
- Show that the countable family of seminorms
$$p_n(u) = |u_n|$$
generate the same topology: give an explicit invariant metric d on X.
- Show that
$$X = \bigcup_{i \in \mathbb{N}} C_i,$$
with C_i closed and with empty interior. Deduce that d is not complete.
- Show directly with an example that d is not complete.

(3) Let X be the same vector space as before, but with the norm
$$\|u\| = \sup_{n \in \mathbb{N}} |u_n|.$$
- Prove that the completion of X is
$$c_0 = \Big\{u = (u_1, u_2, u_3, \dots), \lim_{n \to \infty} u_n = 0\Big\},$$
which is a Banach space.
- Let $\ell \in (c_0)^*$ be a linear continuous functional. Show that
$$\|\ell\|_{(c_0)^*} = \sum_{n=1}^{\infty} |\ell_n|,$$
where $\ell_n = \ell e_n$, with
$$(e_n)_i = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$$
- Show that the vector base $\{e_n\}_{n \in \mathbb{N}}$ is not a linear Hamel base.

(4) Let $B(S)$ be the set of uniformly bounded functions from a set S into $\mathbb{R}$. Consider a subspace Y and a functional $\ell : Y \mapsto \mathbb{R}$ such that
$$y \geq 0 \text{ on } S \quad \Longrightarrow \quad \ell y \geq 0.$$

- Show that if Y contains the function 1 (i.e. the function with constant value 1), then ℓ can be extended to a positive functional on all $B(S)$, i.e.

$$x(t) \geq 0,\ t \in S \quad \Longrightarrow \quad \ell x \geq 0.$$

(Consider the function $p(x) = C \sup_{t\in S} x(t)$, with C has to be chosen.)

- Find an example where such an extension is not possible.

(5) Let X as in exercise (2) with the sup norm

$$\|u\| = \sup_{n\in\mathbb{N}} |u_n|,$$

and consider the two sets

$$A = \Big\{u \in X : u_{2n} = 0, n \in \mathbb{N}\Big\}, \quad B = \Big\{u \in X : u_{2n-1} = 2^n u_n\Big\}.$$

- Show that A, B are closed, and that $A \cap B = \emptyset$, $X = A + B$.
- Prove that P_A, P_B defined by the equation

$$x = P_A x + P_B x, \quad P_A x \in A,\ P_B x \in B,$$

are not continuous. Deduce that X is not complete.

- Consider the norm

$$\|u\|_1 = \sup_{n\in\mathbb{N}} \Big\{2^n |u_{2n}| + |u_{2n-1}|\Big\}.$$

Prove with this norm P_A, P_B are continuous. As a consequence, show that are continuous also in the completion of X with $\|\cdot\|_1$.